ARMA Models as Linear Processes with Abs. Summability: Theory and Exercises

We have already defined the concept of a linear causal process constructed upon a white noise process and a sequence of absolutely summable coefficients. The interest in such a process lies in the following Theorem.

Wold Decomposition Theorem. If $y=(y_t)_{t\in\mathbb{Z}}$ is weakly stationary then $y=z+l$ where z is a deterministic process and l is a causal linear process constructed upon a white noise process and a sequence of square summable coefficients. □

Hence a (weaker than we have for expositional convenience version of a) linear process is the stochastic component of an arbitrary weakly stationary process, the remaining component being a real double sequence $z=(z_t)_{t\in\mathbb{Z}}$, $z_t\in\mathbb{R}$ $\forall t\in\mathbb{Z}$.

In what follows we are interested in a subclass of such models, the so called ARMA models with absolutely summable coefficients.

Power Series w.r.t. the Lag Operator

Given that the typical element of such a process has the form $y_t=\sum_{j=0}^{\infty}B_j\varepsilon_{t-j}=\sum_{j=0}^{\infty}B_jL^j\varepsilon_t:=\Phi(L)\varepsilon_t$ where $\Phi(L)=\sum_{j=0}^{\infty}B_jL^j$ which is an operator defined as a power series w.r.t. L.

Definition. Given a real sequence $(B_j)_{j\in\mathbb{N}}$, $\Phi(L):=\sum_{j=0}^{\infty}B_jL^j$ is called a formal power series w.r.t. L.

If $\exists p \in \mathbb{N}$: $B_j = 0$ $\forall j > p$ then $\Phi(L)$ is termed of p-order (finite order). $\square$ [i.e. p = degree of the polynomial $\Phi(L)$]

Definition. [product]. If $\Phi(L)$ and $\Psi(L)$ are formal power series w.r.t. L, then $\Phi(L)\Psi(L)$ is a formal power series w.r.t. L, defined by convolution of coefficients, i.e. $\Phi(L)\Psi(L) = \sum_{j=0}^{\infty} \gamma_j L^j$ where

$$(*) \quad \gamma_j = \sum_{k=0}^{j} B_k \psi_{j-k} \quad \Phi(L) = \sum_{j=0}^{\infty} B_j L^j,$$

$$\Psi(L) = \sum_{j=0}^{\infty} \psi_j L^j. \ \square$$

Lemma. $\Phi(L)\Psi(L) = \Psi(L)\Phi(L)$.

Proof. Obvious from the definition of γ_j, $j \in \mathbb{N}$. $\square$

Lemma. If $\Phi(L)$ has absolutely summable coefficients and $\Psi(L)$ has finite order, then $\Phi(L)\Psi(L)$ has absolutely summable coefficients. If moreover $\exists A > 0$, $b \in (0,1)$: $|B_j| \le A b^j$, $\forall j \in \mathbb{N}$ then $|\gamma_j| \le A^* c^j$, $A^* > 0$, $c \in (0,1)$, $\forall j \ge 0$.
[Prove it].

Definition. [Multiplicative Inverse]. $\Psi(L)$ is called a multiplicative inverse of $\Phi(L)$ iff $\Phi(L)\Psi(L) = \Psi(L)\Phi(L) = \underline{1}$.

Lemma. If a multiplicative inverse of $\Phi(L)$ exists, then it is unique. Hence it is denoted by $\Phi^{-1}(L)$.

Proof. If $\Psi_1(L)$, $\Psi_2(L)$ are multiplicative inverses, then

$$\left.\begin{array}{l} \Psi_2(L)\Phi(L)\Psi_1(L) = \Psi_1(L) \\ \Psi_2(L)\Phi(L)\Psi_1(L) = \Psi_2(L) \end{array}\right\} \Rightarrow \Psi_1(L) = \Psi_2(L).$$

Lemma. $\Phi^{-1}(L)$ exists iff $b_0 \neq 0$. Furthermore if $\Phi^{-1}(L) := \sum_{j=0}^{\infty} \varphi_j L^j$ then

$$\varphi_0 = 1/b_0,$$
$$\varphi_j = -1/b_0 \sum_{k=1}^{j} b_k \varphi_{j-k}, \quad j > 0. \square$$

[Prove it].

