Some Indicative Further Topics on Conditionally Heteroskedastic Models

1. An AR(1)-ARCH(1) Process

Consider $|b_0|<1$, $\omega>0$, $\alpha\geq 0$, and $(z_t)_{t\in\mathbb{Z}}$ iid with $E(z_0)=0$, $E(z_0^2)=1$, $E(z_0^3)=0$, $E(z_0^4):=u<+\infty$, $\alpha<\frac{1}{\sqrt{u}}$, and the processes

$(y_t)_{t\in\mathbb{Z}}$, $(\varepsilon_t)_{t\in\mathbb{Z}}$, $(h_t)_{t\in\mathbb{Z}}$, defined by

$$\begin{cases} y_t = b_0 y_{t-1}+\varepsilon_t, & t\in\mathbb{Z} \\ \varepsilon_t = z_t h_t^{1/2}, & t\in\mathbb{Z} \\ h_t = \omega+\alpha\varepsilon_{t-1}^2, & t\in\mathbb{Z} \end{cases} \quad (*)$$

whence due to the relevant theories previously established (explain) the triplet $[(y_t)_{t\in\mathbb{Z}},(\varepsilon_t)_{t\in\mathbb{Z}},(h_t)_{t\in\mathbb{Z}}]$ is the unique stationary and ergodic solution of $(*)$, has the form

$$y_t=\sum_{i=0}^{\infty}b_0^i\varepsilon_{t-i}, \quad t\in\mathbb{Z}$$

$$\varepsilon_t = z_t h_t^{1/2}, \quad t\in\mathbb{Z}$$

$$h_t=\omega\Big[1+\sum_{p=1}^{\infty}\alpha^p\prod_{m=1}^{p}z_{t-m}^2\Big], \quad t\in\mathbb{Z},$$

and $(\varepsilon_t)_{t\in\mathbb{Z}}$ is also an s.m.d. process* with $E(\varepsilon_0^2)=\frac{\omega}{1-\alpha}$, and simultaneously $(\varepsilon_t^2)_{t\in\mathbb{Z}}$ is the unique stationary and ergodic solution of $\varepsilon_t^2=\omega+\alpha\varepsilon_{t-1}^2+v_t$ where $(v_t)_{t\in\mathbb{Z}}$ is a stationary and ergodic s.m.d. process defined by $v_t:=(z_t^2-1)h_t$, $t\in\mathbb{Z}$. For the obvious reason $(y_t)_{t\in\mathbb{Z}}$ is called an AR(1)-ARCH(1) process (this would obviously be also true, if we weaken the assumption framework above, e.g. excluding the condition $E(z_0^3)=0$). We are interested in the limit theory of the OLSE for b_0, b_T, in this context.

1A. (Strong Consistency)

We have that $$b_T=\frac{\sum_{t=1}^{T}y_t y_{t-1}}{\sum_{t=1}^{T}y_{t-1}^2}=b_0+\frac{1/T\sum_{t=1}^{T}\varepsilon_t y_{t-1}}{1/T\sum_{t=1}^{T}y_{t-1}^2}$$

* w.r.t. $(G_t)_{t\in\mathbb{Z}}$ as previously established.

and from the general theory we have already established, due to the strong stationarity and ergodicity properties of $(\varepsilon_t)_{t\in\mathbb{Z}}$, $(y_t)_{t\in\mathbb{Z}}$, that $(\varepsilon_t)_{t\in\mathbb{Z}}$ is an s.m.d. process, $E(y_0^2) = \frac{\omega}{1-\alpha}\frac{1}{1-b_0^2} > 0$, and Birkhoff's LLN,

$$\frac{1}{T}\begin{pmatrix}\sum_{t=1}^{T} \varepsilon_t y_{t-1}\\ \sum_{t=1}^{T} y_{t-1}^2\end{pmatrix} \to \begin{pmatrix} E(\varepsilon_0 y_{-1})\\ E(y_0^2)\end{pmatrix} =$$

$$= \begin{pmatrix}0\\ \frac{\omega}{1-\alpha}\frac{1}{1-b_0^2}\end{pmatrix} \quad \mathbb{P}\text{ a.s.}$$

and thereby due to the CMT

$b_T \to b_0$ $\mathbb{P}$ a.s. (explain the details)

1.B Rate and Asymptotic Normality.

