The MA(1) process
Stationarity Ergodicity and Estimation

Remember that, a linear process $y = (y_t)_{t \in \mathbb{Z}}$ defined upon a strictly stationary and ergodic white noise process $\epsilon = (\epsilon_t)_{t \in \mathbb{Z}}$ and a sequence of coefficients $(\theta_j)_{j \in \mathbb{N}}$ which are absolutely summable, is defined by
$y_t := \sum_{j=0}^{\infty} \theta_j \epsilon_{t-j} \quad \forall\, t \in \mathbb{Z}$.

Consider now a linear process where $\cancel{\theta_j = \begin{cases} 1, & j=0 \\ 0, & j \geq 1 \end{cases}}$

θj = 1, j=0
θj = 0, j=1
θj = 0, j>1

Given all the above, the MA(1) process is defined as the process $(y_t)_{t \in \mathbb{Z}}$ where $y_t = \theta \epsilon_{t-1} + \epsilon_t$, for some $\theta \in \mathbb{R}$

- Given the results in the paragraph Transformations and Heridarity and the ones about Causal Linear process, since $(\epsilon_t)_{t \in \mathbb{Z}}$ is strictly stationary and ergodic, $(y_t)_{t \in \mathbb{Z}}$ is also a strictly stationary and ergodic process.

- Remember that, for a linear process built on a stationary ergodic $WN(E\epsilon_0^2)$ we have already proved:
  - $E(y_t) = 0 \quad \forall\, t \in \mathbb{Z}$, independent of t
  - $E(y_t y_{t-k}) = E(\epsilon_0^2) \sum_{j=0}^{\infty} \theta_j \theta_{j+k} = \begin{cases} E(\epsilon_0^2)(1+\theta^2), & k=0 \\ E(\epsilon_0^2)\theta, & k=1 \\ 0, & k>1 \end{cases}$ independent of t, $\forall\, t \in \mathbb{Z}$, $\forall\, k \in \mathbb{Z}$

thus, $(y_t)_{t \in \mathbb{Z}}$ is weakly stationary.

Notice that,

▷ The stationarity ergodicity property and the weak stationarity do not depend on θ.

▷ Strict stationarity and ergodicity holds if $(\epsilon_t)_{t \in \mathbb{Z}}$ is simply stationary ergodic

▷ Weak stationarity property holds if $(\epsilon_t)_{t \in \mathbb{Z}}$ is simply $WN(E\epsilon_0^2)$.

- GMM Estimation of the MA(1) parameter

In the case of an AR(1) process since essentially we had a linear model we did OLS, however, in the context of MA(1), $y_t = \theta_0 \epsilon_{t-1} + \epsilon_t$, the process $(\epsilon_{t-1})_{t \in \mathbb{Z}}$ is non-observable, so we can't derive the OLS estimator.
Suppose for the sake of simplicity that: $(\epsilon_t)_{t\in\mathbb{Z}}$ is comprised by iid random variables such that: $E(\epsilon_0^2) = 1$, $K_4 = E(\epsilon_0^4) < +\infty$.
We know that $(y_t)_{t\in\mathbb{Z}}$ satisfies $y_t = \theta_0 \epsilon_{t-1} + \epsilon_t$ for $\theta_0 \in \mathbb{R}$ and our statistical problem is the estimation of θ_0.
Thus, the semi-parametric statistical model is:
$\left\{ (y_t(\theta))_{t\in\mathbb{Z}},\ y_t = \theta\epsilon_{t-1} + \epsilon_t,\ \theta \in \mathbb{R},\ \epsilon_t \text{ iid},\ E(\epsilon_0) = 0,\ E(\epsilon_0^2) = 1,\ K_4 = E(\epsilon_0^4) < +\infty \right\}$

Do we know anything about a moment of y_t that includes θ_0?
Remember that: for $E(\epsilon_0^2) = 1$, $E(y_t y_{t-1}) = \theta_0$
Thus, by the analogy principle we can consider $M_T(\omega,\theta) = \frac{1}{T}\sum_{t=1}^{T}(y_t y_{t-1} - \theta)$ which is well-defined and the identification condition holds.
Hence, we can define the objective function as:
$$Q_T(\omega,\theta) = \left(\frac{1}{T}\sum_{t=1}^{T}(y_t y_{t-1} - \theta)\right)^2$$

Obviously, $\theta_T = \underset{\theta\in\mathbb{R}}{\operatorname{argmin}}\, Q_T(\omega,\theta) = \underset{\theta\in\mathbb{R}}{\operatorname{argmin}} \left(\frac{1}{T}\sum_{t=1}^{T}(y_t y_{t-1} - \theta)\right)^2 = \frac{1}{T}\sum_{t=1}^{T} y_t y_{t-1}$.