Lemma [Multiplicative Inverse and Absolute Summability]. Suppose that $\Phi(L)$ is of order $p>0$. Consider the characteristic polynomial of $\Phi(L)$, $\Phi(z) = B_0 + B_1 z + \ldots + B_p z^p$, for z a complex variable. If each of the p roots of $\Phi(z)$ lie outside the unit disc, i.e. each of the roots has norm greater than one, then $\exists\, A>0,\ b \in (0,1)$: $|\varphi_j| \leq A b^j$, $j \in \mathbb{N}$, i.e. and since $\sum_{j=0}^{\infty} |\varphi_j| \leq \sum_{j=0}^{\infty} A b^j = \frac{A}{1-b}$, hence $\Phi^{-1}(L)$ has absolutely summable coefficients. (refer to the roots condition as UDC)

[Try to prove it!]

Example. $\Phi(L) = 1 - B_1 L$, $p=1$, $\Phi(z) = 1 - B_1 z$ with a single root $1/B_1$. $\quad$ UDC $\Leftrightarrow \left|\frac{1}{B_1}\right| > 1 \Leftrightarrow |B_1| < 1$

implies that $\varphi_j = B_1^j$, and $A=1$, $b = B_1$. $\square$

Example. $\Phi(L) = 1 - B_2 L^2$, $\Phi(z) = 1 - B_2 z^2$ with $\Delta = 4B_2$. If $B_2 > 0$ then there exist two real roots $\frac{\pm 2\sqrt{B_2}}{-2B_2} = \pm \frac{1}{\sqrt{B_2}}$ and the UDC is equivalent to $\frac{1}{\sqrt{B_2}} > 1$ $\Leftrightarrow \sqrt{B_2} < 1 \Leftrightarrow B_2 \in (0,1)$. When $B_2 < 0$ then we have two imaginary roots $\frac{\pm 2\sqrt{|B_2|}\, i}{-2B_2} = \pm \frac{1}{\sqrt{|B_2|}}\, i$ and the condition is again equivalent to $1/\sqrt{|B_2|} > 1 \iff b_2 \in (-1,0)$. In both

cases and using the lemma above on the form of the ψ_j we have that $\psi_0 = 1$, $\psi_j = \begin{cases} 0, & j \text{ odd} \\ B_2^{j/2}, & j \text{ even}. \end{cases}$

Hence $\Phi^{-1}(L) = \sum_{j=0}^{\infty} \psi_j L^j = \sum_{j=0,2,\ldots} B_2^{j/2} L^j = \sum_{j=0}^{\infty} B_2^j L^{2j}$.

Notice that $A = 1$, $b = B_2^2$,

ARMA Processes as Linear Processes with Absolutely Summable Coefficients

Definition. Suppose that $(\varepsilon_t)_{t\in\mathbb{Z}} \sim WN(\sigma^2)$. Let $\Phi(L) = 1 + \sum_{j=1}^{p} B_j L^j$ and $\Theta(L) = 1 + \sum_{j=1}^{q} \theta_j L^j$ are of order p and q respectively. Then the ARMA(p,q) process constructed on $(\varepsilon_t)_{t\in\mathbb{Z}}$, $\Phi(L)$, $B(L)$ is defined as a solution of

$$[\Delta] \quad \Phi(L) y_t = \Theta(L) \varepsilon_t, \quad t \in \mathbb{Z}. \square$$

$[\Delta]$ defines a p-th order linear stochastic difference equation.