Analogously, from the general theory we have already established (remember the paragraph on the CLT for stationary and ergodic s.m.d. processeses), $\sqrt{T}$ rate and asymptotic normality for $\sqrt{T}(b_T - b_0)$ would follow from the existence of

I. $\sum_{i,j=0,\, i\neq j}^{\infty} b_0^i b_0^j E(\varepsilon_t^2 \varepsilon_{t-1-i}\varepsilon_{t-1-j})$, and

II. $\sum_{i=0}^{\infty} b_0^{2i} E(\varepsilon_t^2 \varepsilon_{t-1-i}^2)$.

For I. we have that $E|\varepsilon_t^2 \varepsilon_{t-1-i}\varepsilon_{t-1-j}| \le E^{1/2}(\varepsilon_t^4)\left[E|\varepsilon_{t-i-1}\varepsilon_{t-1-j}|^2\right]^{1/2}$ $\le E^{1/2}(\varepsilon_t^4)\, E^{1/4}(\varepsilon_{t-1-i}^4)\, E^{1/4}(\varepsilon_{t-1-j}^4) = E(\varepsilon_0^4) = E(z_0^4 h_0^2) = u E(h_0^2) < +\infty$ due to succesive applications of the Cauchy-Schwarz inequality, and

$$E(\varepsilon_t^2 \varepsilon_{t-1-i}\varepsilon_{t-1-j}) = E(z_t^2 h_t \varepsilon_{t-1-i}\varepsilon_{t-1-j}) \underset{\text{(why?)}}{\overset{\text{ind.}}{=}} E(z_t^2) E(h_t \varepsilon_{t-1-i}$$

$\varepsilon_{t-1-j}) = E(h_t \varepsilon_{t-1-i}\varepsilon_{t-1-j})$. Suppose without loss of generality that $i<j$ and due to LIE which is applicable from the

moment existence argument above, we have that

$$E(h_t \varepsilon_{t-1-i} \varepsilon_{t-1-j}) = E\left[E(h_t \varepsilon_{t-1-i} \varepsilon_{t-1-j} / \mathcal{F}_{t-1-i})\right].$$

Notice now that $E\left[(h_t \varepsilon_{t-1-i} \varepsilon_{t-j-1}) / \mathcal{F}_{t-1-i}\right] = E\left(h_t z_{t-1-i} h_{t-1-i}^{1/2} \varepsilon_{t-1-j} / \mathcal{F}_{t-1-i}\right) \underset{\text{(why?)}}{=} h_{t-1-i}^{1/2} \varepsilon_{t-1-j} E(h_t z_{t-1-i} / \mathcal{F}_{t-1-i})$ and

$$E(h_t z_{t-1-i} / \mathcal{F}_{t-1-i}) = E\left[\omega \left(1 + \sum_{p=1}^{i} \alpha^p \prod_{u=1}^{p} z_{t-u}^2 + \alpha^{i+1} \prod_{u=1}^{i+1} z_{t-u}^2 h_{t-i-1}\right) z_{t-i-1} / \mathcal{F}_{t-i-1}\right] = \omega E(z_{t-i-1} / \mathcal{F}_{t-i-1}) + \omega E\left(z_{t-i-1} \sum_{p=1}^{i} \alpha^p \prod_{u=1}^{p} z_{t-u}^2 / \mathcal{F}_{t-1-i}\right)$$

$$+ \omega \alpha^{i+1} E\left(z_{t-1-i} \prod_{u=1}^{i+1} z_{t-u}^2 h_{t-1-i} / \mathcal{F}_{t-i-1}\right) \underset{\text{(why?)}}{\overset{\text{ind}}{=}}$$

$$= \omega E(z_{t-i-1}) + E(z_{t-1-i}) E\left(\sum_{p=1}^{i} \alpha^p \prod_{u=1}^{p} z_{t-u}^2\right) + \omega \alpha^{i+1} h_{t-i-1} E(z_{t-1-i}^3) E\left(\prod_{u=1}^{i} z_{t-u}^2\right) = 0 + 0 \sum_{p=1}^{i} \alpha^p + \omega \alpha^{i+1} h_{t-i-1} 0 \cdot 1 = 0.$$

