MSc Course: Econometrics 2 - Optional Exercises-2018

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- The following set of exercises is optional.
- The proposed solutions can only be submited at the day of the final exam of June's 2018 exam period. Those should be written in a booklet that contains in its first page your identification data and the number of the written pages. You are to deliver this only at the time that you deliver your exam paper. In the case that you do, you must also make a note in the first page of your exam paper that you are delivering optional exercises. You must make sure that the exam supervisor first inspects the correctness of the information in the first page of the booklet, second signs the first page of the booklet and the note on your exam paper and then encloses the booklet in your exam paper and accepts them.
- The proposed solutions will be graded according to the relevant anouncement.
- The exercises' grade will be valid for any repeater to June's 2018 exam period, exam.
- The instructor retains the right to ask for clarifications on the proposed solutions.

Exercise 1. Find the linear process solution of an ARMA $(1, 2)$ recursion assuming the UDC. Derive the autocovariance and autocorrelation functions.

Exercise 2. For an arbitrary invertible MA (2) process derive the sequence $\left(\rho_j\right)_{j\in\mathbb{N}}$ appearing in the representation $\varepsilon_t = \sum_{j=0}^{\infty} \rho_j y_{t-j}$.

Exercise 3. Consider the process $(y_t)_{t\in\mathbb{Z}}$, defined by

$$y_t = (1 + \theta_1 L)\left[u_t \cos(t) + v_t \sin(\lambda t)\right],\ t \in \mathbb{Z},\ \lambda \in \{-1, 0, 1\},\ \theta_1 \in \mathbb{R},$$

where $\begin{pmatrix} u_t \\ v_t \end{pmatrix} \sim N\left(\mathbf{0}_{2\times 1}, \mathsf{Id}_{2\times 2}\right),\ t \in \mathbb{Z}$, $(v_t)_{t\in\mathbb{Z}}$ and $(u_t)_{t\in\mathbb{Z}}$ are i.i.d. and mutually independent. Is the process strongly and/or weakly stationary? In the case that it is weakly stationary derive its autocovariance function.

Exercise 4. In the context of an MA(1) process w.r.t. an i.i.d. zero mean and unit variance white noise process with parameter value $\theta_{1_0} \notin [-1, 1]$, derive the indirect inference estimator based on $\beta_T = \frac{\sum_{t=1}^{T} x_t x_{t-1}}{\sum_{t=1}^{T} x_{t-1}^2}$ as well as its limit theory.

Exercise 5. Consider the process $(y_t)_{t\in\mathbb{Z}}$, defined by $y_t = \beta_0 y_{t-1} + \varepsilon_t$, $\varepsilon_t = z_t\sqrt{h_t}$, $t \in \mathbb{Z}$, $|\beta_0| < 1$, where $(z_t)_{t\in\mathbb{Z}}$ is i.i.d., with $\mathbb{E}\left(z_0^3\right) = 0$ and $\mathbb{E}\left(z_0^4\right) = \kappa < +\infty$, and $(\varepsilon_t)_{t\in\mathbb{Z}}$ is a GARCH $(1,1)$ process such that $\left(\varepsilon_t^2\right)_{t\in\mathbb{Z}}$ has the relevant ARMA $(1,1)$ representation w.r.t. the s.m.d. process $(v_t)_{t\in\mathbb{Z}}$, $v_t := \left(z_t^2 - 1\right) h_t$, $t \in \mathbb{Z}$. Derive the autocovariance function of the $\left(\varepsilon_t^2\right)_{t\in\mathbb{Z}}$ process. Derive the limit in distribution of $\sqrt{T}\left(\beta_T - \beta_0\right)$ as $T \to \infty$, where β_T is the OLSE of β_0 in the context of the relevant linear model.

