Further Remarks on Metrics Comparison and Boundness.

Suppose that d^*, d are well defined metrics on X and that for some $f : \mathbb{R}_+ \to \mathbb{R}_+$ that is strictly increasing we have that

$$[1] \qquad \forall x,y \in X, \quad d^*(x,y) \leq f(d(x,y)) \quad (\text{i.e. } d^* \leq f(d)).$$

Then $\forall x \in X, \varepsilon > 0$ we have that

$$y \in O_d(x,\varepsilon) \Leftrightarrow d(x,y) < \varepsilon \overset{f \text{ s.i.}}{\Longrightarrow} f(d(x,y)) < f(\varepsilon)$$

$$\overset{[1]}{\Longrightarrow} d^*(x,y) < f(\varepsilon) \Rightarrow y \in O_{d^*}(x, f(\varepsilon)).$$

Thereby $\forall x, \varepsilon > 0, \quad O_d(x,\varepsilon) \subseteq O_{d^*}(x, f(\varepsilon)).$

Hence if $X \supseteq A$ is d-bounded then it is also d^*-bounded. Note that what we have already established corresponds to the case that $f(z) = cz$ for $c > 0$.

Exercise. Would it be possible to obtain the result on d^*-boundness by allowing f to be increasing (along perhaps with other properties) that do not imply strict monotonicity?