Hence, $E(h_t \varepsilon_{t-1-i} \varepsilon_{t-1-j} / \mathcal{F}_{t-1-i}) = 0$ which then implies that $E(\varepsilon_t^2 \varepsilon_{t-i-1} \varepsilon_{t-j-1}) = 0$ when $i < j$. Interchanging i with j in the argument above, we obtain that the previous is also 0 when $i > j$ hence $\sum_{\substack{i,j=0 \\ i \neq j}}^{\infty} b_0^i b_0^j E(\varepsilon_t^2 \varepsilon_{t-1-i} \varepsilon_{t-1-j}) = 0$.

[Remember that the unique stationary and ergodic solution $(h_t)_{t \in \mathbb{Z}}$ is obtained as a limit of backward substitutions in the ARCH(1) [GARCH(1,0)] recursion. Iterating the substitutions i-times we obtain $h_t = \omega\left[1 + \sum_{p=1}^{i} \alpha^i \prod_{u=1}^{p} z_{t-u}^2 + \alpha^{i+1} \prod_{u=1}^{i+1} z_{t-u}^2 h_{t-i-1}\right]$.

Furthermore in an analogous fashion to the previous calculation, we have can show that when the distribution of z_0 is symmetric $E(\varepsilon_t^2\varepsilon_{t-j})=0$ $\forall j>0$, a property known as dynamic symmetry for the process, which does not correspond to the empirical dynamic asymmetry property that appears as a stylized fact to aforementioned series of financial returns]. $\square$

II. From the AR(1) representation of $(\varepsilon_t^2)_{t\in\mathbb{Z}}$ and previous calculations we obtain that

$$\sum_{i=0}^{\infty} b_0^{2i} E(\varepsilon_t^2\varepsilon_{t-1-i}^2) =$$

$$=\sum_{i=0}^{\infty} b_0^{2i}\left[\frac{\alpha^{i+1}}{1-\alpha^2}(u-1)\frac{P(\omega,\alpha,0)}{1-A(\alpha,0,u)}+\left(\frac{\omega}{1-\alpha}\right)^2\right] = \frac{\alpha(u-1)}{1-\alpha^2}\frac{P(\omega,\alpha,0)}{1-A(\alpha,0,u)}\sum_{i=0}^{\infty}(b_0^2\alpha)^i$$

$$+\left(\frac{\omega}{1-\alpha}\right)^2\sum_{i=0}^{\infty} b_0^{2i} = \frac{\alpha(u-1)}{1-\alpha^2}\frac{P(\omega,\alpha,0)}{1-A(\alpha,0,u)}\frac{1}{1-b_0^2\alpha}+\frac{\omega^2}{(1-\alpha)^2}\frac{1}{1-b_0^2} := C(\omega,\alpha,u,b_0).$$

$\square$

Hence using our general results we obtain that

$$\sqrt{T}(b_T-b_0)\xrightarrow{d} N\left(0, v(\omega,\alpha,u,b_0)\right)$$

where $v(\omega,\alpha,u,b_0) := C(\omega,\alpha,u,b_0)\left[E(y_0^2)\right]^{-2}$

$$= C(\omega,\alpha,u,b_0)\left(\frac{1}{1-b_0^2}\frac{\omega}{1-\alpha}\right)^{-2} = E(\varepsilon_0^2 y_{-1}^2)\left(E(y_0^2)\right)^{-2}. \square$$

Notice that when $\alpha=0$, $C(\omega,0,u,b_0)=\omega^2(1-b_0^2)^{-1}$, whence $v(\omega,0,u,b_0)=1-b_0^2$.

I.c. A non-parametric estimator of $v(\omega,\alpha,u,b_0)$.

