Some further comments on the EGARCH(1,1) Process

[With Corrections - 30/07/18]

Remember that the process adheres to the general form of conditional heteroskedasticity with $(h_t)_{t\in\mathbb{Z}}$ satisfying the exponential recursion

$$(*) \qquad h_t = \exp\left[\omega + \alpha\left(|z_{t-1}| - E|z_{t-1}|\right) + \gamma z_{t-1} + \beta \ln h_{t-1}\right],\ t\in\mathbb{Z}$$

with $\omega, \alpha, \gamma, \beta \in \mathbb{R}$.

i. Positivity: the presence of the exponential ensures positivity without restrictions on the parameters.

ii. Stationarity and ergodicity: Define $(u_t)_{t\in\mathbb{Z}}$ by $u_t := \alpha(|z_{t-1}| - E|z_{t-1}|) + \gamma z_{t-1}$. It is an iid process with $E(u_t)=0$, $E(u_t^2) = \alpha^2\left(1-\left(E(|z_0|)\right)^2\right) + 2\alpha\gamma E(z_0|z_0|) + \gamma^2 < +\infty$. Notice that when the distribution of z_0 is symmetric around zero then $E(z_0|z_0|)=0$. Given this $(*)$ is equivalent to

$$(**) \qquad \ln h_t = \omega + \beta \ln h_{t-1} + u_t, \quad \text{i.e. an } AR(1)$$

recursion with non zero constant for the logarithm of the conditional variance. Hence (why?) if $|\beta|<1$ there exists a unique stationary and ergodic solution to $(**)$, given by

$$\ln h_t = \frac{\omega}{1-\beta} + \sum_{i=0}^{\infty} \beta^i u_{t-i}, \quad \text{and thereby (why?)}$$

a unique stationary and ergodic solution to $*)$ given by

$$h_t = \exp\left(\frac{\omega}{1-\beta} + \sum_{i=0}^{\infty} \beta^i u_{t-i}\right) =$$

$$= \exp\left(\frac{\omega^*}{1-\beta} + \sum_{i=0}^{\infty} \beta^i \left(\alpha|z_{t-1-i}| + \gamma z_{t-1-i}\right)\right) \quad \text{with } \omega^* = (\omega + \alpha E|z_0|).$$

Hence, if $|\beta|<1$ the EGARCH(1,1) process $(y_t)_{t\in\mathbb{Z}}$ is stationary ergodic (why?).

iii. Measurability. The stationary solution above is obviously adapted to the filtration prescribed by our general definition (why?).

iv. Moments. In what follows suppose for simplicity that $Z_0 \sim N(0,1)$ and $\alpha = 0$ (the latter restriction can be relaxed at the expense of tedious calculations; more abstract conside-rations would be involved if the former is also relaxed).

Notice that the standard normality assumption implies that

$$\mathbb{E}\exp(kZ_0) = \exp(k^2/2)$$ ($k \in \mathbb{R}$, see the mgf of $N(0,1)$)

$$\mathbb{E}\left(Z_0 \exp(kZ_0)\right) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} x e^{kx} e^{-x^2/2}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} x e^{-1/2(x^2-2kx+k^2)}\,dx\, e^{k^2/2} =$$

$$= e^{k^2/2}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} x e^{(x-k)^2/2}\,dx \underset{du=dx}{\overset{u=x-k}{=}}$$

$$= e^{k^2/2}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}(u+k)e^{-u^2/2}\,du = k e^{k^2/2}, \qquad k \in \mathbb{R}.$$

$$\mathbb{E}\left(Z_0^2 \exp(kZ_0)\right) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} x^2 e^{kx} e^{-x^2/2}\,dx$$

$$= e^{k^2/2}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} x^2 e^{-(x-k)^2/2}\,dx =$$

$$= e^{K^2/2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} (u+K)^2 e^{-u^2/2} du = (1+K^2) e^{K^2/2}, \quad K \in \mathbb{R}.$$

