(X,d) has by construction a topology constructed by d, i.e. a consistent collection of open subsets, upon which notions of convergence, continuity, etc, can be defined.

Definition. $A \subseteq X$ is d-open iff, $\forall y \in A$, $\exists \varepsilon_y > 0 : O_d(y,\varepsilon_y) \subseteq A$, or $A = \phi$.

Definition. $A \subseteq X$ is d-closed iff A' is d-open.

Remarks. 1. Any d-open ball is d-open, since if $y \in O_d(x,\varepsilon)$, see $\varepsilon_y : \varepsilon - d(x,y) > 0$. Now, $O_d(y,\varepsilon_y) \subseteq O_d(x,\varepsilon)$ since if $z \in O_d(y,\varepsilon_y)$ then $d(x,z) \leq d(x,y) + d(y,z) < d(x,y) + \varepsilon_y < d(x,y) + \varepsilon - d(x,y) = \varepsilon$.

2. Any d-closed ball is d-closed, since if $y \in O_d'(x,\varepsilon) \Leftrightarrow d(x,y) > \varepsilon$, then $\varepsilon_y := d(x,y) - \varepsilon$, and $O_d(y,\varepsilon_y) \subseteq O_d'(x,\varepsilon)$, since if $z \in O_d(y,\varepsilon_y) \Rightarrow d(x,y) \leq d(x,z) + d(y,z) \Leftrightarrow d(x,z) \geq d(x,y) - d(y,z) > d(x,y) - (d(x,y) - \varepsilon) = \varepsilon \Rightarrow z \in O_d'(x,\varepsilon)$.

3. X is always d-open. Since by definition ϕ is d-open and $X' = \phi$ and thereby also d-closed which also implies that X is also d-closed. Thereby X, ϕ are simultaneously d-open and d-closed (d-clopen).

4. The collection of d-open sets, say τ_d is termed as a topology on X, and the pair (X,τ_d) is termed as a topological space with topology induced by d. (τ_d is also termed metrizable). It is easy to prove that τ_d contains X and ϕ, and it is closed w.r.t. arbitrary unions and finite intersections. The dual notion τ_d' is the collection of d-closed sets which also contains X, ϕ and it is closed

w.r.t. finite unions and arbitrary intersections. E.g. $\tau = \{\emptyset, X\}$ is called indiscrete topology and it is possible to prove that there exists no d that generates it when X contains more than one elements. Hence there may exist topologies that are non metrizable. The Theorem of Nagata-Smirnov metrization theorem characterizes the issue and it is obviously completely out of the scope of the course.

Corollary. τ_d is the collection of all subsets of X if X is finite or $d = d_\delta$.

Proof. In both cases $\{x\}$ is an open d-ball (if $\varepsilon \le \min\limits_{y \in X} d(x,y)$ in the first case, and $\varepsilon \le 1$ in the second). Hence if $A \subseteq X$, then if $A = \emptyset$ the it is by definition open, w if $A \neq \emptyset$, then if $x \in A$ $\{x\} = O_d(x, \varepsilon)$ (for ε as before) $\subseteq A$. $\square$

Hence in both cases, every subset of X is also closed (why?) hence every subset of X is clopen. Such topological spaces are termed as totally disconected.

Topological Comparison of Metrics

Lemma. If for some $c > 0$, $d_2 < c d_1$ (as functions) then $\tau_{d_2} \subseteq \tau_{d_1}$

Proof. If A is d_2-open then $\forall y \in A$, $\exists \varepsilon_y > 0 : O_{d_2}(y, \varepsilon_y) \subseteq A$. But due to a previous exercise (see Ex. 6 in the second set of exercises) $\exists \delta_y > 0 : O_{d_1}(y, \delta_y) \subseteq O_{d_2}(y, \varepsilon_y)$. Hence A is d_1-open. $\square$

Thereby if $\exists c_1, c_2 > 0 : c_1 d_2 \le d_1 \le c_2 d_2$ then $\tau_{d_1} = \tau_{d_2}$. Hence on $\mathbb{R}^k$ $\tau_{d_{max}} = \tau_{d_I} = \tau_{d_{II}}$. Such metrics are termed as topologically equivalent.

