[With Corrections and complements in Red - 13/5/2017]

# Elements of Metric Fixed Point Theory: Banach FPT

We have now the adequate vocabulary so as to be able to state and prove the fixed point theorem of Banach.

**Definition**. Suppose that $f$ is a self-map on $X$, i.e. $f: X \to X$. $x^* \in X$ is a fixed point of $f$ iff $x^* = f(x^*)$.

**Comments**

1. The issues of the existence of fixed points for a given $f$, the cardinality of the set of fixed points, the detection or the approximation of fixed points, etc, has obvious significance for the solution of (systems of) equations. The fixed point theory is a corpus of results that are occupied with the above given relevant structures and properties for $X$ and/or $f$. The metric fixed point theory assumes at least that $X$ is endowed with a metric.
2. A fixed point of $f$ is an element of $x$ that is left invariant by $f$. E.g. when $f = id_X$ then $\forall x \in X$, $f(x) = x$. If $x^*$ is a fixed point of $f$ then it is also a fixed point of $f^{(n)}: X \to X$ $\forall n > 0$, since: a. the result obviously holds for $n = 1$. b. Suppose that it holds for $n = k$, i.e. $f^{(k)}(x^*) = x^*$. c. Then for $n = k+1$ we have that $f^{(k+1)}(x^*) = f(f^{(k)}(x^*)) = f(x^*) = x^*$.
3. Partially conversely to the previous, if for some $n > 1$, $x^*$ is the unique fixed point of $f^{(n)}: X \to X$, then $f(x^*) = f(f^{(n)}(x^*)) = f^{(n+1)}(x^*) = f^{(n)}(f(x^*))$, hence $f(x^*)$ is a fixed point of $f^{(n)}$. But then uniqueness implies that $x^* = f(x^*)$ and thereby $x^*$ is a fixed point of $f$. It must also be unique since if $x_* \neq x^*$ is also a fixed point of $f$ then it would also, due to the previous be a fixed point of $f^{(n)}$ which is impossible (why?). Such results could be useful for reducing the fixed points issues of $f$ to the ones involving its compositional iterates, which may be easier to solve. $\square$

We will now state and prove Banach's Fixed Point Theorem (BFPT) or contraction mapping FPT.

**Theorem.** (BFPT) Suppose that for some $d$, $(X,d)$ is complete and $f$ is a $d$-contraction. Then $f$ has a unique fixed point, say $x^* \in X$ and we have that $x^* = d\text{-}\lim_{n\to+\infty} f^{(n)}(x)$ $\forall x \in X$. $\square$

**Comments.** BFPT ensures existence and uniqueness for the fixed point. Hence it is a very strong result, obtained however by the consideration of strong properties of $X$ and $f$, i.e. the existence of a metric that completely metrizes $X$ and w.r.t. which $f$ is a contraction. Moreover, it provides by a means of approximation of $x^*$ via the construction of $X$-valued (parts of) sequences. $\square$

**Proof.** The proof is consisted by a list of auxiliary lemmata:

**Lemma $1_{BFPT}$.** If $f$ has a fixed point, then it is unique. $\square$

**Proof of $1_{BFPT}$.** Suppose that $x^*, x_* \in X$ are fixed points of $f$. We have that $d(x^*, x_*) = d(f(x^*), f(x_*)) \leq c\, d(x^*, x_*)$ for $0<c<1$ since $f$ is a contraction. Hence $d(x^*, x_*) \leq c\, d(x^*, x_*) \Leftrightarrow d(x^*, x_*) = 0 \Leftrightarrow x^* = x_*$ since $d$ is a metric. $\square$

**Comment.** The $d$-contractiveness of $f$ implies uniqueness upon existence. Would that remain true if $d$ were a pseudo-metric? $\square$

**Lemma $2_{BFPT}$.** Consider $(x_n)_{n\in\mathbb{N}}$ with $x_n = \begin{cases} x, & n=0 \\ f^{(n)}(x), & n>0 \end{cases}$, for $x$ an arbitrary element of $X$ (!). If $(x_n)_{n\in\mathbb{N}}$ is $d$-convergent then the limit is a fixed point of $f$. $\square$