Given b_T, consider $e_t := y_t - b_T y_{t-1}$, $\forall t=1,\dots,T$. Notice that

$$e_t y_{t-1}^2-\varepsilon_t y_{t-1}^2 = (y_t-b_T y_{t-1})^2 y_{t-1} - \varepsilon_t^2 y_{t-1}^2 = (b_0 y_{t-1}+\varepsilon_t)^2 y_{t-1}^2$$

$-\varepsilon_t^2 y_{t-1}^2 = [(b_0-b_T)y_{t-1}+\varepsilon_t]^2 y_{t-1}^2 - \varepsilon_t^2 y_{t-1}^2 = (b_0-b_T)^2 y_{t-1}^4 + 2(b_0-b_T)$
$\varepsilon_t y_{t-1}^3 + \varepsilon_t^2 y_{t-1}^2 - \varepsilon_t^2 y_{t-1}^2 = (b_0-b_T)^2 y_{t-1}^4 + 2(b_0-b_T)\varepsilon_t y_{t-1}^3.$

Hence due to the triangle inequality

$$\left|\frac{1}{T}\sum_{t=1}^{T} e_t^2 y_{t-1}^2 - \frac{1}{T}\sum_{t=1}^{T} \varepsilon_t^2 y_{t-1}\right| \leq$$

$$(b_0-b_T)^2 \frac{1}{T}\sum_{t=1}^{T} y_{t-1}^4 + 2\,|b_0-b_T|\,\frac{1}{T}\sum_{t=1}^{T}|\varepsilon_t y_{t-1}^3| \quad (*)$$

Since $|b_0|<1$ and $E(\varepsilon_0^4)=\kappa E(h_0^2)<+\infty$ we have that $E(y_0^4)<+\infty$ (Derive it!).

Analogously $E(|\varepsilon_0||y_{t-1}|^3) \leq E^{1/4}(|\varepsilon_0|^4)\, E^{3/4}(|y_{t-1}|^4) < +\infty$ due to Hölder's inequality [$p,q>0: \frac{1}{p}+\frac{1}{q}=1$, $E(|xy|) \leq E^{1/p}(|x|^p)$ $E^{1/q}(|y|^q)$ – The Cauchy-Schwarz is a special case for $p=q=2$]

Hence Birkhoff's LLN (why is it applicable?) implies that

$$\frac{1}{T}\begin{pmatrix} \sum_{t=1}^{T} y_{t-1}^4 \\ \sum_{t=1}^{T} \varepsilon_t y_{t-1}^3 \end{pmatrix} \to \begin{pmatrix} E(y_0^4) \\ E(\varepsilon_t y_{t-1}^3) \end{pmatrix} \quad P \text{ a.s. as } T\to+\infty$$

We already know that $|b_0-b_T|\to 0$ P a.s. as $T\to+\infty$

and due to the CMT, $|b_0-b_T|^2 \to 0$ P a.s. as $T\to+\infty$.

Hence the r.h.s. of $(*)$ converges to 0, P a.s. as $T\to+\infty$

and then (*) implies that $\left|\frac{1}{T}\sum_{t=1}^{T} e_t^2 y_{t-1}^2 - \frac{1}{T}\sum_{t=1}^{T} \varepsilon_t^2 y_{t-1}^2\right| \to 0$

$\mathbb{P}$ a.s. as $T \to +\infty$. Furthermore due to strong stationarity and ergodicity, the previously derived fact that $E(\varepsilon_t^2 y_{t-1}^2) =$

$$= \sum_{i=0}^{\infty} b_0^{2i} E(\varepsilon_t^2 \varepsilon_{t-1-i}^2) + \sum_{\substack{i,j=0 \\ i \neq j}}^{\infty} b_0^i b_0^j E(\varepsilon_t^2 \varepsilon_{t-1-i} \varepsilon_{t-1-j}) = C(\omega, \alpha, u, b_0)$$

$\in \mathbb{R}$, hence Birkhoff's LLN implies that $\frac{1}{T}\sum_{t=1}^{T} \varepsilon_t^2 y_{t-1}^2 \to E(\varepsilon_t^2 y_{t-1}^2)$.

