Total Boundness

A refinement of the Boundness notion can be defined if we substitute the assertion of existence of a covering ball with an assertion of existence of a finite cover given any ε>0. In this respect we obtain the notion of a totally bounded subset of a metric space. This among others will later enable an important characterization of compactness.

As usually in what follows, (X,d) is a metric space, A is a subset of X and ε a positive real number.

Definition. An ε-open (d-cover) of A is a collection of (d-) open balls each of ε-radius, say C_ε, such that $A \subseteq \bigcup C_\varepsilon$.

There is no requirement in the previous definition for the location of the centers of the balls inside A. Obviously, for any $\varepsilon>0$, $C_\varepsilon := \{O_d(x,\varepsilon), x\in A\}$ is such a cover that contains as many elements as the number of elements in A. The notion of total boundness concerns the issue of the existence of a finite ε-open cover for each $\varepsilon>0$. It is easy to define the analogous notion of an ε-closed cover.

Definition. A is termed (d-) totally bounded iff $\forall \varepsilon>0$ A admits a finite ε-open cover, i.e. iff $\forall \varepsilon>0, \exists n(\varepsilon)\in\mathbb{N}$: $\exists x_1, x_2, \dots, x_{n(\varepsilon)} \in X$ such that $A \subseteq \bigcup_{i=1}^{n(\varepsilon)} O_d(x_i,\varepsilon)$.

Remark. The definition above is equivalent to the existence of a finite ε-closed cover for any $\varepsilon>0$. This is due to the following facts. Suppose that A is totally bounded and let $\varepsilon>0$. Let $C_\varepsilon = \{O_d(x_i,\varepsilon), x_i\in X, i=1,\dots,n(\varepsilon)\}$ be an ε-open cover, then since $O_d(x_i,\varepsilon)\subseteq O_d[x_i,\varepsilon]$, $\overline{C}_\varepsilon := \{O_d(x_i,\varepsilon), x_i\in X, i=1,\dots,n(\varepsilon)\}$ is obviously a finite ε-closed cover. Conversely, suppose that $\forall\varepsilon>0$ there exists a finite ε-closed cover, and consider $\varepsilon>0$. There exists $n(\varepsilon/2)\in\mathbb{N}$, and $x_1,x_2,\dots,x_{n(\varepsilon/2)}\in X$ so that $\mathcal{C}_{\varepsilon/2}:=\{O_d[x_i,\varepsilon/2],\ x_i\in X,\ i=1,\dots,n(\varepsilon/2)\}$ is a finite $\varepsilon/2$-closed cover. Since $O_d[x_i,\varepsilon/2)\subseteq O_d(x_i,\varepsilon)$ (why?) the $C_\varepsilon:=\{O_d(x_i,\varepsilon),\ x_i \text{ as in } \mathcal{C}_{\varepsilon/2},\ i=1,\dots,n(\varepsilon/2)\}$ is a finite ε-open cover of A.

Consider $N(\varepsilon,A,d):=\min\{\#C_\varepsilon,\ C_\varepsilon \text{ is a } \varepsilon\text{-open } (d\text{-}) \text{ cover of } A\}$. $N(\varepsilon,A,d)$ is termed as the covering number of A corresponding to ε and d. Its natural logarithm is associated to the notion of the (d-) metric entropy of A. Obviously A is (d-) totally bounded iff $N(\varepsilon,A,d)$ is finite $\forall\varepsilon>0$. $\square$

Definition. (X,d) is totally bounded iff X is a (d-) totally bounded subset of itself (consider the analogy with the relevant escalation of notions in the case of boundness).

Corollary. If X is finite then (X,d) is totally bounded.

Proof. $X=\{x_1,x_2,\dots,x_m\}$ for some $m\in\mathbb{N}$. Since $x_i\in O_d(x_i,\varepsilon)$ $\forall i=1,\dots,m$, $\forall\varepsilon>0$, $C_\varepsilon:=\{O_d(x_i,\varepsilon),\ x_i\in X,\ i=1,\dots,m\}$ is a finite ε-open (d-) cover of X. $\square$

Corollary. (X,d_δ) is totally bounded iff X is finite.

Proof. ($\Leftarrow$) Obvious from the previous corollary.

($\Rightarrow$) Suppose that (X,d_δ) is totally bounded and X is infinite. Let $\varepsilon\leq 1$. Remember that $O_{d_\delta}(x,\varepsilon)=\{x\}$ hence the union of any finite collection of such balls is obviously a strict subset of X. Contradiction. $\square$

Remark. Remember that (X,d_δ) is always bounded, yet it is totally bounded iff X is finite hence the two notions do not generally coincide. The following result implies that total boundness is stronger.

Lemma. If A is (d-) totally bounded then it is (d-) bounded.

Proof. For $\varepsilon = 1$, $\exists\, C_1 := \{ O_d(x_i, 1),\ x_i \in X,\ i = 1, \dots, n(1) \}$ that is a finite ε-open (d-) cover of A. Consider $\delta := \sup_{x \in A} \sup_{i=1,\dots,n(1)} d(x_i, x)$

$$\le \sum_{i=1}^{n(1)} \sup_{x \in O_d(x_j, 1)} \sup_{i=1,\dots,n(1)} d(x, x_i) \le \sum_{j=1}^{n(1)} \sup_{x \in O_d(x_j, 1)} \sup_{i=1,\dots,n(1)} \big(d(x, x_j) + d(x_j, x_i) \big)$$

$$= \sum_{i=1}^{n(1)} \sup_{x \in O_d(x_j, 1)} d(x, x_j) + \sup_{i=1,\dots,n(1)} d(x_j, x_i) \le n(1) \cdot 1 + n(1) \max_{j, i = 1, \dots, n(1)} d(x_j, x_i)$$

$$= n(1) \Big(1 + \max_{i,j=1,\dots,n(1)} d(x_i, x_j) \Big) < +\infty$$

since $n(1) \in \mathbb{N}$. Hence δ exists as a non-negative real number. Consider $O_d[x_1, \delta]$. Then if $y \in A \Rightarrow y \in O_d[x_1, \delta]$, since $d(x_1, y) \le \sup_{x \in A} d(x_1, x)$

