Further Non Topological Notions in Metric Spaces: Completeness

A. Cauchy Sequences

Definition. A sequence $(x_n)_{n\in\mathbb{N}}$, with $x_n\in X$ $\forall n\in\mathbb{N}$ is called d-Cauchy iff $\forall \varepsilon>0$, $\exists n(\varepsilon)$: $\forall n,m\geq n(\varepsilon)$, $d(x_n,x_m)<\varepsilon$.

Remark: the previous is essentially a notion of asymptotic concentration. Obviously, the final part of the definition is equivalent to that $x_n\in O_d(x_m,\varepsilon)$, $\forall n,m\geq n^*(\varepsilon)$ and also equivalent to that $x_m\in O_d(x_n,\varepsilon)$, $\forall n,m\geq n^*(\varepsilon)$.

Example. $X=\mathbb{R}$, $d(x,y)=|x-y|$, $x_n=\frac{1}{n+1}$ $\forall n\in\mathbb{N}$. For $\varepsilon>0$, consider the inequality $(*)\ \left|\frac{1}{n+1}-\frac{1}{m+1}\right|<\varepsilon$. Let $m^*=\max(m,n)$, $n^*=\min(m,n)$. Then $\exists k\in\mathbb{N}$: $m^*=n^*+k$ and $(*)\Leftrightarrow \frac{1}{n^*+1}-\frac{1}{m^*+1}<\varepsilon$ $\Leftrightarrow \frac{1}{n^*+1}-\frac{1}{n^*+1+k}<\varepsilon \Leftarrow \frac{(n^*+1)^2}{k}+(n^*+1)<\varepsilon$ and choose $n(\varepsilon)$ as the smallest natural greater than or equal to $\frac{1}{\varepsilon}-1$. Then for any $n,m\geq n(\varepsilon)$ the final inequality is valid hence also $(*)$. Thus $(\frac{1}{n+1})_{n\in\mathbb{N}}$ is d-Cauchy. □

Example. $X=\mathbb{N}^*$, $d^*(n,m):=\left|\frac{1}{n}-\frac{1}{m}\right|$ (show that it is a metric). $x_n=n$, $\forall n\in\mathbb{N}^*$. Using the derivation in the previous example we have that $(n)_{n\in\mathbb{N}}$ is d^*-Cauchy.

Counter-Example. $X=\mathbb{N}$, $d(n,m)=|n-m|$, $x_n=n$ $\forall n\in\mathbb{N}$. Suppose that $(n)_{n\in\mathbb{N}}$ is d-Cauchy. Then for $\varepsilon=1/2$, $\exists n(1/2)$: if $n\geq n(1/2)$, $|x_n-x_{n+1}|<1/2 \Leftrightarrow 1<1/2$. Contradiction. □

The previous two examples imply that the notion depends on the metric.

Lemma. If $(x_n)_{n\in\mathbb{N}}$ is d-convergent then it is d-Cauchy.
Proof. Let $x = d\text{-}\lim(x_n)$. For $\varepsilon>0$, $\exists\, n^*(\varepsilon/2) : d(x,x_n)<\varepsilon/2\ \forall n\geq n^*(\varepsilon/2)$. Then if $n,m\geq n^*(\varepsilon/2)$, $d(x_n,x_m)\leq d(x,x_n)+d(x,x_m) < \varepsilon/2+\varepsilon/2=\varepsilon$. Set $n(\varepsilon):=n^*(\varepsilon/2)$. $\square$

The converse does not hold as the following example implies.

Example. $X=(0,1]$, $d(x,y)=|x-y|$, $x_n=\frac{1}{n+1}$ $\forall n\in\mathbb{N}$. Using the derivation of the first example we can establish that $(x_n)_{n\in\mathbb{N}}$ is d-Cauchy. But $(x_n)_{n\in\mathbb{N}}$ is d-divergent (why?). $\square$

Cauchy Comparison

Lemma. Suppose that $\exists\, c>0 : d_1\leq c d_2$ (as functions). Then if (x_n) is d_2-Cauchy it is also d_1-Cauchy.
Proof. [Exercise]

Corollary. If $\exists\, c_1,c_2>0 :\ c_1 d_2\leq d_1\leq c_2 d_2$ as functions, then $(x_n)_{n\in\mathbb{N}}$ is d_2-Cauchy iff it is d_1-Cauchy.