Hence $\left|\frac{1}{T}\sum_{t=1}^{T} e_t^2 y_{t-1}^2 - E(\varepsilon_t^2 y_{t-1}^2)\right| \leq \left|\frac{1}{T}\sum_{t=1}^{T} e_t^2 y_{t-1}^2 - \frac{1}{T}\sum_{t=1}^{T} \varepsilon_t^2 y_{t-1}^2\right|$
$+ \left|\frac{1}{T}\sum_{t=1}^{T} \varepsilon_t^2 y_{t-1}^2 - E(\varepsilon_t^2 y_{t-1}^2)\right|$. We established that both terms on the

r.h.s. of the last inequality converge to 0, $\mathbb{P}$ a.s. as $T \to +\infty$, hence the inequality implies that $\frac{1}{T}\sum_{t=1}^{T} e_t^2 y_{t-1}^2 \to E(\varepsilon_t^2 y_{t-1}^2)$, $\mathbb{P}$ a.s. as $T \to +\infty$.

Analogously (explain!) $\frac{1}{T}\sum_{t=1}^{T} y_{t-1}^2 \to E(y_0^2) = \frac{\omega}{1-\alpha}\frac{1}{1-b_0^2}$, $\mathbb{P}$ a.s. as $T \to +\infty$.

Since both the considered limits are constant, $\frac{1}{T}\begin{pmatrix} \sum_{t=1}^{T} e_t^2 y_{t-1}^2 \\ \sum_{t=1}^{T} y_{t-1}^2 \end{pmatrix} \to$
$\begin{pmatrix} E(\varepsilon_0^2 y_{-1}^2) \\ E(y_0^2) \end{pmatrix}$, $\mathbb{P}$ a.s. as $T \to +\infty$, and thereby,

due to the CMT (why is it applicable?) $\begin{pmatrix} \frac{1}{T}\sum_{t=1}^{T} e_t^2 y_{t-1}^2 \\ \frac{1}{T^2}\left(\sum_{t=1}^{T} y_{t-1}^2\right)^2 \end{pmatrix} \to$

$\to V(\omega, \alpha, b, u)$, $\mathbb{P}$ a.s. as $T \to +\infty$. Hence $V_T := T\dfrac{\sum_{t=1}^{T} e_t^2 y_{t-1}^2}{\sum_{t=1}^{T} y_{t-1}^2} \to V(\omega, \alpha, b, u)$

$\mathbb{P}$ a.s. as $T\to+\infty$, hence V_T is a strongly consistent estimator of the asymptotic variance. (V_T can be characterized as non-parametric since it is derivable without the need of estimating the parameters appearing in the aforederived expression of the asymptotic variance).

Slutsky's lemma along with the previous limit theories imply that

$$\frac{1}{\sqrt{V_T}}\sqrt{T}(b_T-b_0) \xrightarrow{d} N(0,1) \text{ as } T\to+\infty.$$

Hence,

$$T(b_T-b_0)^2\frac{1}{V_T} = T(b_T-b_0)^2\cdot\frac{\frac{1}{T}\left(\sum_{t=1}^{T}(y_{t-1})^2\right)^2}{\sum_{t=1}^{T}e_t^2y_{t-1}^2} \xrightarrow{d} \chi_1^2 \text{ as } T\to+\infty.$$

We have essentially proven the following proposition.

Proposition. Under the current assumption framework, as $T\to+\infty$

$$W_T(b_0) := (b_T-b_0)^2\frac{\left(\sum_{t=1}^{T}y_{t-1}^2\right)^2}{\sum_{t=1}^{T}e_t^2y_{t-1}^2} \xrightarrow{d} \chi_1^2. \square$$

1.D A Wald test for b_0.

