We will develop a basic vocabulary concerning analysis in Metric spaces. We will use the following notions in our constructions.

1. For $X \neq \phi$ a structure on X is a set consisting of subsets of 2^X, relations involving X, etc. E.g. a σ-algebra on X is a subset of 2^X closed w.r.t. countable unions, complements, that contains X. E.g. algebraic operation is a function $X \times X \to X$. Given a structure, the pair $(X, \text{structure})$ is termed as a structured set or a mathematical space. E.g. measurable space, and e.g. a groupoid respectively. A morphism between spaces with the same structure is any function that "respects" the structures.□ [*]

2. If X, Y are sets, then $X \times Y = \{(x,y), x \in X, y \in Y\}$. Notice that $X \times Y \neq Y \times X$ but there exists an 1-1 and onto (i.e. bijection) function $X \times Y \to Y \times X$ (Specify). The product can be generalized to arbitrary number of factors, i.e. if $X_i, i \in I$ are sets then

$$\prod_{i \in I} X_i = \{ (x_i)_{i \in I} , x_i \in X_i, i \in I \},$$

3. A set is termed countable iff it is finite or in bijective correspondence with the set of natural numbers. Typical examples of countably infinite sets are $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{Z}^2$, etc. If a set is not countable it is termed uncountable. Any uncountable set has greater cardinality than any countable one. Typical examples are $\mathbb{R}$, any "continuous" subset of $\mathbb{R}$, $[0,1]$, etc. Countable products of factors with ↯

[*] X is termed the carrier set of the structure.

countable cardinalities are countable, e.g. $\mathbb{Z}^2$, etc. $\square$

4. A subset A of $\mathbb{R}$ is bounded from above, w.r.t. the usual order, iff $\exists\, M \in \mathbb{R}$: $x \leq M$, $\forall x \in A$. M is called an upper bound, and if it exists it is obviously non-unique. The set of upper bounds has a unique minimal element, $\sup A$. When $\sup A \in A$ then it is the maximum element of A. The dual notions are, bounded from below, lower bound, and greatest lower bound or $\inf A$. When $\inf A$ and $\sup A$ exist then A is called bounded. A function $f: X \to \mathbb{R}$ is bounded iff the image $\{f(x), x \in X\}$ is bounded or equivalently if the set $\{|f(x)|, x \in X\}$ has a supremum (why?), or equivalently $\sup_{x \in X} |f(x)| < +\infty$. The set of bounded real functions on X is denoted with $\mathcal{B}(X, \mathbb{R})$. Notice that $\sup_{x \in X}(f(x)+g(x)) \leq \sup_{x \in X}(f(x)) + \sup_{x \in X}(g(x))$, $\inf_{x \in X}(f(x)+g(x)) \geq \inf_{x \in X}(f(x)) + \inf_{x \in X}(g(x))$. Furthermore, if $\lambda \in \mathbb{R}$, $\sup_{x \in X} |\lambda f(x)| = |\lambda| \sup_{x \in X} |f(x)|$, and the previous imply that $\mathcal{B}(X, \mathbb{R})$ is a vector space over $\mathbb{R}$ (Check all the algebraic properties) $\square$

Any other requisite notion will be locally introduced.

Distance Functions and Metric Spaces

Definition. A metric, or distance function on $X \neq \emptyset$, is any $d: X \times X \to \mathbb{R}$ with the following properties: $\forall x, y, z \in X$

i. $d(x,y) \geq 0$ (positivity)
ii. $d(x,y) = 0 \iff x = y$ (separation)
iii. $d(x,y) = d(y,x)$ (symmetry)
iv. $d(x,y) \leq d(x,z) + d(z,y)$ (triangle inequality). □

Remarks:

1. A metric is essentially a way to attribute a notion of distance on any pair of elements of X. The required properties reflect intuitive properties of any such procedure (are they?). □

2. Properties i + ii are collectively termed as positive definiteness of d. If ii does not hold*, then d is term as a pseudo-metric. □ * and $d(x,x) = 0, \forall x \in X$.

Definition. The pair (X,d) is termed as a metric space. □

Hence a metric space is a structured set, where the structure is consisted of a metric. When d is actually a pseudo-metric then (X,d) is termed a pseudo-metric space. Any pseudo-metric space can be transformed to a metric space via an equivalence relation. Notice that when d_1, d_2 are two different metrics on X, then (X,d_1) is different than (X,d_2).