Remark. The previous condition also implies that $\tau_{d_1}=\tau_{d_2}$ as we have already seen. Beware, it is possible to construct examples for which $\tau_{d_1}=\tau_{d_2}$ (yet the previous condition does not hold) and sequences which are d_1-Cauchy but not d_2-Cauchy, something that implies that "Cauchy-ness" is not a topological notion. $\square$

B. Complete Metric Spaces

Definition. (X,d) is called complete iff every d-Cauchy sequence d-converges inside X.

Example. It is possible to prove that $(\mathbb{R}, d)$ where $d(x,y) = |x-y|$ is complete. The set of irrational numbers is comprised by every possible non-rational limit of of d-Cauchy rational Cauchy sequences that do not have rational d-limits. $\square$

Counter-Example. When $X = (0,1]$ and d as above is an example of an incomplete metric space as a previous example shows.

Remark. It is possible to prove that every (X,d) admits a **completion**, loosely, there exists some $(\hat{X}, \hat{d})$ such that the two spaces have some "tight topological relation".

Example. (X, d_δ) is complete. This is due to the fact that $(x_n)_{n\in\mathbb{N}}$, $x_n \in X, \forall n \in \mathbb{N}$ is Cauchy iff it is eventually constant, i.e. $\exists n^* : x_n = x \ \forall n \geq n^*$. Obviously such a sequence is d_δ-Cauchy (why?). Conversely if a sequence $(y_n)_{n\in\mathbb{N}}$, $y_n \in X \ \forall n \in \mathbb{N}$, then for $\varepsilon \leq 1$ $\exists n(\varepsilon) : d_\delta(x_m, x_n) < 1, \forall n \geq n(\varepsilon)$, but this implies that $x_m = x_n \ \forall n, m \geq n(\varepsilon)$. Hence $(y_n)_{n\in\mathbb{N}}$ is eventually constant. Hence (X, d_δ) is complete. $\square$

Completeness of metric subspaces

Remember that $\emptyset \neq A \subseteq X$ is a metric subspace of (X,d) with d restricted to $A \times A$.

Lemma. If (X,d) is complete then A (as a metric subspace) is complete iff it is closed.

Proof. ($\Rightarrow$) Suppose that $(x_n)_{n\in\mathbb{N}}$, $x_n \in A \ \forall n \in \mathbb{N}$ is d^*-Cauchy (d^* is the restriction of d on $A \times A$). Then $(x_n)_{n\in\mathbb{N}}$ is d-Cauchy since $\forall x, y \in A$, $d^*(x,y) = d(x,y)$ and thereby it d-converges to some $x \in X$ since (X,d) is complete. Since A is closed then $x \in A$.

(⇐) Suppose that $(x_n)_{n\in\mathbb{N}}$ is d^*-Cauchy. Then since (A,d^*) is complete it d^*-converges inside A. □

Completeness and comparison of metrics

Lemma. Suppose that for $c_1,c_2>0$, $c_1 d_2 \le d_1 \le c_2 d_1$ (as functions). Then (X,d_1) is complete, iff (X,d_2) is complete.

Proof. Obvious from the corollary a ve. □

Completeness and finite products

Lemma. Suppose that I is a finite index set and (X_i,d_i) are metric spaces. Then (X,d_f) is d_f-co ete iff (X_i,d_i) is complete $\forall i \in I$, $\forall f = \Pi_{max}, \Pi_I, \Pi_{II}$.

Proof. (Exercise!)

Corollary. $(\mathbb{R}^K, d_f)$ is d_f-complete, $\forall f = max, I, II$.

Proof. (Exercise!)

Completeness and functions spaces

Lemma. Suppose that (E,d_E) is complete and $Y \neq \phi$. Then $(B(Y,E), d_{sup})$ is complete.