**Proof of $2_{BFPT}$.** Suppose that $X \ni y = d\text{-}\lim(x_n)$. Then

$$y = d\text{-}\lim_{n\to+\infty} x_n = d\text{-}\lim_{n\to+\infty} f^{(n)}(x) = d\text{-}\lim_{n\to+\infty} f(f^{(n-1)}(x)) = d\text{-}\lim_{n\to+\infty} f(x_{n-1})$$

$$\underset{\substack{f \text{ is a contraction} \\ \text{hence } d/d\text{-continuous}}}{=} f(d\text{-}\lim_{n\to+\infty}(x_{n-1})) = f(y).$$ Hence $y = f(y)$ if $y$ exists. □

**Lemma $3_{BFPT}$.** The $X$-valued sequence $(x_n)_{n\in\mathbb{N}}$ defined above is $d$-Cauchy. □

**Proof of $3_{BFPT}$.** We have first that $\forall n \geq 0$, $d(x_{n+1}, x_n) \leq c^n d(x_1, x_0)$ where $c$ is the contraction coefficient of $f$. This is due to the following inductive argument. When $n=0$ it obviously holds. Suppose that it holds for $n=k$, i.e. $d(x_{k+1}, x_k) \leq c^k d(x_1, x_0)$. Then for $n=k+1$ we have that $d(x_{k+2}, x_{k+1}) = d(f(x_{k+1}), f(x_k)) \underset{\text{contr.}}{\leq} c\, d(x_{k+1}, x_k) \underset{\text{ind. step}}{\leq} c c^k d(x_1, x_0) = c^{k+1} d(x_1, x_0)$.

Now, suppose that $n > m$ and consider $d(x_n, x_m) \leq d(x_n, x_{n-1}) + d(x_{n-1}, x_m) \leq d(x_n, x_{n-1}) + d(x_{n-1}, x_{n-2}) + d(x_{n-2}, x_m) \leq \ldots \leq d(x_n, x_{n-1}) + d(x_{n-1}, x_{n-2}) + \ldots + d(x_{m+2}, x_{m+1}) + d(x_{m+1}, x_m) \underset{\text{previous}}{\leq} (c^{n-1} + c^{n-2} + \ldots + c^{m+1} + c^m)\, d(x_1, x_0)$

$= c^m (c^{n-1-m} + c^{n-2-m} + \ldots + c + 1)\, d(x_1, x_0) \leq c^m \sum_{i=0}^{\infty} c^i d(x_1, x_0) = \frac{c^m}{1-c} d(x_1, x_0)$.

Thereby we have that $d(x_n, x_m) \leq \frac{c^m}{1-c} d(x_1, x_0)$. (a symmetric argument would imply that if $m > n$, $d(x_n, x_m) \leq \frac{c^n}{1-c} d(x_1, x_0)$, hence the previous assumption that $m < n$ can be considered without loss of generality)

Since $c \in (0,1)$, $c^m \to 0$ as $m \to +\infty$, hence, if $\varepsilon > 0$, $\exists\, n(\varepsilon) : \frac{c^m}{1-c} d(x_1, x_0) < \varepsilon$, $\forall m \geq n(\varepsilon)$, and thereby $d(x_n, x_m) < \varepsilon$ $\forall n, m \geq n(\varepsilon)$. □

Since $(x_n)_{n\in\mathbb{N}}$ is $d$-Cauchy, then $y = d\text{-}\lim x_n$ exists in $X$ due to the fact that $(X,d)$ is complete. Due to $2_{BFTP}$, $y$ is a fixed point of $f$, which then must be unique due to $1_{BFTP}$. □

**Comment.** The proof is partially based on the limiting behavior of $(x_m)_{m\in\mathbb{N}}$, which is constructed using an *arbitrary* initial value in $X$! This provides the basis for the construction of approximating algorithms for $x^*$ in quite complex frameworks. Given limitations in computing resources, the choice of $x_0$ could be of importance w.r.t. issues of the speed of convergence, etc.