Examples

Example 1. [Discrete Space]

Define d_δ by $d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}, \quad x,y \in X.$

d_δ is obviously a well defined real function that satisfies i, ii and iii. For the triangle inequality we have that for $x,y,z \in X$ then

$$d(x,z)+d(z,y)=\begin{cases} 0 & \text{iff } x=y=z & (A) \\ 1 & \text{iff } x=z \text{ and } z\neq y \\ & \quad \text{or } x\neq z \text{ and } z=y & (B) \\ 2 & \text{iff } x\neq z\neq y & (\Gamma) \end{cases}$$

Notice that $(A) \Rightarrow d(x,y)=0$,

$(B) \Rightarrow d(x,y)=1$,

$(\Gamma) \Rightarrow d(x,y) \leq 1$. (i.e. $=0$ if $x=y$, $=1$ if $x\neq y$)

The metric space (X, d_δ) is called discrete. The previous imply that any non empty set can be considered as a metric space, when endowed with the discrete metric. This and Russel's paradox imply that the collection of metric spaces is not a set! $\square$

Example 2. $\mathbb{R}$ with the usual metric.

For $x,y\in\mathbb{R}$, $d_u(x,y)=|x-y|$. $\square$

Example 3. $\mathbb{R}$ with a metric defined by an injection.

Let $f:\mathbb{R}\to\mathbb{R}$ be 1-1 (injective). Define

d_f by $d_f(x,y)=|f(x)-f(y)|$. d_f is obviously a real function. $d_f(x,y)=0 \Leftrightarrow f(x)=f(y) \underset{1\text{-}1}{\Longleftrightarrow} x=y$, hence separation follows from injectivity. The other properties are obvious (show them!). Hence d_f is a metric. Notice that $d_u=d_f$ for $f(x)=x$ (i.e. the identity on $\mathbb{R}$, $id_{\mathbb{R}}$). For a further example and $f(x)=\exp(x)$, we obtain the metric $d_e(x,y)=|e^x-e^y|$. $\square$

The previous suggest that the set of metrics that can be defined on $X \neq \emptyset$, is always non empty, and it can contain more than one elements. Each different metric defines on X a different metric space, possibly with some common and some differing properties. The resulting taxonomy is essentially one of the objects of study of the theory of metric spaces.

Example 1. [$\mathbb{R}^k, k \geq 1$ and A a p.d. matrix]

Consider $X = \mathbb{R}^k$, $k \geq 1$ and let A be a p.d. $k \times k$ matrix. Define d_A by $d_A(x,y) = \sqrt{(x-y)'A(x-y)}$.

Notice that: $(x-y)'A(x-y) \geq 0 \Leftarrow A$ pd. hence i. is satisfied

$(x-y)'A(x-y) = 0 \Rightarrow x-y=0$ since A is pd. (would this remain true if A was semi-pd.?), hence ii holds.

$(x-y)'A(x-y) = (-1)(y-x)'A(-1)(y-x) = (y-x)'A(y-x)$ hence iii holds.

$$(x-y)'A(x-y) = [(x-z)+(z-y)]'A[(x-z)+(z-y)]$$

$$= [(x-z)'+(z-y)']A[(x-z)+(z-y)] = (x-z)'A(x-z) + (z-y)'A(z-y)$$

$$+ (x-z)'A(z-y) + (z-y)'A(x-z) =$$

$$(x-z)'A(x-z) + (z-y)'A(z-y) + \underbrace{2(x-z)'A(z-y)}_{\text{why?}}, \quad \text{whence}$$

$$d_A^2(x,y) = d_A^2(x,z) + d_A^2(z,y) + 2(x-z)'A(z-y).$$

Due to the Cauchy-Schwarz inequality we have that for any $z_1, z_2 \in \mathbb{R}^k$, $z_1'Az_2 \leq (z_1'Az_1)^{1/2}(z_2'Az_2)^{1/2}$

hence for $z_1=(x-z)$, $z_2=(z-y)$ we have that $(x-z)'A(z-y)\leq d_A(x,z)\,d_A(z,y)$ and thereby

$$d_A^2(x,y)\leq d_A^2(x,z)+d_A^2(z,y)+2d_A(x,z)d_A(z,y)$$
$$=\left(d_A(x,z)+d_A(z,y)\right)^2$$ and thereby iv follows

for d_A due to the monotonicity of $x\to\sqrt{x}$.

Notice that for $A=I_{k\times k}$ we obtain the usual Euclidean metric on $\mathbb{R}^k$, $d_I(x,y)=\sqrt{(x-y)'(x-y)}$ $=\sqrt{\sum_{i=1}^{k}(x_i-y_i)^2}$. When $k=1$, then $A=\alpha>0$ and $d_A(x,y)=\sqrt{\alpha(x-y)^2}=\sqrt{\alpha}\,|x-y|=|\sqrt{\alpha}x-\sqrt{\alpha}y|$, and thereby d_A can be perceived as an extension w.r.t. k of d_f for $f(x)=\sqrt{\alpha}\,x$. $\square$

Example 5. [$\mathbb{R}^k$, $k\geq 1$ with d_{11}]

