The utility if I buy from firm 1

$$U = V - t * (a - x)^2 - P1$$

The utility if I buy from firm 2

$$U = V - t * (b - x)^2 - P2$$

The marginal consumer

$$\text{Simplify}\left[\text{Solve}\left[V - t * (a - x)^2 - P1 - \left(V - t * (b - x)^2 - P2\right) == 0, x\right]\right]$$

$$\left\{\left\{x \to \frac{P1 - P2 + a^2\,t - b^2\,t}{2\,a\,t - 2\,b\,t}\right\}\right\}$$

The FOC for firm 1

$$\text{Simplify}\left[D\left[P1 * \left(\frac{P1 - P2 + a^2\,t - b^2\,t}{2\,a\,t - 2\,b\,t}\right), P1\right]\right]$$

$$\frac{2\,P1 - P2 + a^2\,t - b^2\,t}{2\,a\,t - 2\,b\,t}$$

$$\text{Simplify}\left[\text{Solve}\left[\frac{2\,P1 - P2 + a^2\,t - b^2\,t}{2\,a\,t - 2\,b\,t} == 0, P1\right]\right]$$

$$\left\{\left\{P1 \to \frac{1}{2}\left(P2 + \left(-a^2 + b^2\right)\,t\right)\right\}\right\}$$

The FOC for firm 2

$$\text{Simplify}\left[D\left[P2 * \left(1 - \frac{P1 - P2 + a^2\,t - b^2\,t}{2\,a\,t - 2\,b\,t}\right), P2\right]\right]$$

$$-\frac{P1 - 2\,P2 + \left(-2\,a + a^2 - (-2 + b)\,b\right)\,t}{2\,(a - b)\,t}$$

$$\text{Simplify}\left[\text{Solve}\left[-\frac{P1 - 2\,P2 + \left(-2\,a + a^2 - (-2 + b)\,b\right)\,t}{2\,(a - b)\,t} == 0, P2\right]\right]$$

$$\left\{\left\{P2 \to \frac{1}{2}\left(P1 + \left(-2\,a + a^2 - (-2 + b)\,b\right)\,t\right)\right\}\right\}$$

the prices are

$$P2 = \frac{1}{2}\left(P1 + \left(-2\,a + a^2 - (-2 + b)\,b\right)\,t\right)$$

$$\frac{1}{2}\left(P1 + \left(-2\,a + a^2 - (-2 + b)\,b\right)\,t\right)$$

$$\text{Simplify}\left[\text{Solve}\left[P1 - \left(\frac{1}{2}\left(P2 + \left(-a^2 + b^2\right)\,t\right)\right) == 0, P1\right]\right]$$

$$\left\{\left\{P1 \to -\frac{1}{3}\left(2\,a + a^2 - b\,(2 + b)\right)\,t\right\}\right\}$$

Clear[P2, a, b, P1]

P1 = $-\frac{1}{3}\left(2\,a + a^2 - b\,(2 + b)\right)\,t$

$-\frac{1}{3}\left(2\,a + a^2 - b\,(2 + b)\right)\,t$

Simplify$\left[\frac{1}{2}\left(P1 + \left(-2\,a + a^2 - (-2 + b)\,b\right)\,t\right)\right]$

$\frac{1}{3}\left(-4\,a + a^2 - (-4 + b)\,b\right)\,t$

P2 = $\frac{1}{3}\left(-4\,a + a^2 - (-4 + b)\,b\right)\,t$

$\frac{1}{3}\left(-4\,a + a^2 - (-4 + b)\,b\right)\,t$

The profit function of the first firm is

Simplify$\left[P1 * \left(\frac{P1 - P2 + a^2\,t - b^2\,t}{2\,a\,t - 2\,b\,t}\right)\right]$

$-\frac{1}{18}\,(2 + a + b)\left(2\,a + a^2 - b\,(2 + b)\right)\,t$

The FOC wrt location

Simplify$\left[D\left[-\frac{1}{18}\,(2 + a + b)\left(2\,a + a^2 - b\,(2 + b)\right)\,t,\ a\right]\right]$

$\frac{1}{18}\left(-4 - 3\,a^2 + b^2 - 2\,a\,(4 + b)\right)\,t$

Simplify$\left[Solve\left[\frac{1}{18}\left(-4 - 3\,a^2 + b^2 - 2\,a\,(4 + b)\right)\,t == 0,\ a\right]\right]$

$\left\{\{a \to -2 - b\},\ \left\{a \to \frac{1}{3}\,(-2 + b)\right\}\right\}$

the SOC

Simplify$\left[D\left[\frac{1}{18}\left(-4 - 3\,a^2 + b^2 - 2\,a\,(4 + b)\right)\,t,\ a\right]\right]$

$-\frac{1}{9}\,(4 + 3\,a + b)\,t$

when a = -2 - b

a = -2 - b

$-2 - b$

Simplify$\left[-\frac{1}{9}\,(4 + 3\,a + b)\,t\right]$

$\frac{2}{9}\,(1 + b)\,t$

is a min

when a $= \frac{1}{3}(-2+b)$

$a = \frac{1}{3}(-2+b)$

$\frac{1}{3}(-2+b)$

`Simplify`$\left[-\frac{1}{9}(4+3a+b)\,t\right]$

$-\frac{2}{9}(1+b)\,t$

is a max

Then, $a = \frac{1}{3}(-2+b)$

$a = \frac{1}{3}(-2+b)$

$\frac{1}{3}(-2+b)$

The profit function of the second firm is

`Simplify`$\left[P2 * \left(1 - \frac{P1 - P2 + a^2\,t - b^2\,t}{2\,a\,t - 2\,b\,t}\right)\right]$

$-\frac{1}{18}(-4+a+b)\left(-4a+a^2-(-4+b)\,b\right)t$

the FOC of the profit function of the second firm is

`Simplify`$\left[D\left[-\frac{1}{18}(-4+a+b)\left(-4a+a^2-(-4+b)\,b\right)t,\ b\right]\right]$

$-\frac{1}{18}\left(-16+a^2+16b-2ab-3b^2\right)t$

`Simplify`$\left[\text{Solve}\left[-\frac{1}{18}\left(-16+a^2+16b-2ab-3b^2\right)t == 0,\ b\right]\right]$

$\left\{\{b \to 4-a\},\ \left\{b \to \frac{4+a}{3}\right\}\right\}$

the SOC

`Simplify`$\left[D\left[-\frac{1}{18}\left(-16+a^2+16b-2ab-3b^2\right)t,\ b\right]\right]$

$\frac{1}{9}(-8+a+3b)\,t$

when b is

$b = 4 - a$

$4 - a$

Simplify$\left[\frac{1}{9}\ (-8+a+3\,b)\ t\right]$

$-\frac{2}{9}\ (-2+a)\ t$

is a min

when b is

$b=\frac{4+a}{3}$

$\frac{4+a}{3}$

Simplify$\left[\frac{1}{9}\ (-8+a+3\,b)\ t\right]$

$\frac{2}{9}\ (-2+a)\ t$

is a max

$b=\frac{4+a}{3}$

$\frac{4+a}{3}$

From $a=\frac{1}{3}\ (-2+b)$ **we have if I insert b from above :**

Simplify$\left[\text{Solve}\left[a-\left(\frac{1}{3}\ (-2+b)\right)==0,\ a\right]\right]$

$\left\{\left\{a\rightarrow-\frac{1}{4}\right\}\right\}$

For

$a=-\frac{1}{4}$

$-\frac{1}{4}$

we find the optimal b

$b=\frac{4+a}{3}$

$\frac{5}{4}$