Proof. Consider $(f_n)_{n\in\mathbb{N}}$, $f_n \in B(Y,E)$, $\forall n \in \mathbb{N}$ is d_{sup}-Cauchy. For any $x \in Y$, the E valued sequence $(f_n(x))_{n\in\mathbb{N}}$ is d_E-Cauchy since, for $\varepsilon>0$

$$\exists\, n(\varepsilon): \qquad \sup_{x\in Y} d_E(f_n(x), f_m(x)) < \varepsilon, \ \forall n,m \ge n(\varepsilon)$$

and
$$d_E(f_n(x), f_m(x)) \le \sup_{x\in Y} d_E(f_n(x), f_m(y)).$$

Since (E,d_E) is complete $f_n(x)$ d_E-converges to some $e_x \in E$. This holds for any $x \in Y$, and thereby we can define

$f : Y \to E$ by $f(x) := e_x$. If $f \in B(Y,E)$ and $d_{sup}(f_n, f) \to 0$ then we would have proven what is needed.

a. $d_{sup}(f_n, f) \to 0$.

We have that $d_{sup}(f_n, f) := \sup_{x \in Y} d_E(f_n(x), f(x)) \leq$

$\sup_{x \in Y} d_E(f_n(x), d_E\text{-}\lim_{u \to +\infty} f_u(x)) \underset{\text{conti-nuity of } d_E}{=} \sup_{x \in Y} d_E \lim_{u \to +\infty} d_E(f_n(x), f_u(x))$

$\leq \sup_{x \in Y} \sup_{u \geq n} \sup_{x \in Y} d_E(f_n(x), f_u(x)) = \sup_{u \geq n} \sup_{x \in Y} d_E(f_n(x), f_u(x))$

$= \sup_{u \geq n} d_{sup}(f_n, f_u)$. Since by assumption $(f_n)_{n \in \mathbb{N}}$ is d_{sup}-Cauchy, for arbitrary $\varepsilon > 0$, $\exists n(\varepsilon) : d_{sup}(f_n, f_u) < \varepsilon$ $\forall n, u \geq n(\varepsilon) \Rightarrow \sup_{u \geq n} d_{sup}(f_n, f_u) < \varepsilon$ $\forall n \geq n(\varepsilon)$. Thereby $d_{sup}(f_n, f) < \varepsilon$ $\forall n > n(\varepsilon)$, establishing the needed convergence.

b. $f \in B(Y,E)$

We have that $\sup_{x,y} d_E(f(x), f(y)) \leq \sup_{x} d_E(f_n(x), f(x))$

$+ \sup_{y \in Y} d_E(f_n(y), f(y)) + \sup_{x,y \in Y} d_E(f_n(x), f_n(y)) =$

$$\underbrace{2\, d_{sup}(f_n, f)}_{A} + \underbrace{\sup_{x,y \in Y} d_E(f_n(x), f_n(y))}_{B}$$

A converges to zero as $n \to +\infty$ by a., hence for $\varepsilon > 0$ $\exists\, n(\varepsilon)$: $d_{sup}(f_n, f) < \varepsilon$ $\forall n \geq n(\varepsilon)$, hence $\delta := \max_{n < n(\varepsilon)} (\varepsilon, d_{sup}(f_n, f))$

$< +\infty$. Thereby $A < \delta$. For B observe that since $(f_n)_{n \in \mathbb{N}}$ is d_{sup}-convergent due to the previous argument it must also be d-sup bounded since $f_n \in O_{d_{sup}}(f, \delta)$ $\forall n \in \mathbb{N}$ which is equivalent (why?) to that $B < +\infty$. Hence $A + B < +\infty$ and thereby $\sup_{x,y \in Y} d_E(f(x), f(y)) < +\infty$. □

Corollary. $(B(Y, \mathbb{R}), d_{sup})$ is complete.

Proof. Combine the previous lemma with that $\mathbb{R}$ with the usual metric is complete.

Lemma. Suppose that (Y, d_Y) is totally bounded and complete. Let (E, d_E) be complete and consider $C(Y, E) = \{f: Y \to E,$ f is d_E/d_Y-continuous$\}$. Then $(C(Y,E), d_{sup})$ is complete.

Proof. It suffices (why?) that $C(Y,E)$ is a d_{sup}-closed subset of $B(Y,E)$. Show it!

[The notes are in a state of perpetual correction. They do not substitute the lectures. Please report any typos to stelios@aueb.gr or the course's e-class.]