**Example.** Consider the equation $x=\sin(x)$. Obviously $x=0$ is a solution, yet is it unique? Any solution is a fixed point of $f:\mathbb{R}\to\mathbb{R}$, $f(x)=\sin(x)$. Consider $d=d_I$ (i.e. the usual metric on $\mathbb{R}$). Consider also $g(x):=f^{(2)}(x)=\sin(\sin(x))$. We have that $\sup_{x\in\mathbb{R}}\left|{dy}/{dx}\right| = \sup_{x\in\mathbb{R}}|\cos(\sin x)\cos(x)|<1$. Hence $g$ is a $d_I$-contraction. Furthermore $(\mathbb{R},d_I)$ is complete and thereby due to the BFPT we have that $g=f^{(2)}$ has a unique fixed point. Due to a previous comment (explain) $f$ also has a unique fixed point and thereby $x=0$ is the unique solution to $x=\sin(x)$ (why didn't we use directly $f$ instead of $f^{(2)}$?).

The example framework is explained in the following corollary.

**Corollary 1$_{BFTP}$.** Suppose that $(X,d)$ is complete and $f^{(m)}$ is a $d$-contraction for some $m>0$. Then $f$ has a unique fixed point. $\square$

**Proof.** Due to the BFPT $f^{(m)}$ has a unique fixed point which due to a previous comment is also the unique fixed point of $f$. $\square$

The BFPT does not generally find $x^*$. However using results such as the following, more information on its location could be acquired.

**Corollary $2_{BFTP}$.** Suppose that the assumptions of the BFPT hold and that, $Y$ is a non-empty $d$-closed subset of $X$, and $f(Y)=\{f(x): x\in Y\}\subseteq Y$. Then $x^*\in Y$. □

**Proof.** Since $Y$ is a $d$-closed subset of $X$ it is also $d$-complete (explain). The condition $f(Y)\subseteq Y$ implies that $f$ is a self-map when restricted to $Y$. Since $f$ is a $d$-contraction on $X$ it remains so when restricted to $Y$ (why?). Due to the BFPT $f|_Y$ has a unique fixed point, say $y\in Y$. But $x^*\in X$ is the unique (on $X$) solution of $x=f(x)$. If $x^*\neq y$ then since $y\in Y\subseteq X$ $y=f(y)$ would be a second solution. Contradiction. Hence $x^*=y$. □

**Comment.** If $Y^*$ is another non empty $d$-closed subset of $X$ such that $Y^*\cap Y=\emptyset$, then the previous corollary implies that $\exists x\in Y^*:$ $f(x)\notin Y^*$ (why?) □

In the previous example $Y$ could be selected as $[-1,1]$.

## Application of the BFPT: Blackwell's Lemma and Bellman's Equation

In what follows $X$ will typically be a space of functions while the self map, that in this context will be usually denoted by $\Phi$ (as $f$ will typically denote a point in $X$).

For a non empty set $Y$, consider $X:=B(Y,\mathbb{R})$, $d=d_{sup}$ and define the following partial order on $X$:

$$\text{if } f,g\in X \text{ then } f\geq g \text{ iff } f(x)\geq g(x)\ \forall x\in Y$$

(prove that this is a well-defined partial order)

Suppose that $\Phi$ is a self-map on X, i.e. $\forall f \in B(Y,\mathbb{R})$, $\Phi(f) \in B(Y,\mathbb{R})$.

**Definition.** $\Phi$ is said to satisfy the BL hypotheses iff it satisfies:

1. $f \geq g \Rightarrow \Phi(f) \geq \Phi(g)$ (monotonicity), and
2. $\exists \delta \in (0,1): \forall f \in X, a > 0$ (in what follows $f+a$ is defined by $f(x)+a$, $\forall x \in Y$, whence $f+a \in B(Y,\mathbb{R})$ since $\sup_{x\in Y} |f(x)+a| \leq \sup_{x \in Y} |f(x)| + |a| < +\infty$) $\Phi(f) + \delta a \geq \Phi(f+a)$. $\square$

The following result, called Blackwell's lemma (BL), concerns the consideration of whether $\Phi$ is a $d_{sup}/d_{sup}$-contraction. Such a result, given the fact that $(B(Y,\mathbb{R}), d_{sup})$ is complete would greatly facilitate the application of the BFPT.

