An Example of Inconsistency for the OLSE in the ARMA Context

Remember that w.r.t. the statistical inference issue in the context of an ARMA(p,q) process and given that the sample is consisted of

$$(*) \quad (x_t)_{t=1-p^*,\dots,T}\,,$$

i.e. of $T+p^*$ consequtive random variables from the process (for $p^* \geq p$), the OLSE is computationally infeasible when the MA component is non-trivial (i.e. $q>0$). In such a case it could be possible that the OLSE can be used for the estimation of the true values of the AR parameters (given that p is known, or some lower bound of p is known). The following example shows that the non incorporation of the non trivial MA component in the regressor matrix (something that renders the OLSE feasible) can lead to inconsistent estimation for the AR parameters.

In this respect suppose that $(x_t)_{t\in\mathbb{Z}}$, is the already studied solution of the ARMA(1,1) recursion

$$(**) \quad \phi(L)\,x_t = \Theta(L)\,\varepsilon_t\,, \quad t\in\mathbb{Z}\,,$$

$$\phi(L) = 1-\varphi_{1_0}L\,, \quad |\varphi_{1_0}|<1\,, \quad \Theta(L) = 1+\theta_{1_0}L\,, \quad \theta_{1_0}\neq 0$$

and that also $(\varepsilon_t)_{t\in\mathbb{Z}}$ is a stationary and ergodic white noise process with $\sigma^2=1$ (for simplicity). Suppose furthermore that it is known that $p=1$, $(*)$ holds with $p^*=p=1$, and that either i. q is mistakenly thought to equal zero, or ii, q is known to equal one but the determination of θ_{1_0} is at present not of interest. In the context of the linear model $(*)$, $(**)$ and either i, ii the relation $Y = X\beta + \varepsilon$ holds with

$$Y = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_T \end{pmatrix}, \quad X = \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_{T-1} \end{pmatrix}, \quad b = \varphi_{1_0} \in \mathbb{R}\,, \quad \varepsilon = \begin{pmatrix} \varepsilon_1 + \theta_{1_0}\varepsilon_0 \\ \varepsilon_2 + \theta_{1_0}\varepsilon_1 \\ \vdots \\ \varepsilon_T + \theta_{1_0}\varepsilon_{T-1} \end{pmatrix} \quad \boxed{\substack{*\\**}}\,.$$

Notice that in the context of ii, $\boxed{\substack{*\\**}}$ can be interpreted as a choice that renders the OLSE for φ_{1_0} computationally feasible. Furthermore, notice that since $\theta_{1_0}\neq 0$, every row of X is correlated with the same row of ε (why?), and this essentially implies the inconsistency of the aforementioned OLSE.

As mentioned above the OLSE for φ_{10} is computationally feasible since given [*] we have that

$$b_T = (X'X)^{-1}X'Y = \frac{\sum_{t=1}^{T} x_t x_{t-1}}{\sum_{t=1}^{T} x_{t-1}^2} = \frac{1/T \sum_{t=1}^{T} x_t x_{t-1}}{1/T \sum_{t=1}^{T} x_{t-1}^2}.$$

Our previous derivations and Birkhoff's LLN (why is it applicable?) imply that:

a. $\frac{1}{T}\sum_{t=1}^{T} x_{t-1}^2 \to E(x_0^2)$ a.s. where $E(x_0^2) = \gamma_0 \overset{\sigma^2=1}{=} \left(1 + \frac{(\theta_{10}+\varphi_{10})^2}{1-\varphi_{10}^2}\right)$ (x),

b. $1/T \sum_{t=1}^{T} x_t x_{t-1} = \underbrace{\frac{1}{T}\sum_{t=1}^{T} \varphi_{10} x_{t-1}^2}_{A} + \underbrace{\frac{1}{T}\sum_{t=1}^{T} \varepsilon_t x_{t-1}}_{B} + \underbrace{\frac{1}{T}\sum_{t=1}^{T} \theta_{10}\varepsilon_{t-1} x_{t-1}}_{\Gamma}$ (why?),

with $A \to \varphi_{10}^2 \gamma_0$ a.s., $B \to E(\varepsilon_t x_{t-1}) = E\left(\sum_{i=0}^{\infty} \lambda_i \varepsilon_t \varepsilon_{t-1-i}\right) = \sum_{i=0}^{\infty} \lambda_i E(\varepsilon_t \varepsilon_{t-1-i}) = 0$ since $t \neq t-1-i \ \forall i \in \mathbb{N}$, $\Gamma \to E(\varepsilon_t x_t) = E\left(\sum_{i=0}^{\infty} \lambda_i \varepsilon_t \varepsilon_{t-i}\right) = \sum_{i=0}^{\infty} \lambda_i E(\varepsilon_t \varepsilon_{t-i}) = \lambda_0$ since $E(\varepsilon_t \varepsilon_{t-i}) = \begin{cases} 1, & i=0 \\ 0, & i>0 \end{cases}$. Remember that

$$\lambda_i = \begin{cases} 1, & i=0 \\ \left(1+\frac{\theta_{10}}{\varphi_{10}}\right)\varphi_{10}^i, & i>0 \end{cases},$$ hence $\Gamma \to 1$ a.s.

Thereby the CMT implies that $b_T \to \varphi_{10}\frac{\gamma_0}{\gamma_0} + \theta_{10}\cdot\frac{1}{\gamma_0} + \frac{0}{\gamma_0} = \varphi_{10} + \theta_{10}/\gamma_0$.

Hence since $\theta_{10} \neq 0$, b_T is an inconsistent estimator of φ_{10}.

1. Would an analogous result hold true in the general ARMA(p,q) case where $q>0$, for the OLSE for the AR parameter vector?
2. Would it be possible to obtain an inconsistency corrected version of the OLSE if we had a consistent estimator for the θ_{10}? Explain.