# PS2,answers

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## PROBLEM SET 2

For all allowed values of the parameters, find all global maxima of the following maximization problem, or show that none exist.

Objective function $x_1 + 2\sqrt{x_2}$

Constraints $p_1 x_1 + p_2 x_2 \leq m, x_1 \geq 0, x_2 \geq 0$

variables $x_1, x_2$

parameters $p_1, p_2, m$

conditions on parameters $p_1 > 0, p_2 > 0, m > 0$

STEP1 : NORMAL FORM

MAX $f(x)$

$M - P_1 x_1 - P_2 x_2 \geq 0$

$x_1 \geq 0, \ x_2 \geq 0$

STEP2 : LAGRANGIAN

$L = \lambda_0 \cdot f + \lambda_1 [M - P_1 x_1 - P_2 x_2]$

STEP 3 : WHICH THEOREMS APPLY ?

WEIERITRASS YES

FRITZ JOHN YES EXCEPT AT $x_2 = 0$

ARROW-ENTHOVEN YES BUT WE DON'T KNOW YET HOW TO PROVE IT

CANDIDATE LOCAL MAXIMA = FJ POINTS AND

ALL POINTS $(x_1, 0)$, $x_1 \geq 0$

STEP 4: WRITE DOWN THE FJ NECESSARY CONDITIONS FOR $x_\varepsilon > 0$

$$\frac{\partial L}{\partial x_1} = \lambda_0 - \lambda_1 P_1 \leq 0, \quad \frac{\partial L}{\partial x_1} \cdot x_1 = 0$$

$$\frac{\partial L}{\partial x_\varepsilon} = \frac{\lambda_0}{\sqrt{x_\varepsilon}} - \lambda_1 P_\varepsilon = 0$$

$$\frac{\partial L}{\partial \lambda_1} = M - P_1 x_1 - P_\varepsilon x_\varepsilon \geq 0, \quad \frac{\partial L}{\partial \lambda_1} \cdot \lambda_1 = 0$$

$$x_i \geq 0, \qquad \lambda_0, \lambda_1 \geq 0$$

$$\lambda_0 = 0 \text{ OR } 1 \qquad \text{NOT ALL } \lambda_j = 0$$

STEP 5: SEARCH FOR SOLUTIONS (COOKBOOK PROCEDURE)

HYPOTHESIS: $\lambda_0 = 0$

SOLUTION: $\lambda_1 > 0$, $M = P_1 x_1 + P_\varepsilon x_\varepsilon$

$$-\lambda_1 P_\varepsilon = 0 \Rightarrow \lambda_1 = P_\varepsilon = 0$$

CONSISTENCY CHECK FAIL.

FROM NOW ON, $\lambda_0 = 1$

HYPOTHESIS: $\lambda_0 = 1$, $x_1 > 0$

SOLUTION: $\lambda_1 = 1/P_1 > 0$, $M = P_1 x_1 + P_\varepsilon x_\varepsilon$

$$\frac{1}{\sqrt{x_2}} = \frac{P_2}{P_1} \Rightarrow x_\varepsilon = \left(\frac{P_1}{P_2}\right)^2$$

$$x_1 = \frac{M}{P_1} - \frac{P_1}{P_2}$$

CONSISTENCY CHECK: $M > \dfrac{P_1^2}{P_2}$

WE RECORD OUR FINDING

FJ POINT $x_A$

$$\boxed{\begin{aligned} x_A &= \left[\frac{M}{P_1} - \frac{P_1}{P_2},\ \left(\frac{P_1}{P_2}\right)^2\right] \quad \text{if } M > \frac{P_1^2}{P_2} \\ f(x_A) &= \frac{M}{P_1} + \frac{P_1}{P_2} \end{aligned}} \qquad (A)$$

HYPOTHESIS: $\lambda_0 = 1,\ x_1 = 0$

SOLUTION: $\lambda_1 > 0,\quad M = P_1 x_1 + P_2 x_2 = P_2 x_2$

$$x_2 = \frac{M}{P_2}$$

$$\lambda_1 = \frac{1}{P_2 \sqrt{x_2}} = \frac{1}{\sqrt{P_2} \cdot \sqrt{M}}$$

$$1 \le \lambda_1 P_1 = \frac{P_1}{\sqrt{P_2} \cdot \sqrt{M}}$$

CONSISTENCY CHECK $M \le \dfrac{P_1^2}{P_2}$

WE RECORD OUR FINDINGS

FJ POINTS $x_B$

FJ POINTS $x_B$

$$\boxed{\begin{array}{l} x_B = \left[0, \frac{M}{p_2}\right] \quad \text{if } M \le \frac{p_1^2}{p_2} \\ f(x_B) = 2\sqrt{\frac{M}{p_2}} \end{array}} \qquad (B)$$

HYPOTHESIS: $x_2 = 0$

THEN THE PROBLEM BECOMES

MAX $f(x) = x_1$ SUBJECT TO $p_1 x_1 \le M$, $x_1 \geq 0$

SOLUTION $x_1 = M/p_1$

CONSISTENCY CHECK OK

CANDIDATE LOCAL MAX $x_C$

$$\boxed{\begin{array}{l} x_C = \left[\frac{M}{p_1}, 0\right] \\ f(x_C) = M/p_1 \end{array}}$$

**END OF THE SEARCH FOR CANDIDATE LOCAL MAXIMA** BECAUSE WE HAVE EXHAUSTED ALL CASES

**STEP 6: SELECT THE BEST OUT OF THE CANDIDATE LOCAL MAXIMA**

$x_A$ AND $x_B$ ARE NOT COMPARABLE, BECAUSE THEY OBTAIN UNDER INCOMPATIBLE PARAMETER

VALUES.

CASE $M > \frac{P_1^2}{P_2}$ WE COMPARE $x_A$ TO $x_C$

$$f(x_A) - f(x_C) = \frac{M}{P_1} + \frac{P_1}{P_2} - \frac{M}{P_1} = \frac{P_1}{P_2} > 0.$$

WE RECORD OUR FINDING

GLOBAL MAXIMA

$$x = \begin{cases} x_A & \text{if } M > P_1^2/P_2 \\ ? & \text{if } M \le P_1^2/P_2 \end{cases}$$

CASE $M \le P_1^2/P_2$

WE COMPARE $x_B$ TO $x_C$

$$f(x_B) - f(x_C) = 2\sqrt{\frac{M}{P_2}} - \frac{M}{P_1} > 0$$

WE RECORD OUR FINDINGS

GLOBAL MAXIMA

$$\boxed{x = \begin{cases} x_A & \text{if } M > P_1^2/P_2 \\ x_B & \text{if } M \le P_1^2/P_2 \end{cases}}$$