**Lemma (Blackwell's lemma).** If $\Phi$ satisfies the BL hypotheses then it is a $d_{sup}/d_{sup}$-contraction. $\square$

**Proof.** Let $f,g \in X$. We have that
$$f(x)-g(x) \leq |f(x)-g(x)| \ \forall x \in Y$$
$$\leq \sup_{x\in Y} |f(x)-g(x)| \ \forall x \in Y$$
$\cap$
$\mathbb{R}$

Hence $g + d_{sup}(f,g) \geq f$ and interchanging $f$ with $g$ we also obtain $f + d_{sup}(f,g) \geq g$. Define $g^* := g + d_{sup}(f,g)$, $g_* := f + d_{sup}(f,g)$, and due to a previous comment we have that $g^*, g_* \in X$ (explain!).

Hence $g^* \geq f$ and $g_* \geq g$. Now due to BL-1
$$\begin{cases} \Phi(g^*) \geq \Phi(f) \\ \Phi(g_*) \geq \Phi(g). \end{cases} \quad (A)$$

Due to BL-2
$$\begin{matrix} \Phi(g) + \delta d_{sup}(f,g) \geq \Phi(g^*) \\ \Phi(f) + \delta d_{sup}(f,g) \geq \Phi(f^*) \end{matrix} \quad (B).$$

Combining (A) and (B) we obtain that due to transitivity
$$\begin{cases} \Phi(g) + \delta d_{sup}(f,g) \geq \Phi(f) \\ \Phi(f) + \delta d_{sup}(f,g) \geq \Phi(g) \end{cases} \iff$$

$$\Phi(f)(x)-\Phi(g)(x)\leq \delta d_{sup}(f,g) \quad \forall x\in Y$$
$$\Phi(g)(x)-\Phi(f)(x)\leq \delta d_{sup}(f,g) \quad \forall x\in Y \iff$$

$$|\Phi(f)(x)-\Phi(g)(x)|\leq \delta d_{sup}(f,g)$$ and since the r.h.s. is independent of $x$ it follows that
$$\sup_{x\in Y}|\Phi(f)(x)-\Phi(g)(x)|\leq \delta d_{sup}(f,g) \iff$$

$$d_{sup}(\Phi(f),\Phi(g))\leq \delta d_{sup}(f,g)$$ and the result follows from the fact that $\delta\in(0,1)$. $\square$

## Some Intermediate results (IR) [See also the optional exercises]

1. If $Y$ endowed with $d_Y$ is $d_Y$-totally bounded and $d_Y$-complete then $(Y,d_Y)$ is called compact (as a topological space). Compactness is equivalent to that if $(x_n)_{n\in\mathbb{N}}: x_n\in Y$ $\forall n\in\mathbb{N}$, then it has a subsequence that $d_Y$-converges inside $Y$.

2. If $(Y,d_Y)$ compact, then $C(Y,\mathbb{R}):=\{f:Y\to\mathbb{R},\ f \text{ is } d_I/d_Y\text{-continuous}\}$ is a $d_{sup}$-closed subset of $B(Y,\mathbb{R})$ and thereby it is $d_{sup}$-complete.

3. If $(Y,d_Y)$ compact and $f\in C(Y,\mathbb{R})$ then $\arg\max_{x\in Y} f(x)\neq\emptyset$.

Hence (review the relevant result - $C(Y,\mathbb{R})\subseteq K^0$). Hence $\sup: C(Y,\mathbb{R})\to\mathbb{R}$ is $d_I/d_{sup}$-continuous. $\square$

## Bellman Equation

Suppose that $(Y,d_Y)$ is compact and let $X=C(Y,\mathbb{R})$, $d=d_{sup}$.

Furthermore let $w: Y\times Y \to \mathbb{R}$ be $d_I/d_{C\Pi_I}$-continuous and $\delta \in (0,1)$.

(Notice that this implies that $\forall x\in Y$ $w(x,\cdot): Y \to \mathbb{R}$ is $d_I/d_Y$-continuous - why?).

**Definition.** In the framework above the functional equation

$$f \in C(Y,\mathbb{R}), \qquad (*) \quad f(x) = \sup_{y\in Y}[w(x,y)+\delta f(y)], \forall x\in Y$$

is called Bellman Equation.