We have that:

- $E_{\theta_0}\theta_T = E_{\theta_0}\left(\frac{1}{T}\sum_{t=1}^{T} y_t y_{t-1}\right) = \frac{1}{T}\sum_{t=1}^{T} E_{\theta_0}(y_t y_{t-1}) = \frac{1}{T}\sum_{t=1}^{T}\theta_0 = \theta_0$

therefore, the estimator is unbiased.

- $Var_{\theta_0}\theta_T = E_{\theta_0}\left[\left(\frac{1}{T}\sum y_t y_{t-1} - \theta_0\right)^2\right] = E_{\theta_0}\left[\left(\frac{1}{T}\sum_{t=1}^{T} y_t y_{t-1}\right)^2\right] - \theta_0^2$

$$= \frac{1}{T^2} E_{\theta_0}\left(\sum_{t,s=1}^{T} y_t y_{t-1} y_s y_{s-1}\right) - \theta_0^2 = \frac{1}{T^2}\sum_{t,s=1}^{T} E\left(y_t y_{t-1} y_s y_{s-1}\right) - \theta_0^2$$

We have that

$$
\begin{aligned}
E_{\theta_0}(y_t^2 y_{t-1}^2) &= E_{\theta_0}\left[(\epsilon_t + \theta_0 \epsilon_{t-1})^2 (\epsilon_{t-1} + \theta_0 \epsilon_{t-2})^2\right] \\
&= E_{\theta_0}\left[(\epsilon_t^2 + 2\theta_0 \epsilon_t \epsilon_{t-1} + \theta_0^2 \epsilon_{t-1}^2)(\epsilon_{t-1}^2 + 2\theta_0 \epsilon_{t-1}\epsilon_{t-2} + \theta_0^2 \epsilon_{t-2}^2)\right] \\
&= E_{\theta_0}(\epsilon_t^2 \epsilon_{t-1}^2) + 2\theta_0 E_{\theta_0}(\epsilon_t^2 \epsilon_{t-1}\epsilon_{t-2}) + \theta_0^2 E_{\theta_0}(\epsilon_t^2 \epsilon_{t-2}^2) + 2\theta_0 E_{\theta_0}(\epsilon_t \epsilon_{t-1}^3) \\
&\quad + 4\theta_0^2 E_{\theta_0}(\epsilon_t \epsilon_{t-1}^2 \epsilon_{t-2}) + 2\theta_0^3 E_{\theta_0}(\epsilon_t \epsilon_{t-1} \epsilon_{t-2}^2) + \theta_0^2 E_{\theta_0}(\epsilon_{t-1}^4) \\
&\quad + 2\theta_0^3 E_{\theta_0}(\epsilon_{t-1}^3 \epsilon_{t-2}) + \theta_0^4 E(\epsilon_{t-1}^2 \epsilon_{t-2}^2) \\
&\overset{\text{indep.}}{=} E(\epsilon_t^2)E(\epsilon_{t-1}^2) + 2\theta_0 E(\epsilon_t^2)E(\epsilon_{t-1})E(\epsilon_{t-2}) + \theta_0^2 E(\epsilon_t^2)E(\epsilon_{t-2}^2) \\
&\quad + 2\theta_0 E(\epsilon_t)E(\epsilon_{t-1}^3) + 4\theta_0^2 E(\epsilon_t)E(\epsilon_{t-1}^2)E(\epsilon_{t-2}) + 2\theta_0^3 E(\epsilon_t)E(\epsilon_{t-1})E(\epsilon_{t-2}^2) \\
&\quad + \theta_0^2 E(\epsilon_{t-1}^4) + 2\theta_0^3 E(\epsilon_{t-1}^3)E(\epsilon_{t-2}) + \theta_0^4 E(\epsilon_{t-1}^2)E(\epsilon_{t-2}^2) \\
&\overset{\text{homog.}}{=} 1 + 0 + \theta_0^2 + 0 + 0 + 0 + \theta_0^2 \kappa_4 + 0 + \theta_0^4 \\
&= 1 + (1+\kappa_4)\theta_0^2 + \theta_0^4
\end{aligned}
$$

Consider now the computation of $E_{\theta_0}(y_t y_{t-1} y_s y_{s-1})$.