Topology on Metric Subspaces

It is not difficult to show that if $\emptyset \neq Y \subseteq X$ and $d^* := d|_{Y\times Y}$, then $\tau_{d^*} = \{A \cap Y : A \in \tau_d\}$.

Hence for example if Y is a finite subset of X, then τ_{d^*} is the set of all subsets of Y (show it!)

Product Metric Spaces

If as previously $\mathcal{I}$ is a finite index set and (X_i, d_i) the factor metric spaces, then $\tau_{d_{\Pi_{max}}} = \tau_{d_{\Pi_1}} = \tau_{d_{\Pi_{11}}}$ (why?) and it contains $A = \prod_{i \in \mathcal{I}} A_i$, for $A_i \in \tau_{d_i}$, $\forall i \in \mathcal{I}$.

Local Open (Closed) Neighborhoods

Definition. For $x \in X$, the collection $\tau_d(x) := \{A \in \tau_d : x \in A\}$ is termed as the collection of open neighborhoods of x. Similarly $\bar{\tau}_d(x) := \{A \in \tau_d', x \in A\}$ is termed as the collection of closed neighborhoods of x.

The previous codify "local information" on topological properties of (X,d) around X. Obviously if $A \in \tau_d(x)$ then $\exists\ \varepsilon_x > 0 : O_d(x, \varepsilon_x) \subseteq A$. Furthermore due to the demuoc First countability of (X,d), $\exists\ n^*_{\varepsilon_x} \in \mathbb{N} : O_d(x, 1/n^*_{\varepsilon_x}) \subseteq O_d(x, \varepsilon_x)$. This implies that there exists a countable subcollection of $\tau_d(x)$ that codifies the aforementioned information, something not nessecarilly true in general topological spaces.

Sequential Convergence

Definition. A sequence of elements of X, $(x_n)_{n\in\mathbb{N}}$ converges w.r.t. d to an element of X, x, termed as the d-limit of $(x_n)_{n\in\mathbb{N}}$, and abbreviated as $x = d\text{-}\lim(x_n)$ or $x = \lim x_n$ or $x_n \to x$, iff $\forall \varepsilon > 0$, $\exists n(\varepsilon) \in \mathbb{N}$, $x_n \in O_d(x,\varepsilon)$ $\forall n \geq n(\varepsilon)$.

Remarks. 1. $x_n \to x$ iff for any open ball centered at x, almost every member of the sequence lies inside the ball, in the sense that at most a finite of elements can be outside, whilst this number can depend on the ball (i.e. the ball radius).

2. The definition can be equivalently be expressed w.r.t. closed balls centered at x, due to the first lemma that we have proven concerning the inclusions between balls. (prove it!)

3. $x_n \to x$ iff $d(x,x_n) \to 0$, since $(d(x,x_n))_{n\in\mathbb{N}}$ is a real sequence which can be considered to have its values in $\mathbb{R}$ equiped with the usual metric.

Lemma. $x_n \to x$, iff $\forall A \in \tau_d(x)$, almost every member of the sequence lies inside A, in the sense that at most a finite number of elements of the sequence are allowed to lie outside A, and this number can depend on A.

Proof. ($\Leftarrow$) Suppose that the condition concerning the members of $\tau_d(x)$ holds. $\forall \varepsilon > 0$, $O_d(x,\varepsilon) \in \tau_d(x)$ (why?) hence $x_n \to x$.