The previous imply (compare it the GARCH(1,1) case) that for any $K \in \mathbb{R}$, $\mathbb{E}(h_t^K) = \mathbb{E} \exp\left(\frac{K\omega}{1-\beta} + K\gamma \sum_{i=0}^{\infty} \beta^i z_{t-1-i}\right) \overset{\text{indep.}}{=}$

$$= \exp\left(\frac{K\omega}{1-\beta}\right) \prod_{i=0}^{\infty} \mathbb{E} \exp\left(K\gamma\beta^i z_{t-1-i}\right) = \exp\left(\frac{K\omega}{1-\beta}\right) \prod_{i=0}^{\infty} \exp\left(K^2\gamma^2\beta^{2i}/2\right)$$

$$= \exp\left(\frac{K\omega}{1-\beta} + \frac{K^2\gamma^2}{2} \sum_{i=0}^{\infty} \beta^{2i}\right) = \exp\left(\frac{K\omega}{1-\beta} + \frac{K^2\gamma^2}{2(1-\beta^2)}\right).$$

Hence for $K=1$, and in this restricted framework, we obtain the $\mathbb{E}(h_t) = \exp\left(\frac{\omega}{1-\beta} + \frac{\gamma^2}{2(1-\beta^2)}\right) < +\infty$ required by our general framework.

v. Dynamic Asymmetry

$$\mathrm{Cov}(h_t z_{t-1}) = \mathbb{E}(h_t z_{t-1}) = \mathbb{E}\left[\exp\left(\frac{\omega}{1-\beta} + \gamma \sum_{i=0}^{\infty} \beta^i z_{t-1-i}\right) z_{t-1}\right]$$

$$\overset{\text{indep.}}{=} \exp\left(\frac{\omega}{1-\beta}\right) \mathbb{E}\left(z_{t-1} \exp\left(\gamma z_{t-1}\right)\right) \prod_{i=1}^{\infty} \mathbb{E}\left[\exp\left(\gamma\beta^i z_{t-1-i}\right)\right]$$

$$= \exp\left(\frac{\omega}{1-\beta}\right) \gamma \exp\left(\gamma^2/2\right) \prod_{i=1}^{\infty} \exp\left(\frac{\gamma^2\beta^{2i}}{2}\right)$$

$$= \gamma \exp\left(\frac{\omega}{1-\beta} + \frac{\gamma^2}{2(1-\beta^2)}\right) < 0$$ iff $\gamma < 0$, and thereby

in this respect the model can reproduce the relevant stylized fact (remember that in this restricted framework $\mathbb{E}(z_0^3) = 0$).

Exercise: Derive $\mathrm{Cov}(h_t, z_{t-k})$ for any $k \in \mathbb{N}$.

vi. Autocovariance of the squares.

Remember that due to iii-iv, $(y_t)_{t\in\mathbb{Z}}$ is a stationary ergodic s.m.d. w.r.t. the relevant filtration. What about the autocovariance function of the squares? (given that we do not have an obvious ARMA-type representation). For $k \geq 1$

$$\mathrm{Cov}(y_t^2, y_{t-k}^2) \overset{(\text{why?})}{=} \mathbb{E}(h_t h_{t-k} z_{t-k}^2) - [\mathbb{E}(h_t)]^2 \text{ and}$$

$$\mathbb{E}(h_t h_{t-k} z_{t-k}^2) = \mathbb{E}\left[\exp\left(\frac{\omega}{1-\beta} + \gamma \sum_{i=0}^{\infty} \beta^i z_{t-1-i}\right) \exp\left(\frac{\omega}{1-\beta} + \gamma \sum_{i=0}^{\infty} \beta^i z_{t-1-k-i}\right) z_{t-k}^2\right]$$

$$\overset{(\text{why?})}{=} \exp\left(\frac{2\omega}{1-\beta}\right) \mathbb{E}\exp\left(\gamma \sum_{i=0}^{k-2} \beta^i z_{t-1-i}\right) \mathbb{E}\left(z_{t-k}^2 \exp\left(\gamma \beta^{k-1} z_{t-k}\right)\right) \times$$

$$\times \mathbb{E}\left[\exp\left(\gamma \sum_{i=k}^{\infty} \beta^i z_{t-1-i} + \gamma \sum_{i=0}^{\infty} \beta^i z_{t-1-k-i}\right)\right] \overset{j=i-k}{=}$$

$$= \exp\left(\frac{2\omega}{1-\beta}\right) \prod_{i=0}^{k-2} \mathbb{E}\left(\exp(\gamma \beta^i z_{t-1-i})\right) \left(1+\gamma^2\beta^{2(k-1)}\right) e^{\frac{\gamma^2\beta^{2(k-1)}}{2}} \prod_{j=0}^{\infty} \mathbb{E}\left(\exp\left(\gamma(\beta^j+\beta^{j+k}) z_{t-1-k-j}\right)\right)$$

