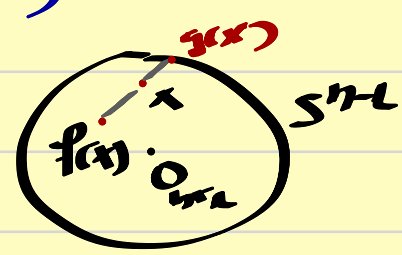
[With correction in green - 27/05/2017]

Brouwer's FPT and Existence of Nash Equilibria

We are now ready to state and prove Brouwer's FPT.

Theorem [BrFPT 1]. $O_{d_2}[0_{n\times 1},1]$ has the fpp. $\square$

Proof. Suppose that it does not. Then $\exists f: X \to X$ continuous, such that $x \neq f(x)$, $\forall x \in X := O_{d_2}[0_{n\times 1},1]$. This implies that $g(x) := f(x) + \lambda(x)(x - f(x))$, for $\lambda(x) > 0$, $\forall x \in X$, and $\lambda(x) = 1$ iff $x \in S^{n-1}$, and continuous is well defined (e.g. when $n=1$, $\lambda(x) = \frac{1}{|x-f(x)|} - \frac{f(x)}{x-f(x)}$ whence $g(x) = f(x) + \frac{x-f(x)}{|x-f(x)|} - f(x)$ $= \frac{x-f(x)}{|x-f(x)|} = \begin{cases} -1, & x<f(x) \\ 1, & x>f(x) \end{cases}$). "Geometrically" g can be defined



by: 1. construct the unique ray pon which x and $f(x)$ lie upon (it is unique since $x \neq f(x)$). 2. $g(x)$ is the unique point of the ray that lies on S^{n-1} and such that x is weakly between $f(x)$ and $g(x)$. (Analytically $\lambda(x)$ is defined as the root that satisfies the previous restriction of $\|f(x) - \lambda(x)(x-f(x))\| = 1$ - try to derive it! Existence and continuity follows by that $x \neq f(x)$, f is continuous and that the roots of 2^{nd} order equations are continuous functions of the parameters. It is obvious that when $\|x\| = 1$, $\lambda(x) = 1$). Hence $g: X \to S^{n-1}$, continuous and such that $g(x) = x$, $x \in S^{n-1}$. I.e. g is a retraction, which is impossible by Borsuk's Lemma. $\square$

Theorem [BrFPT-2]. Suppose that X is a compact, convex non empty subset of $\mathbb{R}^n$. Then X has the fpp. $\square$

Proof. X is compact, thus totally bounded, thus bounded. Hence $\exists \alpha > 0$: $X \subseteq O_{d_2}[0_{n\times 1}, \alpha]$. $O_{d_2}[0_{n\times 1}, \alpha]$ is homeomorphic to

$O_{d_I}[0_{m},1]$ (Consider $f: O_{d_I}[0_{m},1] \to O_{d_I}[0_m,\alpha]$, $f(x)=\alpha x$. It is obviously a bijection – $f^{-1}(y)=\frac{1}{\alpha}y$ and f, f^{-1} are continuous). Hence by Proposition [Hom] and Theorem $O_{d_I}[0_m,\alpha]$ has the fpp. The result follows by Proposition [CC]. $\square$

Remark. We cannot extend the BrFPT to infinite dimensional spaces without further restrictions. E.g. consider $Y=(C([0,1],\mathbb{R}),d_{sup})$ and $X=\{f\in Y: \sup_{x\in[0,1]}|f(x)|\leq 1 \text{ and } f(0)=0, f(1)=1\}$. Prove that X is d_{sup}-complete, d_{sup}-totally bounded and convex. Consider $\Phi: X\to X$, $\Phi(f)=f^2$. (Show that Φ is well defined and d_{sup}/d_{sup}-continuous). Notice that $\forall f\in X$, $\exists x_f\in(0,1): 0<f(x_f)<1$ due to that $f(0)=0$, $f(1)=1$ and f is continuous. If f is a fixed point of Φ, then $f(x)=f^2(x)$ $\forall x\in[0,1] \Rightarrow f(x_f)=f^2(x_f)$ which is impossible. In infinite dimensional cases restrictions such as the ones of BFPT become relevant. $\square$