Again for $X=\mathbb{R}^k$, $k\geq 1$, define d_{11} by $d_{11}(x,y)=\sum_{i=1}^{k}|x_i-y_i|$. Since $|x_i-y_i|\geq 0$ with equality iff $x_i=y_i$, i and ii follow easily. iii, follows from that $|x_i-y_i|=|y_i-x_i|$ while

$$d_{11}(x,y)=\sum_{i=1}^{k}|x_i-z_i+z_i-y_i|\leq\sum_{i=1}^{k}\left(|x_i-z_i|+|z_i-y_i|\right)$$

$= \sum_{i=1}^{k} |x_i - z_i| + \sum_{i=1}^{k} |z_i - y_i| = d_{11}(x,z) + d_{11}(z,y)$ hence iii holds. For $k=1$, $d_u = d_{11}$ hence d_{11} can also be perceived as an extension of d_u w.r.t. k. □

Lemma 1. $d_I(x,y) \leq d_{11}(x,y)$, $\forall x,y \in \mathbb{R}^k$.

Proof. $d_I^2(x,y) = \sum_{i=1}^{k} (x_i - y_i)^2 \leq \left(\sum_{i=1}^{k} |x_i - y_i|\right)^2$ and the result follows from the monotonicity of $x \to \sqrt{x}$. □

Such relations between metrics on the same carrier may imply relations between the properties of the different metric spaces as we will later in the course examine.

Example 6. [$\mathbb{R}^k, k \geq 1$ with d_{max}]

Again for $X = \mathbb{R}^k$, $k \geq 1$, define d_{max} by

$d_{max}(x,y) = \max_{i=1,\dots,k} |x_i - y_i|$. d_{max} is a well defined real function since k is finite (why?). Furthermore

$\max |x_i - y_i| \geq 0$ with equality iff $|x_i - y_i| = 0$ $\forall i = 1, \dots, k$. Hence i and ii hold. iii holds since $|x_i - y_i| = |y_i - x_i|$, while

$$
\begin{aligned}
d_{max}(x,y) &= \max |(x_i - z_i) + (z_i - y_i)| \\
&\leq \max [|x_i - z_i| + |z_i - y_i|] \quad \text{(why?)} \\
&\leq \max |x_i - z_i| + \max |z_i - y_i| \quad \text{(why?)} \\
&= d_{max}(x,z) + d_{max}(z,y)
\end{aligned}
$$

hence iii holds. Notice again that when $k=1$ $d_u = d_{max}$, hence the d_{max} can be perceived as another extension of d_u w.r.t. k. $\square$

Lemma 2. $d_{max}(x,y) \le d_I(x,y)$, $\forall x,y \in \mathbb{R}^k$.

Proof. $d^2_{max}(x,y) = \left(\max |x_i - y_i|\right)^2 \overset{mon.}{=} \max (x_i-y_i)^2$

$$\le \sum_{i=1}^{k} (x_i-y_i)^2$$ and the

result follows from the monotonicity of $x \to \sqrt{x}$. $\square$

Putting together the previous lemmata we obtain

$$d_{max} \le d_I \le d_{II}.$$

Remark. Remember that a real sequence is a function $\mathbb{N} \to \mathbb{R}$. Denote the space of real valued sequences by $\mathbb{R}^{\mathbb{N}}$, it is a vector space over $\mathbb{R}$.

Examples of linear subspaces are $B(\mathbb{N},\mathbb{R})$,

$AS = \{(x_n)_{n\in\mathbb{N}}, \sum_{i=0}^{\infty} |x_n| < +\infty\}$, $SS = \{(x_n)_{n\in\mathbb{N}}, \sum_{i=0}^{\infty} x_i^2 < +\infty\}$

Notice that $SS \subseteq AS \subseteq B(\mathbb{N},\mathbb{R})$. Notice that d_{max} can be extended to $B(\mathbb{N},\mathbb{R})$, d_I to SS

and d_{11} to AS in the obvious way. Provide the details! $\square$

Example 7. [$X=\mathbb{Q}$, p-adic Metric]

Let $X=\mathbb{Q}$. Given a prime number p it can be proven that if $q\in\mathbb{Q}^*$, $\exists! k\in\mathbb{Z}$ and $r\in\mathbb{Z}, s\in\mathbb{N}$ such that $q=p^k\frac{r}{s}$, and $p\nmid r$ and $p\nmid s$. Notice that

$$\text{if } \mathbb{Q}^*\ni x-y \text{ hence } x-y=p^k\frac{r}{s}, y-x=p^k\frac{(-r)}{s}$$