Exercise 6. Consider the process $(y_t)_{t\in\mathbb{Z}}$, defined by $y_t = z_t\sqrt{h_t}$, $t \in \mathbb{Z}$, where the process $(z_t)_{t\in\mathbb{Z}}$ is i.i.d., with $z_0 \sim N(0,1)$, and $(y_t)_{t\in\mathbb{Z}}$ is a stationary and ergodic EGARCH $(1,1)$ process. Derive γ_k for any $k \geq 0$ for the $\left(y_t^2\right)_{t\in\mathbb{Z}}$ process.

Exercise 7. For the process defined in Exercise 5, show that

$$\mathcal{W}_T\left(\beta_0\right) := \left(\beta_T - \beta_0\right)^2 \frac{\left[\sum_{t=1}^T y_{t-1}^2\right]^2}{\sum_{t=1}^T e_t^2 y_{t-1}^2} \overset{d}{\to} \chi_1^2,$$

as $T \to \infty$, where $e_t := y_t - \beta_T y_{t-1}$. Given that

$$\frac{1}{T}\left|\sum_{t=1}^T e_t^2 y_{t-1}^2 - \sum_{t=1}^T \varepsilon_t^2 y_{t-1}^2\right| \leq \frac{2\left|\beta_T - \beta_0\right|}{T}\sum_{t=1}^T \varepsilon_t y_{t-1}^3 + \frac{\left|\beta_T - \beta_0\right|^2}{T}\sum_{t=1}^T y_{t-1}^4,$$

show that the r.h.s. converges a.s. to zero due to the strong consistency of the OLSE, the CMT, strong stationarity and ergodicity for $(y_t)_{t\in\mathbb{Z}}$, Birkhoff's LLN and the fact that $\mathbb{E}\left(y_0^4\right) < +\infty$. Conclude that for $|\beta^\star| < 1$, significance level $\alpha \in (0,1)$, and the hypothesis structure

$$\begin{cases} \mathbb{H}_0 : \beta_0 = \beta^\star \\ \mathbb{H}_1 : \beta_0 \neq \beta^\star \end{cases},$$

the (Wald-type) testing procedure that rejects $\mathbb{H}_0$ iff $\mathcal{W}_T\left(\beta^\star\right) > q_{\chi_1^2}(1-\alpha)$, where $q_{\chi_1^2}(1-\alpha)$ is the $1-\alpha$ quantile of the χ_1^2 distribution is asymptotically exact, i.e.

$$\lim_{T\to\infty} \mathbb{P}\left(\mathcal{W}_T\left(\beta^\star\right) > q_{\chi_1^2}(1-\alpha) / \mathbb{H}_0\right) = \alpha.$$

Exercise 8. Consider the process $(y_t)_{t\in\mathbb{N}}$, defined by $y_t = \beta_0 y_{t-1} + \varepsilon_t$, $t \in \mathbb{Z}$, where $y_0 = 0$, $\beta_0 = 1$, and the process $(\epsilon_t)_{t\in\mathbb{N}}$ is i.i.d. with mean 0 and variance $\sigma^2 > 0$. Furthermore, let β_T denote the OLSE for β_0 in the context of the relevant linear model, and define $X_T(r) = \frac{1}{T}\sum_{t=1}^{[Tr]} \varepsilon_t$, $0 \leq r \leq 1$, where $[x]$ denotes the integer part of a real number x. Given the FCLT, $\sqrt{T}X_T(\cdot) \overset{d}{\to} \sigma W(\cdot)$ as $T \to \infty$, where $W(\cdot)$ denotes a standard Wiener process on $[0,1]$:

1. Show that $\left(\frac{1}{T^2}\sum_{t=1}^T y_{t-1}^2, \frac{1}{T}\sum_{t=1}^T \varepsilon_t y_{t-1}\right) \overset{d}{\to} \sigma^2\left(\int_0^1 W(r)^2 dr, \frac{1}{2}\left(W^2(1) - 1\right)\right)$, as $T \to \infty$.
2. Derive the limiting distribution of $T\left(\beta_T - 1\right)$ as $T \to \infty$.