$$= \exp\left(\frac{2\omega}{1-\beta}\right) \exp\left(\frac{\gamma^2}{2} \sum_{i=0}^{k-2} \beta^{2i}\right) \left(1+\gamma^2\beta^{2(k-1)}\right) e^{\frac{\gamma^2\beta^{2(k-1)}}{2}} \exp\left(\frac{\gamma^2(1+\beta^k)^2}{2} \sum_{j=0}^{\infty} \beta^{2j}\right) \quad [*]$$

when $k \geq 2$

$$= \exp\left(\frac{2\omega}{1-\beta} + \frac{\gamma^2}{1-\beta^2}\right) \left(1+\gamma^2\beta^{2(k-1)}\right) \exp\left(\frac{\gamma^2\beta^k}{1-\beta^2}\right)$$

and thereby when $k \geq 2$

$$\mathrm{Cov}(y_t^2, y_{t-k}^2) = \exp\left(\frac{2\omega}{1-\beta} + \frac{\gamma^2}{1-\beta^2}\right) \left[\left(1+\gamma^2\beta^{2(k-1)}\right) e^{\frac{\gamma^2\beta^k}{1-\beta^2}} - 1\right]$$

and $\lim_{k\to+\infty} \mathrm{Cov}(y_t^2, y_{t-k}^2) = 0$, hence the squared process is regular (why is it weakly stationary? - Derive $\mathrm{Cov}(y_t^2, y_t^2)$ $(k=0)$).

Thereby the particular instance of the Model is also consistent with the levels-squares autocovariance functions stylized facts.

Further Comments: Analogously to the GARCH case the Model can be extended to arbitrary (finite) orders, while the Gaussian QMLE is also a "natural choice" for semi-parametric estimation.

[*] Notice that when $k=1$ due to that $\prod_{i=0}^{k-2} E\exp(\gamma\beta^i z_{t-1-i}) = \prod_{i=0}^{-1} E\exp(\gamma\beta^i z_{t-1-i})$ which is an empty product that by convention equals to 1 (equivalently $\prod_{i=0}^{-1} E\exp(\gamma\beta^i z_{t-1-i}) = \prod_{i=0}^{-1}\exp\left(\frac{\gamma^2}{2}\beta^{2i}\right) = \exp\left(\frac{\gamma^2}{2}\sum_{i=0}^{-1}\beta^{2i}\right)$ which contains an empty sum which by convention equals zero), we have that

$$[*] = \exp\left(\frac{2\omega}{1-\beta}\right)(1+\gamma^2)\exp\left(\frac{\gamma^2}{2}\right)\exp\left(\frac{\gamma^2}{2}\frac{(1+\beta)^2}{1-\beta^2}\right) = (1+\gamma^2)\exp\left(\frac{2\omega}{1-\beta}+\frac{\gamma^2}{2}\left(1+\frac{(1+\beta)^2}{1-\beta^2}\right)\right) =$$

$$(1+\gamma^2)\exp\left(\frac{2\omega}{1-\beta}+\frac{\gamma^2}{2}\left[1+\frac{1+\beta}{1-\beta}\right]\right) = (1+\gamma^2)\exp\left(\frac{2\omega}{1-\beta}+\frac{\gamma^2}{2}\,\frac{1-\beta+1+\beta}{1-\beta}\right) = (1+\gamma^2)\exp\left(\frac{2\omega}{1-\beta}+\frac{\gamma^2}{1-\beta}\right),$$ hence ($k=1$)

$$\operatorname{Cov}(y_t^2, y_{t-1}^2) = E(y_t^2 y_{t-1}^2) - E(y_t^2)E(y_{t-1}^2) =$$

$$= (1+\gamma^2)\exp\left(\frac{2\omega}{1-\beta}+\frac{\gamma^2}{1-\beta}\right) - \exp\left(\frac{\omega}{1-\beta}+\frac{\gamma^2}{2}\frac{1}{1-\beta^2}\right)\exp\left(\frac{\omega}{1-\beta}+\frac{\gamma^2}{2}\frac{1}{1-\beta^2}\right)$$

$$= \exp\left(\frac{2\omega}{1-\beta}\right)\left[(1+\gamma^2)\exp\left(\frac{\gamma^2}{1-\beta}\right) - \exp\left(\gamma^2\frac{1}{1-\beta^2}\right)\right].$$