When $|t-s| = 1$:

$$
\begin{aligned}
E_{\theta_0}(y_t y_{t-1} y_s y_{s-1}) &= E_{\theta_0}(y_t y_{t-1} y_{t-1} y_{t-2}) = E(y_t y_{t-1}^2 y_{t-2}) = E\left[(\epsilon_t + \theta_0 \epsilon_{t-1}) y_{t-1}^2 y_{t-2}\right] \\
&= E_{\theta_0}(\epsilon_t y_{t-1}^2 y_{t-2}) + \theta_0 E_{\theta_0}(\epsilon_{t-1} y_{t-1}^2 y_{t-2}) \\
&\overset{\text{indep.}}{=} E(\epsilon_t)E(y_{t-1}^2 y_{t-2}) + \theta_0 E\left[\epsilon_{t-1}(\epsilon_{t-1} + \theta_0 \epsilon_{t-2})^2 y_{t-2}\right] \\
&= 0 + \theta_0 E\left[\epsilon_{t-1}(\epsilon_{t-1}^2 + 2\theta_0 \epsilon_{t-1}\epsilon_{t-2} + \theta_0^2 \epsilon_{t-2}^2) y_{t-2}\right] \\
&= \theta_0 E(\epsilon_{t-1}^3 y_{t-2}) + 2\theta_0^2 E(\epsilon_{t-1}^2 \epsilon_{t-2} y_{t-2}) + \theta_0^3 E(\epsilon_{t-1}\epsilon_{t-2}^2 y_{t-2}) \\
&\overset{\text{indep.}}{=} \theta_0 E(\epsilon_{t-1}^3)E(y_{t-2}) + 2\theta_0^2 E(\epsilon_{t-1}^2)E(\epsilon_{t-2} y_{t-2}) + \theta_0^3 E(\epsilon_{t-1})E(\epsilon_{t-2}^2 y_{t-2}) \\
&\overset{\text{homog.}}{=} 0 + 2\theta_0^2 E\left[\epsilon_{t-2}(\epsilon_{t-2} + \theta_0 \epsilon_{t-3})\right] + 0 \\
&= 2\theta_0^2 E(\epsilon_{t-2}^2) + 2\theta_0^3 E(\epsilon_{t-2}\epsilon_{t-3}) = 2\theta_0^2 + 0 = 2\theta_0^2.
\end{aligned}
$$

When $|t-s| = 2$:

$$
\begin{aligned}
E_{\theta_0}(y_t y_{t-1} y_s y_{s-1}) &= E_{\theta_0}(y_t y_{t-1} y_{t-2} y_{t-3}) = E_{\theta_0}\left[(\epsilon_t + \theta_0 \epsilon_{t-1}) y_{t-1} y_{t-2} y_{t-3}\right] \\
&= E_{\theta_0}(\epsilon_t y_{t-1} y_{t-2} y_{t-3}) + \theta_0 E_{\theta_0}(\epsilon_{t-1} y_{t-1} y_{t-2} y_{t-3}) \\
&\overset{\text{indep.}}{=} E_{\theta_0}(\epsilon_t) E_{\theta_0}(y_{t-1} y_{t-2} y_{t-3}) + \theta_0 E_{\theta_0}\left[\epsilon_{t-1}(\epsilon_{t-1} + \theta_0 \epsilon_{t-2}) y_{t-2} y_{t-3}\right] \\
&= 0 + \theta_0 E(\epsilon_{t-1}^2 y_{t-2} y_{t-3}) + \theta_0^2 E(\epsilon_{t-1}\epsilon_{t-2} y_{t-2} y_{t-3}) \\
&\overset{\text{indep.}}{=} \theta_0 E(\epsilon_{t-1}^2) E(y_{t-2} y_{t-3}) + \theta_0^2 E(\epsilon_{t-1}) E(\epsilon_{t-2} y_{t-2} y_{t-3}) \\
&= \theta_0 \cdot \theta_0 + 0 = \theta_0^2.
\end{aligned}
$$

