Athens University of Economics and Business
Department of Economics

Postgraduate Program - Master's in Economic Theory
Course: Econometrics II
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EGARCH(1,1) model

$$h_t = \exp\{\omega + \gamma z_{t-1} + \beta \ln h_{t-1}\}\ ,\ y_t = z_t\sqrt{h_t}\ ,\ \{z_t\} \sim \text{i.i.d } N(0,1)$$

Requirement : Calculate $\text{Cov}(y_t^2, y_{t-1}^2)$

SOLUTION

$$\text{Cov}(y_t^2, y_{t-1}^2) = E(y_t^2 y_{t-1}^2) - E(y_t^2)E(y_{t-1}^2) = E(y_t^2 y_{t-1}^2) - \left[E(y_t^2)\right]^2 \qquad [1]$$

We have (see notes) $h_t = \exp\left\{\dfrac{\omega}{1-\beta}\right\}\exp\left\{\gamma\sum_{j=0}^{\infty}\beta^j z_{t-1-j}\right\}$ so

a) $E(y_t^2) = E(z_t^2 h_t) = E\left(z_t^2 \exp\left\{\dfrac{\omega}{1-\beta}\right\}\exp\left\{\gamma\sum_{j=0}^{\infty}\beta^j z_{t-1-j}\right\}\right)$

$$= \exp\left\{\frac{\omega}{1-\beta}\right\}E(z_t^2)E\left(\exp\left\{\gamma\sum_{j=0}^{\infty}\beta^j z_{t-1-j}\right\}\right) = \exp\left\{\frac{\omega}{1-\beta}\right\}\cdot 1\cdot E\left(\exp\{\gamma z_{t-1} + \gamma\beta z_{t-2} + \gamma\beta^2 z_{t-3} + \dots\}\right)$$

$$= \exp\left\{\frac{\omega}{1-\beta}\right\}\exp\left\{\frac{\gamma^2}{2}+\frac{\gamma^2}{2}\beta^2+\frac{\gamma^2}{2}\beta^4+\frac{\gamma^2}{2}\beta^6+\ldots\right\}$$

$$\Rightarrow E\left(y_t^2\right)=\exp\left\{\frac{\omega}{1-\beta}\right\}\exp\left\{\frac{\gamma^2}{2\left(1-\beta^2\right)}\right\}\Rightarrow\left[E\left(y_t^2\right)\right]^2=\exp\left\{\frac{2\omega}{1-\beta}\right\}\exp\left\{\frac{\gamma^2}{1-\beta^2}\right\} \qquad [2]$$

b) $E\left(y_t^2y_{t-1}^2\right)=E\left(z_t^2h_tz_{t-1}^2h_{t-1}\right)=E\left(z_t^2\right)E\left(h_tz_{t-1}^2h_{t-1}\right)=1\cdot E\left(z_{t-1}^2h_th_{t-1}\right)$

$$= E\left(z_{t-1}^2\exp\left\{\frac{\omega}{1-\beta}\right\}\exp\left\{\gamma\sum_{j=0}^{\infty}\beta^jz_{t-1-j}\right\}\exp\left\{\frac{\omega}{1-\beta}\right\}\exp\left\{\gamma\sum_{j=0}^{\infty}\beta^jz_{t-2-j}\right\}\right)$$

$$= \exp\left\{\frac{2\omega}{1-\beta}\right\}E\left(z_{t-1}^2\exp\left\{\gamma\sum_{j=0}^{\infty}\beta^jz_{t-1-j}\right\}\exp\left\{\gamma\sum_{j=0}^{\infty}\beta^jz_{t-2-j}\right\}\right)$$

$$= \exp\left\{\frac{2\omega}{1-\beta}\right\}E\left(z_{t-1}^2\exp\left\{\gamma z_{t-1}+\gamma\beta z_{t-2}+\gamma\beta^2z_{t-3}+\gamma\beta^3z_{t-4}+\ldots\right\}\exp\left\{\gamma z_{t-2}+\gamma\beta z_{t-3}+\gamma\beta^2z_{t-4}+\gamma\beta^3z_{t-5}+\ldots\right\}\right)$$

$$= \exp\left\{\frac{2\omega}{1-\beta}\right\}E\left(z_{t-1}^2\exp\left\{\gamma z_{t-1}\right\}\right)E\left(\exp\left\{\gamma\beta z_{t-2}+\gamma z_{t-2}+\gamma\beta^2z_{t-3}+\gamma\beta z_{t-3}+\gamma\beta^3z_{t-4}+\gamma\beta^2z_{t-4}+\ldots\right\}\right)$$

$$= \exp\left\{\frac{2\omega}{1-\beta}\right\}\left(1+\gamma^2\right)\exp\left\{\frac{\gamma^2}{2}\right\}E\left(\exp\left\{\gamma\left(1+\beta\right)z_{t-2}+\gamma\left(1+\beta\right)\beta z_{t-3}+\gamma\left(1+\beta\right)\beta^2z_{t-4}+\ldots\right\}\right)$$

$$= \exp\left\{\frac{2\omega}{1-\beta}\right\}\left(1+\gamma^2\right)\exp\left\{\frac{\gamma^2}{2}\right\}\exp\left\{\frac{\gamma^2\left(1+\beta\right)^2}{2}\left(1+\beta^2+\beta^4+\beta^6+\ldots\right)\right\}$$

$$= \exp\left\{\frac{2\omega}{1-\beta}\right\}\left(1+\gamma^2\right)\exp\left\{\frac{\gamma^2}{2}\right\}\exp\left\{\frac{\gamma^2\left(1+\beta\right)^2}{2\left(1-\beta^2\right)}\right\} = \exp\left\{\frac{2\omega}{1-\beta}\right\}\left(1+\gamma^2\right)\exp\left\{\frac{\gamma^2}{2}\left(1+\frac{1+2\beta+\beta^2}{1-\beta^2}\right)\right\}$$

$$= \exp\left\{\frac{2\omega}{1-\beta}\right\}\left(1+\gamma^2\right)\exp\left\{\frac{\gamma^2}{2}\left(\frac{1-\beta^2+1+2\beta+\beta^2}{1-\beta^2}\right)\right\} = \exp\left\{\frac{2\omega}{1-\beta}\right\}\left(1+\gamma^2\right)\exp\left\{\frac{2\gamma^2}{2}\left(\frac{1+\beta}{1-\beta^2}\right)\right\}$$

$$\Rightarrow E\left(y_t^2 y_{t-1}^2\right) = \exp\left\{\frac{2\omega}{1-\beta}\right\}\left(1+\gamma^2\right)\exp\left\{\frac{\gamma^2}{1-\beta}\right\} \qquad [3]$$

Inserting [3] and [2] in [1] we get

$$\text{Cov}\left(y_t^2, y_{t-1}^2\right) = \exp\left\{\frac{2\omega}{1-\beta}\right\}\left(1+\gamma^2\right)\exp\left\{\frac{\gamma^2}{1-\beta}\right\} - \exp\left\{\frac{2\omega}{1-\beta}\right\}\exp\left\{\frac{\gamma^2}{1-\beta^2}\right\}$$

$$\text{Cov}\left(y_t^2, y_{t-1}^2\right) = \exp\left\{\frac{2\omega}{1-\beta}\right\}\left[\left(1+\gamma^2\right)\exp\left\{\frac{\gamma^2}{1-\beta}\right\} - \exp\left\{\frac{\gamma^2}{1-\beta^2}\right\}\right]$$

For $\beta < 1$ this is always positive.

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