**Lemma.** There exists a unique $f^* \in C(Y,\mathbb{R})$ that satisfies $[*]$.

**Proof.** Consider for $f\in C(Y,\mathbb{R})$ $\Phi(f) := \max_{y\in Y}[w(x,y)+\delta f(y)]$.

$\forall x\in Y$, $w(x,y)+f(y) \in C(Y,\mathbb{R})$ (why?), thereby due to IR-1 (or 3) $\sup_{y\in Y}[w(x,y)+f(y)]\in\mathbb{R}$. Hence $\Phi: C(Y,\mathbb{R}) \to \mathbb{R}^Y$.

Furthermore, due to IR-1 $\forall x\in Y$, $\forall x_n\in Y: x_n\to x$,

$$\sup_{y\in Y}|w(x_n,y)+\delta f(y) - w(x,y)-\delta f(y)| = \sup_{y\in Y}|w(x_n,y)-w(x,y)|$$

$\to 0$ (This is due to the fact that since $(Y,d_Y)$ is compact, $(Y\times Y, d_{\Pi_I})$ is compact, and since $w$ is continuous $w(Y\times Y)$ is a compact subset of $(\mathbb{R},d_I)$. Hence, the $\mathbb{R}$-valued sequence $V_n := \sup_{y\in Y}|w(x_n,y)-w(x,y)|$ lies inside a compact subset of $(\mathbb{R},d_I)$ and thereby has a convergent subsequence say $V_{k_n} := \sup_{y\in Y}|w(x_{k_n},y)-w(x,y)|$ with $(x_{k_n})_{n\in\mathbb{N}}$ being a subsequence of $(x_n)_{n\in\mathbb{N}}$. But $x_n\to x \Rightarrow x_{k_n}\to x$ and thereby due to continuity $V_{k_n}$ must converge to 0. An analogous reduction to the remaining subsequences of $(v_n)_{n\in\mathbb{N}}$ implies the result). Hence due to IR-3, $\sup_{y\in Y}[w(x_n,y)+\delta f(y)] \to \sup_{y\in Y}[w(x,y)+\delta f(y)]$ establishing

that $\Phi: C(Y,\mathbb{R}) \to C(Y,\mathbb{R})$ i.e. $\Phi$ is a self-map. Furthermore, $(C(Y,\mathbb{R}), d_{sup})$ is complete and thereby the result would follow from BFPT if $\Phi$ is proven a $d_{sup}/d_{sup}$-contraction. From Blackwell's lemma it suffices that satisfies the BL-hypotheses. We have that:

BL-1. Suppose that $f,g \in C(Y,\mathbb{R})$ and $f \geq g \Leftrightarrow$

$f(y) \geq g(y)\ \forall y \in Y \overset{\delta>0}{\Rightarrow} w(x,y)+\delta f(y) \geq w(x,y)+\delta g(y)\ \forall x,y \in Y$

$\Rightarrow \max_{y\in Y}[w(x,y)+\delta f(y)] \geq \max_{y\in Y}[w(x,y)+\delta g(y)]\ \forall x \in Y \Leftrightarrow$

$\Phi(f) \geq \Phi(g)$.

BL-2. If $a>0$, then $(\Phi(f)+\delta a)(x) = \max_{y\in Y}[w(x,y)+\delta f(y)]$

$+\delta a = \max_{y\in Y}[w(x,y)+\delta(f+a)] = (\Phi(f+a))(x)\ \forall x \in Y$,

$\Leftrightarrow \Phi(f)+\delta a = \Phi(f+a)$. $\square$

Comment: The issue of the approximation of $f^*$ via the use of the Banach sequences prescribed in the BFPT lies obviously in the focus of vast literatures in particular problems in economic theory.

## Application of the BFPT: Picard's Theorem

Our analysis will lie in the following framework:

- $Y = [a, b] \subseteq \mathbb{R}$, $Y$ is a compact (why?) subset of $(\mathbb{R}, d_1)$.

