Further Non Topological Notions in Metric Spaces: Lipschitz Continuity

Remember that topological continuity is "equivalent" to "preservation" of convergence of sequences. Since d-convergent sequences are d-Cauchy, "Cauchyness" is preserved for convergent sequences by continuous functions. Is this true for general Cauchy sequences?

Example. $X=(0,1]$, $Y=\mathbb{R}$, $f:X\to Y$, $f(x)=\frac{1}{x}$, d_X is the usual metric restricted to $(0,1]$ and d_Y is the usual metric. For $(x_n)_{n\in\mathbb{N}}$, $x_n=\frac{1}{n+1}$ $\forall n\in\mathbb{N}$, we have that it is d_X-Cauchy (why?). Furthermore f is d_Y/d_X-continuous (why?). However, $(f(x_n))_{n\in\mathbb{N}}$, with $f(x_n)=\frac{1}{1/n+1}=n+1$ $\forall n\in\mathbb{N}$ is not d_Y-Cauchy. $\square$

Hence "Cauchyness" is not generally preserved by topological continuity. The following stronger notion of continuity, is strong enough so as for "Cauchy-preservation" to hold.

Definition. $f:X\to Y$ is d_Y/d_X-Lipschitz continuous iff $\exists c>0$:

$$\forall x_1,x_2\in X,\quad d_Y(f(x_1),f(x_2))\leq c\,d_X(x_1,x_2).\ \square$$

Obviously when c exists, it is non-unique (why?). $\inf\{c>0: d_Y(f(x_1),f(x_2))\leq c\,d_X(x_1,x_2),\ \forall x_1,x_2\in X\}$ is called the Lipschitz-coefficient of f (why does this exist?)

Lemma. If f is d_Y/d_X-Lipschitz, then it is d_Y/d_X-continuous.

Proof. Let $x\in X$, $(x_n)_{n\in\mathbb{N}}$, $x_n\in X$, $\forall n\in\mathbb{N}$, with $x_n\to x$. Then $d_Y(f(x_n),f(x))\leq c\,d_X(x_n,x)\to 0$. Hence $d_Y(f(x_n),f(x))\to 0$ and the result follows from the arbitrariness of $(x_n)_{n\in\mathbb{N}}, x$. $\square$

Hence Lipschitz-continuity implies topological continuity. Consider the following intermediate result.

Lemma. If $(x_n)_{n\in\mathbb{N}}$ is d_X-Cauchy, then $(f(x_n))_{n\in\mathbb{N}}$ is d_Y-Cauchy.

Proof. Let $\varepsilon>0$. Since $(x_n)_{n\in\mathbb{N}}$ is $(x_n)_{n\in\mathbb{N}}$ Lipschitz, $\exists\, n^*(\varepsilon/C)\in\mathbb{N}$: $d_X(x_n,x_m)<\varepsilon/C$ $\forall n,m\in n^*(\varepsilon/C)$. Since f is d_Y/d_X-Lipschitz, $d_Y(f(x_n),f(x_m))\leq C\,d_X(x_n,x)<\varepsilon\frac{C}{C}=\varepsilon$ $\forall n,m\geq n^*(\varepsilon/C)$. The result follows from that ε is arbitrary. $\square$

Corollary. Topological continuity does not imply Lipschitz continuity.

Proof. Consider the framework of the **Example**. Since f does not preserve Cauchyness it cannot be Lipschitz continuous. $\square$

Hence Lipschitz continuity is indeed stronger than topological continuity.

Lipschitz Continuity and Metrics Comparison.

Lemma. If $\exists\, C_1,C_2,C_1^*,C_2^*>0$: $C_1 d_{X_2}\leq d_{X_1}\leq C_2 d_{X_2}$ and $C_1^* d_{Y_2}\leq d_{Y_1}\leq C_2^* d_{Y_2}$, then f is d_{Y_i}/d_{X_j}-Lipschitz iff it is $d_{Y_{i^*}},d_{X_{j^*}}$-Lipschitz, $i,i^*,j,j^*=1,2$.

Proof. (Provide the details!)

Examples

A. For (X,d) a metric space and $z\in X$, consider $f_z : X\to\mathbb{R}$ defined by $f_z(x):=d(z,x)$. Due to the triangle inequality, if $x_1,x_2\in X$, $|f_z(x_1)-f_z(x_2)|=|d(z,x_1)-d(z,x_2)|\leq d(x_1,x_2)$, hence f_z is d_u/d-Lipschitz for d_u the usual metric on $\mathbb{R}$. $\square$

B. For the following example if A is a real $p\times q$ matrix denote by $\|A\| = \left(\sum_{i=1}^{p}\sum_{j=1}^{q} a_{ij}^2\right)^{1/2}$ ($\|\cdot\|$ is termed as Frobenius matrix norm while when $q=1$ it reduces to the standard Euclidean norm on $\mathbb{R}^p$).
If x is $q\times 1$ real vector the via the representation of the Ax operation as a linear combination of the collumns of A, it is possible to prove that

$$\underbrace{\|Ax\|}_{\text{Euc. norm}} \leq \underbrace{\|A\|}_{\text{Frob. norm}} \underbrace{\|x\|}_{\text{Euc. norm}}$$

Suppose now that $X=\mathbb{R}^q$, $d_X = d_I$, $Y=\mathbb{R}^p$, $d_Y = d_I$, and f is everywhere differentiable with bounded partial derivatives, i.e. the Jacobian function $\frac{\partial f(x)}{\partial x'}$ has bounded w.r.t. x Frobenius norm, that is $\sup_{x\in\mathbb{R}^p}\left\|\frac{\partial f(x)}{\partial x'}\right\| < +\infty$. Then f is d_I/d_I - Lipshitz.

