Addendum on Boundness and Relations Between Metrics

Lemma A. Suppose that d_1, d_2 are both well defined metrics with which X can be endowed and for which $\exists\, c_1 > 0 : \forall x,y \in X \quad d_1(x,y) \le c_1 d_2(x,y)$, i.e.

$$d_1 \le c_1 d_2 \text{ as funcions.}$$

Then if $A \subseteq X$ is d_2-bounded it is also d_1-bounded.

Proof. A d_2-bounded $\Leftrightarrow \exists x \in X, \varepsilon > 0 : A \subseteq O_{d_2}(x,\varepsilon) \Leftrightarrow$

$$\left(y \in A \Rightarrow d_2(x,y) < \varepsilon\right) \Rightarrow \left(y \in A \Rightarrow d_1(x,y) < c_1\varepsilon\right) \Leftrightarrow A \subseteq O_{d_1}(x, c_1\varepsilon)$$

and thereby A is d_1-bounded. $\square$

Exercise: Prove the dual, i.e. if A is not d_1-bounded then it is not also d_2-bounded.

Remember that for $X = \mathbb{R}^k$,

$$d_{max} \le d_2 \le d_{11} \quad (*)$$

Notice that $d_{11}(x,y) = \sum_{i=1}^{k} |x_i - y_i| \le \sum_{i=1}^{k} \max_i |x_i - y_i|$

$= \max_i |x_i - y_i| \sum_{i=1}^{k} 1 = k \max_i |x_i - y_i| = k\, d_{max}(x,y)$,

hence since x,y were arbitrary, we obtain

$$d_{11} \le k\, d_{max} \quad (**).$$

Hence due to $(*) + (**)$ $\quad d_{max} \le d_{11} \le k\, d_{max}$.

Furthermore from (*)+(**), $d_{max} \leq d_I \leq k\, d_{max}$, $d_I \leq d_{II} \leq k\, d_I$, $\frac{1}{k} d_{II} \leq d_{max} \leq d_{II}$, $\frac{1}{k} d_I \leq d_{max} \leq d_I$, and $\frac{1}{k} d_{II} \leq d_I \leq d_{II}$. Using those inequalities with Lemma A we obtain the following result.

Lemma B. $A \subseteq \mathbb{R}^k$ is d_a-bounded iff it is d_b-bounded for $a, b = max, II, I$.

Proof. (Provide the details).

[The notes are in a state of perpetual correction. They do not substitute the lectures. Please report any typos to stelios@aueb.gr or the course's e-class.]