In the previous framework and for $|b^*|<1$ consider the hypothesis structure

$$H_0: b_0=b^*$$
$$H_1: b_0\neq b^*.$$

If H_0 is true, the previous proposition implies that

$$W_T(b^*) = (b_T-b^*)^2\frac{\left[\sum_{t=1}^{T}y_{t-1}^2\right]^2}{\sum_{t=1}^{T}e_t^2y_{t-1}^2} \xrightarrow{d} \chi_1^2 \text{ as } T\to+\infty$$

Hence, using the Wald-type statistic $W_T(b^*)$ and this limit theory under H_0, and for $\alpha\in(0,1)$ the significance level, and $q_{\chi_1^2}(1-\alpha) := \inf\{x\in\mathbb{R}: P(u\leq x)=1-\alpha\}$ where

$u \sim \chi^2_1$, which is a well defined positive real number $\forall \alpha \in (0,1)$, we can design the following testing procedure:

Testing Procedure: reject $\mathcal{H}_0$ iff $W_T(\beta^*) > q_{\chi^2_1}(1-\alpha)$.

Then we obtain the following result on asymptotic properties of the procedure.

Lemma. Under the current framework for the testing procedure defined above we have:

$$a. \lim_{T\to+\infty} \mathbb{P}[\text{reject } \mathcal{H}_0 / \mathcal{H}_0] = \alpha, \text{ i.e}$$

the procedure is asymptotically exact, and

$$b. \lim_{T\to+\infty} \mathbb{P}[\text{reject } \mathcal{H}_0 / \mathcal{H}_1] = 1,$$

i.e. the procedure is consistent. $\square$

Proof. Suppose that $\mathcal{H}_0$ is true. Then due to the definition of the procedure and the previous proposition,

$$P(\text{reject } \mathcal{H}_0 / \mathcal{H}_0) = P\left(W_T(\beta^*) > q_{\chi^2_1}(1-\alpha) / \mathcal{H}_0\right)$$

$$\to P(u > q_{\chi^2_1}(1-\alpha)) = 1-(1-\alpha) = \alpha \quad \text{as } T\to+\infty,$$

where $u \sim \chi^2_1$.

Suppose that $\mathcal{H}_1$ is true, hence $W_T(\beta^*) = T(\beta_T - \beta^*)^2 v_T^{-1}$

$$= (\beta_T \pm \beta_0 - \beta^*)^2 v_T^{-1} T = T\left[(\beta_T-\beta_0)^2 + (\beta_0-\beta^*)^2 + 2(\beta_0-\beta^*)(\beta_T-\beta_0)\right] v_T^{-1}.$$

Remember that $T(\beta_T-\beta_0)^2 v_T^{-1} \xrightarrow{d} \chi^2_1$ as $T\to+\infty$.

Furthermore, since $b_0 \neq b^*$ and due to the previous, $T(b_0-b^*)^2 v_T^{-1} \xrightarrow{d} +\infty$ as $T \to +\infty$, while it dominates $2\sqrt{T}(b_0-b^*)\sqrt{T}(b_T-b_0)v_T^{-1}$ which also diverges but at a rate $\sqrt{T}$ (why?). Hence under $\mathcal{H}_1$, $W_T(b^*) \xrightarrow{d} +\infty$, and since $q_{\chi_1^2}(1-\alpha) \in \mathbb{R}$, we obtain that

$$\lim_{T\to+\infty} P(\text{reject } \mathcal{H}_0 / \mathcal{H}_1) = \lim_{T\to+\infty} P\left(W_T(b^*) > q_{\chi_1^2}(1-\alpha)\right) = 1. \square$$

2. Extension of the GARCH(1,1) - The GARCH(p,q) Process

The GARCH(1,1) is a special case of the following GARCH(p,q) recursion, for $p,q \in \mathbb{N}$, $\omega > 0$, $\alpha_i, b_j \geq 0$, $i=1,\dots,p$, $j=1,\dots,q$,

$$\begin{aligned} h_t &= \omega + \sum_{i=1}^{q} \alpha_i y_{t-i}^2 + \sum_{i=1}^{p} b_i h_{t-i}, \quad t\in\mathbb{Z} \\ &= \omega + \sum_{i=1}^{\max(p,q)} (\alpha_i^* z_{t-i}^2 + b_i^*) h_{t-i}, \quad t\in\mathbb{Z} \end{aligned}$$

where $\alpha_i^* = \begin{cases} \alpha_i, & i\leq p \\ 0, & i>p \end{cases}$, $b_i^* = \begin{cases} b_i, & i\leq q \\ 0, & i>q \end{cases}$.