Lemma. If the ULDC hold for the roots of $\Phi(L)$ then the ARMA(p,q) process is a causal linear process, defined by

$$y_t = \Phi^{-1}(L)\Theta(L)\varepsilon_t = \sum_{j=0}^{\infty} \psi_j L^j \varepsilon_t = \sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j}$$

where $(\psi_j)_{j\in\mathbb{N}}$ is an absolutely summable sequence, hence

it is weakly stationary, with $\mathbb{E}(y_t) = 0$, $t \in \mathbb{Z}$, $\gamma_k = \sum_{j=0}^{\infty} \psi_j \psi_{j+k}$, $k \geq 0$. Moreover $(y_t)_{t\in\mathbb{Z}}$ is also regular and short memory. If moreover $(\varepsilon_t)_{t\in\mathbb{Z}}$ is strictly stationary and ergodic then $(y_t)_{t\in\mathbb{Z}}$ is also strictly stationary and ergodic.

Sketch of Proof. Apply the previous lemma to $\Phi^{-1}(L)$. Then apply the previous lemmata to infer the relevant properties on the coefficients of $\Phi^{-1}(L)\Theta(L)$, along with the fact that $\Theta(L)$ is of finite order. Then apply the past results on the definition and properties of linear causal processes. Notice that short memory and hence regularity (why?) follows from that due to the previous, $\exists A^*>0, c\in(0,1)$:

$$|\varphi_j|\leq A^* c^j \Rightarrow \sum_{k=0}^{\infty}|\gamma_k| = \sum_{k=0}^{\infty}\left|\sum_{j=0}^{\infty}\varphi_j\varphi_{j+k}\right| \leq$$

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}|\varphi_j\varphi_{j+k}| \leq \sum_{k=0}^{\infty}\left(\sum_{j=0}^{\infty}A^{*2}c^j c^{j+k}\right)$$

$$= A^{*2}\sum_{k=0}^{\infty}c^k\sum_{j=0}^{\infty}(c^2)^j = \frac{A^{*2}}{1-c^2}\sum_{k=0}^{\infty}c^k = \frac{A^{*2}}{1-c^2}\frac{1}{1-c} < +\infty. \square$$

Convention. The previous hold even in the case where $p=0$, whence $\Phi(L)$ has no roots and thereby ULDC is assumed to hold trivially. Furthermore $ARMA(0,q) := MA(q)$ while analogously $ARMA(p,0) := AR(p)$.

Example. $AR(1)$ [see the previous first example]

Example. [See the second example in the previous section] $\Phi(L) = 1 - B_2 L^2$, $q=0$, i.e. $\Theta(L)=1$. if $\varphi_2\in(-1,1)$ then the previous derivations imply that $\varphi_j = \begin{cases} 0 & , j \text{ odd} \\ B_2^{j/2} & , j \text{ even}\end{cases}$

and $y_t = \sum_{j=0}^{\infty} B_2^j \varepsilon_{t-2j}$, while $\gamma_k = \sum_{j=0}^{\infty}\varphi_j\varphi_{j+k} =$

$$= \begin{cases} 0, & k \text{ odd} \\ \sum_{j=0,2,\dots} B_2^{j/2} B_2^{(j+k)/2} \end{cases} = \begin{cases} 0, & k \text{ odd} \\ B_2^{k/2} \sum_{j=0,2,\dots} B_2^{j}, & k \text{ even} \end{cases}$$

$$= \begin{cases} 0, & k \text{ odd} \\ B_2^{k/2} \sum_{j=0}^{\infty} B_2^{2j}, & k \text{ even} \end{cases} = \begin{cases} 0, & k \text{ odd} \\ \dfrac{B_2^{k/2}}{1-B_2^2}, & k \text{ even} \end{cases}. \square$$

Exercise. Find y_t and γ_k, for AR(2), ARMA(1,1) ARMA(1,2) and MA(q) for $q>0$.

Exercise. Suppose that $\Phi(L)$ obeys the UCD and that $\Phi(L)$ and $\Theta(L)$ have a common root. Show that ARMA(p,q) = ARMA(p-1,q-1) as processes.