Optional Exercises - Exercise 6

Hint: Evaluate the expectation: $E\left[z_0^2 \exp\left[b^i(a|z_0| + \gamma z_0)\right]\right]$

$$E\left[z^2\exp\left[b^i(a|z|+\gamma z)\right]\right] = \int_{-\infty}^{\infty} z^2 \exp\left[b^i(a|z|+\gamma z)\right]\varphi(z)\,dz \xlongequal[\kappa_2=\gamma b^i]{\kappa_1=ab^i}$$

$$\int_{-\infty}^{\infty} z^2 \exp(\kappa_1|z|+\kappa_2 z)\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{z^2}{2}\right)dz =$$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} z^2\exp\left(\kappa_1|z|+\kappa_2 z-\frac{z^2}{2}\right)dz =$$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} z^2\exp\left(-\kappa_1 z+\kappa_2 z-\frac{z^2}{2}\right)dz + \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} z^2\exp\left(\kappa_1 z+\kappa_2 z-\frac{z^2}{2}\right)dz =$$

(For the first integral set $u=-z$, $du=-dz$, when $z\to 0,\ u\to 0$; $z\to-\infty,\ u\to+\infty$.)

$$-\frac{1}{\sqrt{2\pi}}\int_{\infty}^{0} u^2\exp\left[(\kappa_2-\kappa_1)(-u)-\frac{u^2}{2}\right]dz + \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} z^2\exp\left[(\kappa_2+\kappa_1)z-\frac{z^2}{2}\right]dz =$$

$$\frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} u^2\exp\left[(\kappa_1-\kappa_2)u-\frac{u^2}{2}\right]dz + \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} z^2\exp\left[(\kappa_2+\kappa_1)z-\frac{z^2}{2}\right]dz$$

(Using formula 3.462-7, from Gradshteyn and Ryzhik (1994). Table of Integrals, Series and Products)

$$\frac{1}{\sqrt{2\pi}}\left\{-\frac{\frac{\kappa_2-\kappa_1}{2}}{2\left(\frac12\right)^2}+\sqrt{\frac{\pi}{\left(\frac12\right)^5}}\,\frac{2\left(\frac{\kappa_2-\kappa_1}{2}\right)^2+\frac12}{4}\exp\left[\frac{\left(\frac{\kappa_2-\kappa_1}{2}\right)^2}{\frac12}\right]\left[1-\Phi\left(\frac{\frac{\kappa_2-\kappa_1}{2}}{\sqrt{\frac12}}\right)\right]\right\}$$

$$+\frac{1}{\sqrt{2\pi}}\left\{-\frac{\frac{-(\kappa_2+\kappa_1)}{2}}{2\left(\frac12\right)^2}+\sqrt{\frac{\pi}{\left(\frac12\right)^5}}\,\frac{2\left(\frac{-(\kappa_2+\kappa_1)}{2}\right)^2+\frac12}{4}\exp\left[\frac{\left(\frac{-(\kappa_2+\kappa_1)}{2}\right)^2}{\frac12}\right]\left[1-\Phi\left(\frac{\frac{-(\kappa_2+\kappa_1)}{2}}{\sqrt{\frac12}}\right)\right]\right\}=$$

$$\frac{1}{\sqrt{2\pi}}\left\{-(\kappa_2-\kappa_1)+\sqrt{2\pi}\,\frac12\left[(\kappa_2-\kappa_1)^2+1\right]\exp\left[\frac{(\kappa_2-\kappa_1)^2}{2}\right]\left[1-\Phi\left(\frac{\kappa_2-\kappa_1}{2\sqrt2}\right)\right]\right.$$

$$\left.+(\kappa_2+\kappa_1)+\sqrt{2\pi}\,\frac12\left[(\kappa_2+\kappa_1)^2+1\right]\exp\left[\frac{(\kappa_2+\kappa_1)^2}{2}\right]\left[1-\Phi\left(-\frac{\kappa_2+\kappa_1}{2\sqrt2}\right)\right]\right\}=$$

$$\frac{1}{\sqrt{2\pi}}\left[(K_2+K_1)-(K_2-K_1)\right]+\frac{1}{2}\left[(K_2-K_1)^2+1\right]\exp\left[\frac{(K_2-K_1)^2}{2}\right]\Phi\left(-\frac{K_2-K_1}{2\sqrt{2}}\right)$$

$$+\frac{1}{2}\left[(K_2+K_1)^2+1\right]\exp\left[\frac{(K_2+K_1)^2}{2}\right]\Phi\left(\frac{K_2+K_1}{2\sqrt{2}}\right) =$$

$$\frac{1}{\sqrt{2\pi}}\left[b^i(a+j)+b^i(a-j)\right]+\frac{1}{2}\left[b^{2i}(j-a)^2+1\right]\exp\left[\frac{b^{2i}(j-a)^2}{2}\right]\Phi\left(\frac{b^i(a-j)}{2\sqrt{2}}\right)$$

$$+\frac{1}{2}\left[b^{2i}(a+j)^2+1\right]\exp\left[\frac{b^{2i}(a+j)^2}{2}\right]\Phi\left(\frac{b^i(a+j)}{2\sqrt{2}}\right) =$$

$$\frac{1}{\sqrt{2\pi}}(A_i+B_i)+\frac{1}{2}(A_i^2+1)\exp\left(\frac{A_i^2}{2}\right)\Phi\left(\frac{A_i}{2\sqrt{2}}\right)$$

$$+\frac{1}{2}(B_i^2+1)\exp\left(\frac{B_i^2}{2}\right)\Phi\left(\frac{B_i}{2\sqrt{2}}\right)$$

where $A_i = b^i(a-j)$

$B_i = b^i(a+j)$.

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