($\Rightarrow$) Suppose that $x_n \to x$. Let $A \in \tau_d(x)$. Since A is d-open and $x \in A$, $\exists \varepsilon > 0 : O_d(x,\varepsilon) \subseteq A$. Since $x_n \to x$ $\exists n(\varepsilon) : x_n \in O_d(x,\varepsilon)$ $\forall n \geq n(\varepsilon)$. Hence $x_n \in A$ $\forall n \geq n(\varepsilon)$. $\square$

Thereby sequential convergence can be equivalently defined w.r.t. $\tau_d(x)$. This is the case for general topological spaces where the notion of open ball is not available. Notice that the previous lemma can be equivalently stated w.r.t. $\tau_d(x)$. (Prove it!)

If a sequence has a limit it is termed d-convergent. Otherwise it is termed d-divergent. The separation property that we have already proven about metric spaces implies that for a convergent sequence the limit is unique.

Lemma. $(x_n)_{n\in\mathbb{N}}$ can have at most one d-limit.

Proof. Suppose that $x_n \to x_1$, $x_n \to x_2$ and $x_1 \neq x_2$. Then $\exists\, \varepsilon_1, \varepsilon_2 > 0$: $O_d(x_1,\varepsilon_1) \cap O_d(x_2,\varepsilon_2) = \emptyset$. Since $x_n \to x_1$ only a finite number of its elements lie in $O_d'(x_1,\varepsilon_1)$ (i.e. the complement in X of $O_d(x_1,\varepsilon_1)$). Due to the previous, $O_d(x_2,\varepsilon_2) \subseteq O_d'(x_1,\varepsilon_1)$ hence only a finite number of elements lie inside $O_d(x_2,\varepsilon_2)$. Contradiction since $x_n \to x_2$. $\square$

I. Why does the previous proof imply that x_n cannot have more than two limits?

II. Would the previous lemma be true in pseudo-metric spaces?

III. Consider X equiped with the indiscrete topology $\tau = \{\emptyset, X\}$. Prove that every sequence converges to every element of X.

Example. $(x_n)_{n\in\mathbb{N}}$ is termed as eventually constant iff $\exists\, n^* \in \mathbb{N}$ such that $x_n = x$, $\forall n \geq n^*$. $x = d\text{-}\lim x_n$ for every possible d since $x_n \in O_d(x,\varepsilon)$ $\forall n \geq n^*, \forall \varepsilon > 0$. $\square$

Example. Consider (X, d_δ). The $(x_n)_{n\in\mathbb{N}}$ is d_δ-convergent iff it is eventually constant. ($\Rightarrow$) see the previous example. ($\Leftarrow$) if $x_n \to x$, then for $\varepsilon = 1$, $\exists n(1)$ such that $x_n \in O_d(x,1)$ $\forall n \in n(1)$. But $O_d(x,1) = \{x\}$. $\square$

Comparison w.r.t. Convergence

Lemma. If d_2-$\lim(x_n) = x$, and $d_1 < c d_2$ for some $c > 0$ (as functions), then d_1-$\lim(x_n) = x$.

Proof. For $\varepsilon > 0$, $\exists \delta > 0$ such that $O_{d_2}(x,\delta) \subseteq O_{d_1}(x,\varepsilon)$. (why?)

Since d_2-$\lim(x_n) = x$, $\exists n_2(\delta)$: $x_n \in O_{d_2}(x,\delta)$ $\forall n \geq n_2(\delta)$. Due to the previous inclusion, and by setting $n_1(\varepsilon) = n_2(\delta)$: $x_n \in O_{d_1}(x,\varepsilon)$, $\forall n \geq n_1(\varepsilon)$. The result follows from the fact that ε is arbitrary. $\square$

Corollary. If for some $c_1, c_2 > 0$, $c_1 d_2 < d_1 < c_2 d_2$ then $x = d_1$-$\lim(x_n) \iff x = d_2$-$\lim(x_n)$. $\square$

Hence in $\mathbb{R}^k$ a sequence converges w.r.t. d_{max} iff it converges w.r.t. d_I to the same limit iff it converges w.r.t. d_{II} to the same limit.

[The notes are in a state of perpetual correction. They do not substitute the lectures. Please report any typos to stelios@aueb.gr or the course's e-class.]