Analogous, yet possibly more complex, considerations to the $p=q=1$ case, can establish analogous properties. $\square$

3. Another Example: An EGARCH(1,1) process (Exponential GARCH)

The positivity constraints for the $(h_t)_{t\in\mathbb{Z}}$-process in the GARCH(1,1) case as well as properties such like the aforementioned dynamic symmetry property (which can be easily extended to the GARCH(1,1) case) under the symmetry assumption for the distribution of z_0, enforced the examination of models that avoid such constraints and/or exhibit (among others) properties of dynamic assymetry, even when $E(z_0^3)=0$, something consistent with stylized facts of financial returns. Among the plethora of such models a celebrated one is the EGARCH(1,1) process, defined by the recursion (remember that we lie in the general framework of conditional heteroskedasticity) $(\omega, \alpha, \gamma, b \in \mathbb{R})$

$$\ln h_t = \omega + \alpha\left(|z_{t-1}| - E|z_0|\right) + \gamma z_{t-1} + b \ln h_{t-1}, \ t \in \mathbb{Z}$$

$$\Leftrightarrow \qquad h_t = \exp(\omega + u_{t-1} + b \ln h_{t-1}), \ t \in \mathbb{Z}$$

where $u_t := \alpha\left(|z_t| - E|z_0|\right) + \gamma z_t$.

Notice that the exponential formulation of h_t imply that there is no need for positivity constraints, for h_t to be well defined. Notice also that $(u_t)_{t \in \mathbb{Z}}$ is iid (why?) with $E(u_0)=0$, $E(u_0^2) = \alpha^2\left[E(|z_0|^2) - E^2(|z_0|)\right] + \gamma^2 + 2\alpha\gamma E(|z_0| z_0) := \sigma^2 < +\infty$.

Hence $\ln h_t = \omega + b \ln h_{t-1} + u_t$ is an AR(1) - recursion w.r.t. $(u_t)_{t \in \mathbb{Z}}$, and using e.g. the general lemma we know that the latter admits a unique strictly stationary and ergodic solution if (in fact we can prove that it is an iff assertion) $|b|<1$, whence the solution is

$$\ln h_t = \sum_{i=0}^{\infty} b^i (\omega + u_{t-1-i}) =$$

$$= \frac{\omega}{1-b} + \sum_{i=0}^{\infty} b^i u_{t-i-1} \quad \text{and thereby}$$

we can prove the following result.

Lemma. Iff $|b|<1$, the EGARCH(1,1) process $(y_t)_{t \in \mathbb{Z}}$ is strictly stationary and ergodic and is obtained as follows:

$$y_t = z_t h_t^{1/2}, \quad t\in\mathbb{Z}$$

$$h_t = \exp\left(\frac{\omega}{1-b}\right)\prod_{i=0}^{\infty}\exp\left(b^i(\alpha(|z_{t-1-i}|-E|z_0|)+b^i\gamma z_{t-1-i}\right), \quad t\in\mathbb{Z}$$

for $(z_t)_{t\in\mathbb{Z}}$ iid with $E(z_0)=0$, $E(z_0^2)=1$. $\square$

A sufficient condition for the characterization of $(y_t)_{t\in\mathbb{Z}}$ (given the previous) as an strictly stationary and ergodic s.u.d. process is that

[*] $\exists i^*\in\mathbb{N}, c\in(0,1), A>0 : |\ln M(b^i,\alpha,\gamma)|\leq A c^i, \ \forall i\geq i^*$,

and this is due to the fact that $E(h_t) = E\left(\exp\left(\frac{\omega^*}{1-b}\right)\prod_{i=0}^{\infty}\exp(b^i(\alpha|z_{t-1-i}|+\gamma z_{t-1-i}))\right) \overset{ind}{=} \exp\left(\frac{\omega^*}{1-b}\right)\prod_{i=0}^{\infty}E\left(\exp(b^i(\alpha|z_{t-1-i}|+\gamma z_{t-1-i}))\right) \overset{hom}{=}$