When $|t-s| \geq 2$:

$$E_{\theta_0}(y_t y_{t-1} y_s y_{s-1}) = E_{\theta_0}(y_t y_{t-1} y_{t-K} y_{t-K-1}) \overset{\text{indep.}}{=} E_{\theta_0}(y_t y_{t-1}) \cdot E(y_{t-K} y_{t-K-1}) = \theta_0^2$$

Hence,
$$E_{\theta_0}(y_t y_{t-1} y_s y_{s-1}) = \begin{cases} 1 + (1+K_4)\theta_0^2 + \theta_0^4, & |t-s| = 0 \\ 2\theta_0^2, & |t-s| = 1 \\ \theta_0^2, & |t-s| \geq 2 \end{cases}$$

So, now we have that :

$$Var_{\theta_0} \theta_T = \frac{1}{T^2} \sum_{t,s=1}^{T} E_{\theta_0}(y_t y_{t-1} y_s y_{s-1}) - \theta_0^2$$

$$= \frac{1}{T^2} \Big[T \cdot E_{\theta_0}(y_t^2 y_{t-1}^2) + 2(T-1) E_{\theta_0}(y_t y_{t-1}^2 y_{t-2}) + 2(T-2) E_{\theta_0}(y_t y_{t-1} y_{t-2} y_{t-3}) + \dots \Big] - \theta_0^2$$

$$= \frac{1}{T^2} \Big[T[1 + (1+K_4)\theta_0^2 + \theta_0^4] + 2(T-1) 2\theta_0^2 + 2 \sum_{i=2}^{T-1} (T-i) \theta_0^2 \Big] - \theta_0^2$$

$$= \frac{1}{T^2} \Big[T[1 + (1+K_4)\theta_0^2 + \theta_0^4] + 2(T-1)\theta_0^2 + 2 \sum_{i=1}^{T-1} (T-i) \theta_0^2 \Big] - \theta_0^2$$

$$= \frac{1}{T^2} \Big[T[1 + (1+K_4)\theta_0^2 + \theta_0^4] + 2(T-1)\theta_0^2 + T(T-1)\theta_0^2 \Big] - \theta_0^2$$

$$= \frac{1}{T} \Big[1 + (1+K_4)\theta_0^2 + \theta_0^4 \Big] + \frac{(T-1)(T+2)}{T^2} \theta_0^2 - \theta_0^2$$

<u>Asymptotic properties of the GMM estimator</u>

$$\theta_T = \frac{1}{T} \sum_{t=1}^{T} y_t y_{t-1}$$

Since $(y_t)_{t \in \mathbb{Z}}$ is strictly stationary and ergodic, the process $(y_t y_{t-1})_{t \in \mathbb{Z}}$ is also strictly stationary and ergodic. Therefore, by Birkhoff's LLN given that $(y_t y_{t-1})_{t \in \mathbb{Z}}$ is stationary ergodic and $E|y_t y_{t-1}| < +\infty$,

$$\frac{1}{T} \sum_{t=1}^{T} y_t y_{t-1} \longrightarrow E_{\theta_0}(y_t y_{t-1}) = \theta_0 \quad \mathbb{P} \text{ a.s.}$$

i.e. the GMM estimator is a consistent estimator of θ_0.

In order to find the asymptotic distribution of θ_T set :

$$v_t = y_t y_{t-1} - \theta_0 = (\epsilon_t + \theta_0 \epsilon_{t-1})(\epsilon_{t-1} + \theta_0 \epsilon_{t-2}) - \theta_0$$
$$= \epsilon_t \epsilon_{t-1} + \theta_0 \epsilon_t \epsilon_{t-2} + \theta_0 \epsilon_{t-1}^2 + \theta_0^2 \epsilon_{t-1} \epsilon_{t-2} - \theta_0$$
$$= \epsilon_t \epsilon_{t-1} + \theta_0 \epsilon_t \epsilon_{t-2} + \theta_0 (\epsilon_{t-1}^2 - 1) + \theta_0^2 \epsilon_{t-1} \epsilon_{t-2}$$

Assume that we have in hand a CLT for stationary ergodic s.m.d. processes. $(y_t y_{t-1} - \theta_0)_{t \in \mathbb{Z}}$ is not a m.d. process thus the aforementioned wouldn't be applicable.