- $H : [a, b] \times \mathbb{R} \to \mathbb{R}$ is continuous and $\exists \delta > 0 : \forall x \in Y$, $\forall y_1, y_2 \in Y$, $|H(x, y_1) - H(x, y_2)| \leq \delta |y_1 - y_2|$ (i.e. $H$ is uniformly Lipschitz w.r.t. the second argument).
- $x_0 \in Y$, $y_0 \in \mathbb{R}$.

Given the above consider the following boundary value problem BVP:

$$(\Delta) \qquad \begin{cases} f'(x) = H(x, f(x)), \ \forall x \in Y \\ f(x_0) = y_0 \end{cases}$$

**Picard's Theorem.** There exists a unique $f^* \in C(Y, \mathbb{R})$ that solves $(\Delta)$. $\square$

**Proof.** First notice that $f'(x) = H(x, f(x)) \ \forall x \in Y \iff$

$$\int_{x_0}^{x} f'(t)\, dt = \int_{x_0}^{x} \underbrace{H(t, f(t))}_{\text{why exists?}} dt \quad \forall x \in Y \iff$$

$$f(x) - f(x_0) = \int_{x_0}^{x} H(t, f(t))\, dx \quad \forall x \in Y \iff$$

$$f(x) = f(x_0) + \int_{x_0}^{x} H(t, f(t))\, dx \quad \forall x \in Y.$$

Using the condition $y_0 = f(x_0)$ we conclude that $(\Delta)$ is equivalent to:

$$f(x) = y_0 + \int_{x_0}^{x} H(t, f(t))\, dx \quad \forall x \in Y \quad (\Delta').$$

For any $f \in C(Y, \mathbb{R})$ define $(\Phi(f))(x) := y_0 + \int_{x_0}^{x} H(t, f(t))\, dt$.

Obviously (why?) $\forall f \in C(Y,\mathbb{R})$, $\forall x \in Y$, $(\Phi(f))(x) \in \mathbb{R}$, hence $\Phi: C(Y,\mathbb{R}) \to \mathbb{R}^Y$. Due to the properties of the Riemann integral, $\forall x \in Y$, $\forall x_n \to x$ $(\Phi(f))(x_n) = y_0 + \int_{x_0}^{x_n} H(t,f(t))dt \to$

$y_0 + \int_{x_0}^{x} H(t,f(t))dt = (\Phi(f))(x)$. Hence $\Phi: C(Y,\mathbb{R}) \to C(Y,\mathbb{R})$, i.e.

$\Phi$ is a self map on $C(Y,\mathbb{R})$. Furthermore $f^* \in C(Y,\mathbb{R})$ solves $(\Delta)$ iff it solves $(\Delta')$ iff $f^* = \Phi(f^*)$ i.e. $f^*$ is a fixed point of $\Phi$. Hence the theorem will be proven if we can apply the BFPT.

Consider the metric on $C(Y,\mathbb{R})$, $d^*_{sup}$, defined by

$$d^*_{sup}(f,g) := \sup_{x \in Y} e^{-\delta(x-x_0)} |f(x)-g(x)|$$

for any $f,g \in C(Y,\mathbb{R})$ (Prove that $d^*_{sup}$ is a well defined metric). Notice that $\forall f,g$ as above and since $0 < e^{-\delta(b-x_0)} \leq e^{-\delta(x-x_0)} \leq e^{-\delta(a-x_0)}$ $\forall x \in Y$, we have that $\forall x \in X$, $e^{-\delta(b-x_0)}|f(x)-g(x)| \leq e^{-\delta(x-x_0)}|f(x)-g(x)| \leq e^{-\delta(a-x_0)}|f(x)-g(x)|$

(why?) $\Rightarrow e^{-\delta(b-x_0)} \sup_{x\in Y}|f(x)-g(x)| \leq \sup_{x\in Y} e^{-\delta(x-x_0)}|f(x)-g(x)| \leq e^{-\delta(a-x_0)} \sup_{x\in Y}|f(x)-g(x)|$

hence $C^* d_{sup}(f,g) \leq d^*_{sup} \leq C\, d_{sup}(f,g)$ where $C, C^* > 0$ and are independent of $f,g$. Hence, and due to that $C(Y,\mathbb{R})$ is $d_{sup}$-complete, it is also $d^*_{sup}$-complete (explain!).