$$= \exp\left(\frac{\omega^*}{1-b}\right)\prod_{i=0}^{\infty}E\left(\exp\left(b^i(\alpha|z_0|+\gamma z_0)\right)\right) =$$

$$= \exp\left(\frac{\omega^*}{1-b}\right)\exp\left(\sum_{i=0}^{\infty}\ln\left(E(\exp(b^i(\alpha|z_0|+\gamma z_0)))\right)\right)$$

$$= \exp\left(\frac{\omega^*}{1-b}\right)\exp\left(\sum_{i=0}^{\infty}\ln M(b^i,\alpha,\gamma)\right),$$ where $\omega^* = \omega-\alpha E(|z_0|)$, and $M(b^i,\alpha,\gamma) := E(\exp(b^i(\alpha|z_0|+\gamma z_0)))$, hence if [*] holds then (suppose for simplicity that $i^*=0$)

$$0\leq\sum_{i=0}^{\infty}|\ln M(b^i,\alpha,\gamma)| \leq \sum_{i=0}^{\infty}c^i = \frac{1}{1-c} < +\infty,$$

hence $\sum_{i=0}^{\infty}\ln M(b^i,\alpha,\gamma)$ exists. E.g. if $z_0\sim N(0,1)$ it is easy to prove that $M(b^i,\alpha,\gamma) = \exp\left(\frac{b^{2i}}{2}(\alpha-\gamma)^2\right)\Phi(b^i(\alpha-\gamma)) + \exp\left(\frac{b^{2i}}{2}(\alpha+\gamma)^2\right)\Phi(b^i(\alpha+\gamma))$ [L].

Now, by defining $\Gamma = \max\left((\alpha-\gamma), (\alpha+\gamma)\right)$ we obtain that
[1] $\leq \exp\left(\frac{b^{2i}}{2}\Gamma\right) 2\Phi(b^i\Gamma)$.

Hence, and by using series representations for Φ, we obtain that
$$|\ln M(b^i,\alpha,\gamma)| \leq \frac{b^{2i}}{2}|\Gamma| + \left|\ln\left[1+\sqrt{\tfrac{2}{\pi}}\exp\left(-\frac{b^{2i}}{2}\Gamma^2\right)\sum_{j=0}^{\infty}\frac{b^{i(2j+1)}\Gamma^{2j+1}}{(2j+1)!!}\right]\right|$$
$$\leq \frac{b^{2i}}{2}|\Gamma| + \left|\ln\left[1+\sqrt{\tfrac{2}{\pi}}e^{-\frac{b^{2i}\Gamma^2}{2}} b^i|\Gamma| e^{|b|^i\Gamma}\right]\right| \leq \frac{b^{2i}}{2} + M|b|^i \leq \left(M+\tfrac{1}{2}\right)|b|^i \text{ for } M>0.$$
Hence [*] holds for $c=|b|$ and $A = M+1/2$. $\square$ → do not bother with the derivations of c and A in this particular case if you don't want to!

4. Gaussian QMLE in Conditionally Heteroskedastic Processes

It can be proven that the Gaussian QMLE is a semiparametric estimator for the conditional variance in such processes that given stationarity and ergodicity and under "mild" regularity conditions (that in some cases do not ever require weak stationarity) it is strongly consistent, with $\sqrt{T}$ rate and asymptotically normal. $\square$

[The notes are in a state of perpetual correction. They do not substitute the lectures. Please r eport any typos to stelios@aueb.gr or the course's e-class.]

An excellent (among others) reference for conditional heteroskedasticity is:

Francq, C., & Zakoian, J. M. (2011). GARCH models: structure, statistical inference and financial applications. John Wiley & Sons.

Exercise: In the context of the ARCH(1) model, with $\omega=1$, $\alpha\geq 0$ unknown, $E(z_0^4)=\mu<\infty$, and $\alpha<1/\sqrt{\mu}$, derive $\alpha_T = \sum_{t=1}^{T}(y_t^2-1)y_{t-1}^2 \Big/ \sum_{t=1}^{T} y_{t-1}^2$

as an OLSE estimator and show that it is strongly consistent.