Define d_p by $d_p(x,y)=\begin{cases} p^{-k} & \text{if } x-y\neq 0 \\ 0 & \text{if } x-y=0\end{cases}$,

for k the unique exponent in the p-adic representation of $x-y$. We have that it is well defined real function that satisfies i, and ii. iii follows from the previous. Finally, since if $x\neq y$, hence

$$p^k\frac{r}{s}=x-y=(x-z)+(z-y)=\begin{cases} p^{k_1}\frac{r_1}{s_1}+p^{k_2}\frac{r_2}{s_2}, & \text{if } x\neq z\neq y \\ p^k\frac{r}{s}, & \text{if } x=z \text{ or } z=y\end{cases}$$

and thereby $p^k\frac{r}{s}\geq \min\{p^{k_1},p^{k_2}\}\left[\frac{r_1}{s_1}+\frac{r_2}{s_2}\right]$

$$\Rightarrow d_p(x,y)\leq \max\{p^{-k_1},p^{-k_2}\}\leq p^{-k_1}+p^{-k_2}$$

$$=d_p(x,z)+d_p(z,y). \ \square$$

Example 8. [$Y\neq\emptyset$, $X=B(X,\mathbb{R})$ with uniform metric]

Let $Y\neq\emptyset$ and consider $B(X,\mathbb{R})$ ($\neq\emptyset$, why?).

Define d_{sup} by $d_{sup}(f,g)=\sup_{x\in Y}|f(x)-g(x)|$,

$f, g \in B(Y, \mathbb{R})$.

d_{sup} is termed as the uniform metric. d_p is a well defined real function since $d_{sup}(f,g) \leq \sup_{x \in Y} |f(x)| + \sup_{x \in Y} |g(x)| < +\infty$ since $f, g \in B(Y, \mathbb{R})$. Obviously i and iii hold since $\sup_{x \in Y} |f(x) - g(x)| = \sup_{x \in Y} |g(x) - f(x)| \geqslant 0$. Furthermore, $d_{sup}(f,g) = 0 \Rightarrow |f(x) - g(x)| = 0$ $\forall x \in Y \Leftrightarrow f(x) = g(x)$ $\forall x \in Y \Leftrightarrow f = g$, hence ii holds. Finally, if $l \in B(Y, \mathbb{R})$, then $d_{sup}(f,g) = \sup_{x \in Y} |f(x) - g(x)|$

$= \sup_{x \in Y} |f(x) - l(x) + l(x) - g(x)| \leq \sup_{x \in Y} \left[|f(x) - l(x)| + |l(x) - g(x)| \right]$

$\leq \sup_{x \in Y} |f(x) - l(x)| + \sup_{x \in Y} |l(x) - g(x)| = d_{sup}(f, l) + d_{sup}(l, g)$,

hence iii holds. Notice that d_{sup} generalizes d_{max} (see also the previous remark) since any $x \in \mathbb{R}^k$ can be perceived as a function $\{1, 2, \ldots, k\} \to \mathbb{R}$ which is obviously bounded, since it has a finite image (provide the details).

Metric Subspaces

If (X, d) is a metric space and $\emptyset \neq X^* \subseteq X$ then (X^*, d^*) with $d^* = d|_{X^* \times X^*}$ is called a metric subspace of (X,d). An obvious question that we will partially examine concerns the hereditarity of the properties of (X,d) to its metric subspaces.

Product Metric Spaces

Suppose that (X_i, d_i), $i\in I$ is a finite collection of metric spaces. Consider $X=\prod_{i\in I} X_i$. Suppose that $x,y,z\in X$, $x_i, y_i, z_i \in X_i$.

Example P_1.

Define $d_{\Pi_{11}}$ by $d_{\Pi_{11}}(x,y)=\sum_{i\in I} d_i(x_i,y_i)$.

Prove that it is a metric. Notice that $d_{\Pi_{11}}=d_{11}$ for $X_i=\mathbb{R}$, $I=\{1,\dots,k\}$, $d_i(x_i,y_i)=|x_i-y_i|$. $\square$

Example P_2

Define d_{Π_I} by $d_{\Pi_I}(x,y)=\sqrt{\sum_{i\in I} d_i^2(x_i,y_i)}$.

Prove that it is a metric. Notice that $d_{\Pi_I}=d_I$ in the aforementioned framework. $\square$

Example P_3

Define $d_{\Pi_{max}}$ by $d_{\Pi_{max}}(x,y)=\max_{i\in I} d_i(x_i,y_i)$. Prove that it is a metric. Notice that $d_{\Pi_{max}} = d_{max}$ in the previous frameworks.

Lemma 3. $d_{\Pi_{max}}(x,y) \leq d_{\Pi_I}(x,y) \leq d_{\Pi_{II}}(x,y)$, $\forall x,y \in X$.

Proof. Similar to the proofs of lemmata 1 and 2.
(Provide the details)

[The notes are in a state of perpetual correction. They do not substitute the lectures.
Please report any typos to stelios@aueb.gr or the course's e-class.]