Consider, $f_t := \sigma(\epsilon_{t-i}, i \geq 0)$.

$$E(v_t / f_{t-1}) = E\left[\epsilon_t \epsilon_{t-1} + \theta_0 \epsilon_t \epsilon_{t-2} + \theta_0 (\epsilon_{t-1}^2 - 1) + \theta_0^2 \epsilon_{t-1} \epsilon_{t-2} / f_{t-1}\right)$$
$$= E(\epsilon_t \epsilon_{t-1} / f_{t-1}) + \theta_0 E(\epsilon_t \epsilon_{t-2} / f_{t-1}) + \theta_0 E(\epsilon_{t-1}^2 - 1 / f_{t-1}) + \theta_0^2 E(\epsilon_{t-1} \epsilon_{t-2}$$
$$= \epsilon_{t-1} E(\epsilon_t / f_{t-1}) + \theta_0 \epsilon_{t-2} E(\epsilon_t / f_{t-1}) + \theta_0 (\epsilon_{t-1}^2 - 1) + \theta_0^2 \epsilon_{t-1} \epsilon_{t-2}$$
$$\overset{\text{indep.}}{=} \epsilon_{t-1} E(\epsilon_t) + \theta_0 \epsilon_{t-2} E(\epsilon_t) + \theta_0 (\epsilon_{t-1}^2 - 1) + \theta_0^2 \epsilon_{t-1} \epsilon_{t-2}$$
$$= 0 + 0 + \theta_0 (\epsilon_{t-1}^2 - 1) + \theta_0^2 \epsilon_{t-1} \epsilon_{t-2} \neq 0$$

i.e. (v_t) is not a m.d. process w.r.t. $(f_t)_{t \in \mathbb{Z}}$, $f_t = \sigma(\epsilon_{t-i}, i \geq 0)$, thus not a s.m.

Can we find another filtration w.r.t which $(v_t)_{t \in \mathbb{Z}}$ is a s.m.d. process? (Find it

Consider the case where we can't find any filtration w.r.t which the process is

We need a generalization of the CLT for s.m.d that we already know.

<u>Gordin's CLT</u>: Suppose that the stochastic process (v_t) is stationary ergodic. If

1. $E(v_0^2) < +\infty$
2. $\lim_{j \to \infty} E\left[E v_t / f_{t-j}\right]^2 = 0$ for all t
3. for $r_{t,j} = E(v_t / f_{t-j}) - E(v_{t-1} / f_{t-j-1})$, $\sum_{j=0}^{\infty} (E r_{t,j}^2)^{1/2} < +\infty$

then $E(v_0) = 0$, $\sum_{j=-\infty}^{\infty} |\gamma_j| < +\infty$ and $\frac{1}{\sqrt{T}} \sum_{t=1}^{T} v_t \xrightarrow{d} z \sim N(0, \sum_{j=-\infty}^{\infty} \gamma_j)$

Since $(y_t y_{t-1})_{t \in \mathbb{Z}}$ is stationary ergodic process, $(v_t)_{t \in \mathbb{Z}}$ is also stationary ergodic.

1. $E_{\theta_0}(v_0^2) = E_{\theta_0}\left[(y_t y_{t-1} - \theta_0)^2\right] = E_{\theta_0}(y_t^2 y_{t-1}^2 - 2\theta_0 y_t y_{t-1} + \theta_0^2)$
$$= E(y_t^2 y_{t-1}^2) - 2\theta_0 E(y_t y_{t-1}) + \theta_0^2 = 1 + (1 + \kappa_4)\theta_0^2 + \theta_0^4 - 2\theta_0 \theta_0 + \theta_0^2$$
$$= 1 + \kappa_4 \theta_0^2 + \theta_0^4 < +\infty$$

2. $E_{\theta_0}(v_t / f_{t-j}) = E_{\theta_0}(y_t y_{t-1} - \theta_0 / f_{t-j})$

$$= E_{\theta_0}\left[\epsilon_t \epsilon_{t-1} + \theta_0 \epsilon_t \epsilon_{t-2} + \theta_0(\epsilon_{t-1}^2 - 1) + \theta_0^2 \epsilon_{t-1}\epsilon_{t-2} / f_{t-j}\right]$$

$$= \begin{cases} \theta_0(\epsilon_{t-1}^2 - 1) + \theta_0^2 \epsilon_{t-1}\epsilon_{t-2} &, j = 1 \\ 0 &, j > 1 \end{cases}$$

therefore, as $j \to \infty$, $E_{\theta_0}(v_t / f_{t-j}) \to 0$, so the second condition holds.