The Existence of Nash Equilibria

Definition. A finite game $G:=(I, X, V)$ where $I=\{1,2,\dots,n\}$ is the (finite) set of players, $X=\prod_{i\in I} X_i$, where X_i is the strategy set of player $i\in I$, $V=(u_i)_{i\in I}$, $u_i: X\to\mathbb{R}$ is the payoff function of the player $i\in I$. $\square$

Definition. $x^*\in X$ is a Nash equilibrium of G iff

$$V(x^*)=\left(\max_{y\in X_i} u_i((x^*)^{-i},y)\right)_{i\in I}, \text{ where}$$

if $x\in X$, $x^{-i}=(x_1,x_2,\dots,x_{i-1},x_{i+1},\dots,x_n)$, and $u_i(x^{-i},y_i)=u_i(x_1,x_2,\dots,x_{i-1},y,x_{i+1},\dots,x_K)$. The set of Nash equilibria for G is denoted by $N(G)$. $\square$

Assumption. 1. X_i is non empty and compact subset of $\mathbb{R}$, $\forall i \in I$.
2. X_i is convex, $\forall i \in I$.
3. u_i is continuous $\forall i \in I$.
4. $\forall x \in \prod_{i\in I} X_i$, $\underset{y\in X_i}{\operatorname{argmax}}\, u_i(x^{-i}, y)$ is a singleton. $\square$

Remark. 1 implies that X is a compact subset of $\mathbb{R}^k$ (show that either by considering our previous examinations about completeness and boundness w.r.t. finite products, or consider the subsequential definition of compactness). Such a game is called Euclidean Compact.

2. implies that X is convex (why?). Such a game is called convex.

3. and 1. imply that $\forall x \in X$, $\forall i \in I$ $\underset{y\in X_i}{\operatorname{argmax}}\, u_i(x^{-i}, y) \neq \emptyset$ (why?).

4. and 3. and 1. imply that $\forall x \in X$, $\forall x_n \to x$, $\forall i \in I$, $y_{in} := \underset{y\in X_i}{\operatorname{argmax}}\, u_i(x_n^{-i}, y) \to \underset{y\in X_i}{\operatorname{argmax}}\, u_i(x^{-i}, y)$ (why?). 4 is obviously a strong assumption, valid i.e. for example $\forall x \in X$, $\forall i \in I$, $u_i(x^{-i}, y)$ is strictly concave due to 2-3. $\square$

Proposition. Under Assumption the Nash function $b: X \to X$ defined by $b(x) := \left(\underset{y\in X_i}{\operatorname{argmax}}\, u_i(x^{-i}, y)\right)_{i\in I}$ is well defined and continuous. Furthermore $x^* \in \mathcal{N}(G) \iff x^* = b(x^*)$. $\square$

Proof. The first part follows from the previous comment (explain!). For the second one notice that $x^* = b(x^*) \iff x^* = \left(\underset{y\in X_i}{\operatorname{argmax}}\, u_i(x^{-i}, y)\right)_{i\in I}$

$\iff v(x^*) = \left(\max_{y\in X_i} u_i(x^{-i}, y)\right)_{i\in I} \iff x^* \in \mathcal{N}(G)$. $\square$

Remark. If 4. does not hold the b is generally a correspondence. (something like a "multivalued" function).

The following theorem establishes the existence of at least one Nash equilibrium for any finite, Euclidean, compact, convex game that satisfies A.

Theorem. If G satisfies the **Assumption**, then $\mathcal{N}(G) \neq \emptyset$.

Proof. X is a compact and convex, non empty subset of $\mathbb{R}^n$. b is a continuous self map on X, due to the previous proposition. Due to BrFPT-2 b has a fixed point. The result follows from the previous proposition. $\square$

Comment. If A. does not hold an analogous result would follow from an extention of BrFPT-2 for appropriately continuous correspondences, called Kakutani's FPT. $\square$

[The notes are in a state of perpetual correction. They do not substitute the lectures. Please report any typos to stelios@aueb.gr or the course's e-class.]