Hence it suffices that $\Phi$ is $d^*_{sup}/d^*_{sup}$- contraction. We have that $d^*_{sup}(\Phi(f),\Phi(g))$

$$= \sup_{x\in Y} e^{-\delta(x-x_0)} \left| y_0 + \int_{x_0}^{x} H(t,f(t))dt - y_0 - \int_{x_0}^{x} H(t,g(t))dt \right|$$

$$= \sup_{x\in Y} e^{-\delta(x-x_0)} \left| \int_{x_0}^{x} (H(t,f(t)) - H(t,g(t)))dt \right|$$

$$\leq \sup_{x\in Y}\left[e^{-\delta(x-x_0)}\int_{x_0}^{x}|\mathcal{H}(t,f(t))-\mathcal{H}(t,g(t))|\,dt\right]$$

$$\leq \sup_{x\in Y}\left[e^{-\delta(x-x_0)}\int_{x_0}^{x}\delta\,|f(t)-g(t)|\,e^{-\delta(t-x_0)}e^{\delta(t-x_0)}dt\right]$$

$$\leq \sup_{x\in Y}\left[e^{-\delta(x-x_0)}\delta\int_{x_0}^{x}\sup_{t\in Y}\left[|f(t)-g(t)|e^{-\delta(t-x_0)}\right]e^{\delta(t-x_0)}dt\right]$$

$$=\sup_{x\in Y}\left[e^{-\delta(x-x_0)}\sup_{t\in Y}|f(t)-g(t)|e^{-\delta(t-x_0)}\delta\int_{x_0}^{x}e^{\delta(t-x_0)}dt\right]$$

$$\begin{pmatrix} u=\delta(t-x_0) \\ du=\delta dt \end{pmatrix}$$

$$=\sup_{x\in Y}\left[e^{-\delta(x-x_0)}d^*_{\sup}(f,g)\int_0^{\delta(x-x_0)}e^u du\right]$$

$$=\sup_{x\in Y}\left[d^*_{\sup}(f,g)\,e^{-\delta(x-x_0)}\left(e^{\delta(x-x_0)}-1\right)\right]$$

$$=d^*_{\sup}(f,g)\sup_{x\in Y}\left[1-e^{-\delta(x-x_0)}\right]=d^*_{\sup}(f,g)\left(1-e^{-\delta(b-x_0)}\right).$$

Obviously $0\leq 1-e^{-\delta(b-x_0)}<1$ and therefore the result follows. $\square$

Comments:

1. We did not use $d_{\sup}$ because in order for the relevant Lipschitz coefficient to be a contraction, a relation between $x_0, a, b$ and $\delta$ would have to be obeyed (giving rise to the so-called local version of the theorem – try it!). This constitutes of a nice example of the fact that it could be possible to verify the BFPT by suitably choosing $d$.

2. As previously commented the approximation of $f^*$ can be

performed as prescribed by the BFPT, via the use of the so-called Picard iterates. $\square$

**Example.** Consider the following version of $(\Delta)$:

$$\mathcal{H}(x, f(x)) = \sin[\sin(x+f(x))].$$

Since $\sup_{y\in\mathbb{R}} \left| \frac{\partial \mathcal{H}(x,y)}{\partial y} \right| = \sup_{y\in\mathbb{R}} \left| \cos(\sin(x+y))\cos(x+y) \right| < 1$

and hence for $\delta=1$ we have that the Picard's theorem is implementable whatever the choice of $[a,b]$ is. This implies that in this case the Theorem holds for $V=\mathbb{R}$, since it holds, $\forall n\in\mathbb{N}$ for $V_n=[a_n,b_n]$ and $a_m<a_n$, $b_m>b_n$ iff $m>n$, while if $f_m$ solves $(\Delta)$ for $V=V_m$ then $f_m|_{V_n}=f_n$ $\forall n\leq m$, by letting $a_m\to-\infty$, $b_m\to+\infty$ as $m\to+\infty$. $\square$

[The notes are in a state of perpetual correction. They do not substitute the lectures. Please report any typos to stelios@aueb.gr or the course's e-class.]