$\le \sup_{x \in A} \sup_{i=1,\dots,n(1)} d(x_i, x) = \delta$. Hence A is (d-) bounded. □

Total Boundness and Comparison of Metrics

Lemma. Suppose that d_1, d_2 are metrics with which X can be endowed. Suppose that for $c > 0$, $d_1 \le c\, d_2$ (as functions). If A is (d_2-) totally bounded then A is also (d_1-) totally bounded.

Proof. Let $\varepsilon > 0$. Since A is d_2-totally bounded, $\exists\, C^2_{\varepsilon/c} = \{ O_{d_2}(x_i, \varepsilon/c),\ x_i \in X,\ i = 1, \dots, n(\varepsilon/c) \in \mathbb{N} \}$ that is an ε/c-open (d_2-) cover of A. Then $C_\varepsilon := \{ O_{d_1}(x_i, \varepsilon),\ x_i$

as in $C^*_{\varepsilon/c}, i=1,\dots,n(\varepsilon/c)\}$ is an ε-open (d_1-) cover of A. This is due to that if $y \in A \Rightarrow \exists i : y \in O_{d_2}(x_i, \varepsilon/c)$

$\Leftrightarrow d_2(x_i,y) < \varepsilon/c \Rightarrow c d_2(x_i,y) < \varepsilon \Rightarrow d_1(x_i,y) < \varepsilon \Leftrightarrow y \in O_{d_1}(x_i,\varepsilon)$. $\square$

Corollary. If $X = \mathbb{R}^k$, $A \subseteq X$ and A is d_a-totally bounded the it is also d_b-totally bounded for $a,b = 1, \max, 11$.

Proof. Follows by the inequalities provided in the addendum to boundness and the previous lemma. Provide the details.

Corollary. $X = \mathbb{R}^k$, $A \subseteq X$ is d_a-totally bounded iff it is d_a-bounded, $a = 1, \max, 11$.

Proof. ($\Leftarrow$) Follows from the first lemma.

($\Rightarrow$) Prove it. Hint: Show it for $d_{\max}$ and then use the previous corrolary and the analogous result for boundness. To show it for $d_{\max}$, first show that if A is $d_{\max}$-bounded then the covering ball can be chosen to have as center zero. If its radius is $\delta > 0$, and for $\varepsilon > 0$ consider the following: If $\varepsilon \geq \delta$ the covering ball itself can be chosen as the only member of C_ε. If $\varepsilon < \delta$ then consider the vectors

$$\begin{pmatrix}\varepsilon/\delta\\0\\\vdots\\0\end{pmatrix}, \begin{pmatrix}2\varepsilon/\delta\\0\\\vdots\\0\end{pmatrix}, \dots, \begin{pmatrix}m\varepsilon/\delta\\0\\\vdots\\0\end{pmatrix}, \begin{pmatrix}0\\\varepsilon/\delta\\\vdots\\0\end{pmatrix}, \begin{pmatrix}0\\2\varepsilon/\delta\\\vdots\\0\end{pmatrix}, \dots, \begin{pmatrix}0\\m\varepsilon/\delta\\\vdots\\0\end{pmatrix}, \dots, \begin{pmatrix}0\\0\\\vdots\\\varepsilon/\delta\end{pmatrix},$$

$$\begin{pmatrix}0\\0\\\vdots\\2\varepsilon/\delta\end{pmatrix}, \dots, \begin{pmatrix}0\\0\\\vdots\\m\varepsilon/\delta\end{pmatrix},$$ where $m :=$ the largest natural number $\leq \varepsilon/\delta$,

as well as every possible sum of any combination of them consisting of vectors that have no simultaneous non zero elements. The resulting collection of points is finite. Prove that the collection of open $d_{\max}$-balls centered at each member of the collection with ε radius covers A. $\square$

Hence there are examples for which boundness and total boundness coincide. The following is another counter-example.

Example. $X = B(\mathbb{N}, \mathbb{R})$, $d = d_{sup}$, $A = \{(x_n)_{n\in\mathbb{N}} : x_n = \begin{cases} 1, n=i \\ 0, n\neq i \end{cases}$ $i = 0,1,\dots\}$. Prove that A is d_{sup}-bounded yet not d_{sup}-totally bounded. □

Hereditarity

Lemma. If A is (d-)totally bounded and $B \subseteq A$, then B is (d-) totally bounded.

Provide the proof. State and prove the dual (contrapositive).

Products

Lemma. Suppose that I is finite. For (X_i, d_i), $A_i \subseteq X_i$
$\prod_{i\in I} A_i$ is d_a-totally bounded iff A_i is d_i-totally bounded
$\forall i \in I$, $a = \prod_{max}, \prod_{I}, \prod_{1\,1}$.

Proof. Prove it along the lines of the analogous proof for boundness.

Further Remarks:

1. $\ln N(\varepsilon, A, d)$ typically diverges to $+\infty$ as $\varepsilon \downarrow 0^+$. The rate conveys important information on properties of A that are out of the scope of the course.
2. Total boundness, even though it is not a topological property is directly related to compactness. We will later establish the connection, which among others will provide with elabo-

rate alternative arguments for some of the above results.

[Corrections in red]

[The notes are in a state of perpetual correction. They do not substitute the lectures. Please report any typos to stelios@aueb.gr or the course's e-class.]