on $\mathbb{R}^q$ / on $\mathbb{R}^p$

This is due to the fact that due to the Mean Value Theorem we have that $\forall x,y\in\mathbb{R}^p$, $\exists x^*\in\mathbb{R}^p$:

$$f(x)-f(y) = \frac{\partial f}{\partial x'}(x^*)(x-y),$$ and taking norms the previous becomes

$$\|f(x)-f(y)\| = \left\|\frac{\partial f}{\partial x'}(x^*)(x-y)\right\|.$$

It is easy to see that:

$$d_I(f(x), f(y)) = \|f(x)-f(y)\|$$

while due to the submultiplicativity above:

$$\left\|\frac{\partial f(x^*)}{\partial x'}(x-y)\right\| \leq \left\|\frac{\partial f(x^*)}{\partial x'}\right\| \|x-y\|$$

$$= \left\|\frac{\partial f(x^*)}{\partial x'}\right\| d_I(x,y)$$

Finally $\left\|\frac{\partial f(x^*)}{\partial x'}\right\| \leq \sup_{x\in\mathbb{R}^p}\left\|\frac{\partial f(x)}{\partial x'}\right\| < +\infty$, combining the above we obtain that $d_I(f(x), f(y)) \leq \sup_{x\in\mathbb{R}^p}\left\|\frac{\partial f(x)}{\partial x'}\right\| d_I(x,y)$

establishing the result.

As a further example consider the case where $f(x) = Ax$. Then $\sup_{x \in \mathbb{R}^p} \left\| \frac{\partial f(x)}{\partial x'} \right\| = \sup_{x \in \mathbb{R}^p} \|A\| = \|A\|$.

It is possible to prove a partial converse of the above, i.e. if $f : \mathbb{R}^p \to \mathbb{R}^q$ is $d_{\mathbb{I}}/d_{\mathbb{I}}$-Lipschitz continuous then f is Lebesgue almost everywhere differentiable with a bounded w.r.t. x Jacobian in the above sense. Hence for example when $p = q = 1$ and $f(x) = e^x$ then since $\sup_{x \in \mathbb{R}} |f'(x)| = \sup_{x \in \mathbb{R}} |e^x| = +\infty$ the function is not Lipschitz continuous. However notice that if f is restricted to any bounded interval then due to the previous the restrictions are Lipschitz continuous (explain the details). This constitutes an example of a locally Lipschitz continuous function. $\square$

C. Suppose that (X, d_X), (Y, d_Y) are again arbitrary, and for $(f_n)_{n \in \mathbb{N}}$, $f, f_n \in B(X, Y)$, $\forall n \in \mathbb{N}$, $d_{\sup}(f_n, f) \to 0$. Furthermore $\forall n \in \mathbb{N}$ f_n is d_Y/d_X-Lipschitz with Lipschitz coefficient C_n, such that $\exists C^* > 0 : \sup_n C_n < C^*$ (such a sequence can be termed as equi-Lipschitz continuous). Then f is also d_Y/d_X-Lipschitz continuous with Lipschitz coefficient $C \leq C^*$. This is due to that if $x_1, x_2 \in X$, $d_Y(f(x_1), f(x_2)) \leq d_Y(f(x_1), f_n(x_1)) + d_Y(f_n(x_1), f_n(x_2)) + d_Y(f_n(x_2), f(x_2)) \leq 2 \sup_{x \in X} d_Y(f_n(x), f(x)) + d_Y(f_n(x_1), f_n(x_2)) \leq$

$$\leq 2 d_{\sup}(f_n, f) + C_n d_X(x_1, x_2) \leq 2 d_{\sup}(f_n, f) + C^* d_X(x_1, x_2).$$

Hence $d_Y(f(x_1), f(x_2)) \leq 2 d_{\sup}(f_n, f) + C^* d_X(x_1, x_2)$.

Since $d_{\sup}(f_n, f) \to 0$, $\forall \varepsilon > 0$ $\exists n^*(\frac{\varepsilon}{2}) : 2 d_{\sup}(f_n, f) < \varepsilon$ $\forall n \geq n^*(\frac{\varepsilon}{2})$

hence $\forall \varepsilon>0$ by choosing $n=n^*(\varepsilon/2)$,

$d_Y(f(x_1), f(x_2)) \leq \varepsilon + C^* d_X(x_1,x_2)$ hence

$d_Y(f(x_1), f(x_2)) \leq C^* d_X(x_1,x_2)$. The result follows from the fact that x_1, x_2 are arbitrary. $\square$

D. Suppose that (Z, d_Z) is also a metric space and $f: X \to Y$, $g: Y \to Z$ are d_Y/d_X- and d_Z/d_Y-Lipschitz continuous with Lipschitz coefficients C_f, C_g respectively. Then $g \circ f : X \to Z$ is d_Z/d_X-Lipschitz continuous with Lipschitz coef. $C_{gf} \leq C_g C_f$. This is due to that, if $x_1, x_2 \in X$, $d_Z(g(f(x_1)), g(f(x_2))) \leq C_g d_Y(f(x_1), f(x_2)) \leq C_g C_f d(x_1,x_2)$. $\square$

E. A function $f: X \to X$ is called a self map. A d_X/d_X-Lipschitz continuous self map is called a contraction mapping iff its Lipschitz constant is less than one. For $n>0$, $f^{(n)} = \underbrace{f \circ f \circ \ldots \circ f}_{n}$. If f is a contraction mapping the $\forall n>0$, $f^{(n)}$ is also a contraction mapping due to **D** (explain!).

[The notes are in a state of perpetual correction. They do not substitute the lectures. Please report any typos to stelios@aueb.gr or the course's e-class.]