3. From the previous calculations we have that

$$\kappa_{t,j} = E_{\theta_0}(v_t / f_{t-j}) - E_{\theta_0}(v_t / f_{t-j-1}) = \begin{cases} \theta_0(\epsilon_{t-1}^2 - 1) + \theta_0^2 \epsilon_{t-1}\epsilon_{t-2} &, j = 1 \\ 0 &, j > 1 \end{cases}$$

thus,

$$E(\kappa_{t,j}^2) = \begin{cases} E\left[\theta_0(\epsilon_{t-1}^2 - 1) + \theta_0^2 \epsilon_{t-1}\epsilon_{t-2}\right]^2 &, j = 1 \\ 0 &, j > 1 \end{cases}$$

$$E(\kappa_{t,1}^2) = E\left[\left(\theta_0(\epsilon_{t-1}^2 - 1) + \theta_0^2 \epsilon_{t-1}\epsilon_{t-2}\right)^2\right] =$$

$$= E\left[\theta_0^2(\epsilon_{t-1}^2 - 1)^2 + 2\theta_0^3(\epsilon_{t-1}^2 - 1)\epsilon_{t-1}\epsilon_{t-2} + \theta_0^4 \epsilon_{t-1}^2 \epsilon_{t-2}^2\right]$$

$$\overset{\text{indep.}}{=} \theta_0^2 E(\epsilon_{t-1}^4 - 2\epsilon_{t-1}^2 + 1) + 2\theta_0^3 E\left((\epsilon_{t-1}^2 - 1)\epsilon_{t-1}\right) E(\epsilon_{t-2}) + \theta_0^4 E(\epsilon_{t-1}^2) E(\epsilon_{t-2}^2)$$

$$= \theta_0^2(\kappa_4 - 1) + 0 + \theta_0^4$$

$$= \theta_0^2(\kappa_4 - 1) + \theta_0^4$$

Hence, $\sum_{j=1}^{\infty} \left(E_{\theta_0}(\kappa_{t,j}^2)\right)^{1/2} = \left(E_{\theta_0}(\kappa_{t,1}^2)\right)^{1/2} = \sqrt{\theta_0^2(\kappa_4 - 1) + \theta_0^4}$

Since, Gordin's conditions hold for the process $(y_t y_{t-1} - \theta_0)_{t \in \mathbb{Z}}$,

$$\frac{1}{\sqrt{T}} \sum_{t=1}^{T} (y_t y_{t-1} - \theta_0) \xrightarrow{d} z \sim N\left(0, \sum_{j=-\infty}^{\infty} \gamma_j\right)$$

where $\gamma_j = Cov(v_t v_{t-j}) = E_{\theta_0}\left[(y_t y_{t-1} - \theta_0)(y_{t-j} y_{t-j-1} - \theta_0)\right]$

$$= E_{\theta_0}(y_t y_{t-1} y_{t-j} y_{t-j-1}) - \theta_0^2 = \begin{cases} 1 + (1 + \kappa_4)\theta_0^2 + \theta_0^4 - \theta_0^2 &, j = 0 \\ 2\theta_0^2 - \theta_0^2 &, |j| = 1 \\ \theta_0^2 - \theta_0^2 &, |j| > 1. \end{cases}$$

$$\gamma_j = \begin{cases} 1 + \kappa_4 \theta_0^2 + \theta_0^4 , & j = 0 \\ \theta_0^2 , & |j| = 1 \\ 0 , & |j| > 1 \end{cases}$$

and $\displaystyle\sum_{j=-\infty}^{\infty} \gamma_j = 1 + \kappa_4 \theta_0^2 + \theta_0^4 + 2\theta_0^2 = (1+\theta_0^2)^2 + \kappa 4\, \theta_0^2$

Therefore,

$\sqrt{T}\,(\theta_T - \theta_0) \xrightarrow{d} z \sim N\left(0,\ (1+\theta_0^2)^2 + \kappa 4\, \theta_0^2\right)$