Question 4

Consider the (generalized) linear model written as:

$$y = X\beta + \varepsilon, \mathbb{E}[\varepsilon|X] = 0, \mathbb{E}[\varepsilon\varepsilon'|X] = \sigma^2\Omega.$$

In the following suppose Ω is known and in all cases X is taken as given.

1. What is the covariance matrix, $Cov(\tilde{\beta}, \tilde{\beta} - \hat{\beta})$, where $\tilde{\beta}$ is the GLS estimator and $\hat{\beta}$ is the OLS estimator?

Let $(P\Omega P')^{-1} = I_n$ (transformation to attain spherical errors) and $\tilde{X} = PX, \tilde{Y} = PY, \tilde{\beta} = P\beta$.

Our new model is now

$$\tilde{y} = \tilde{X}\tilde{\beta} + \tilde{\varepsilon}$$

$$\tilde{\beta} = (\tilde{X}'\tilde{X})^{-1}\tilde{X}\tilde{y} = \beta + (\tilde{X}'\tilde{X})^{-1}\tilde{X}\tilde{\varepsilon}$$

$$\begin{aligned}\mathrm{Cov}(\tilde{\beta}, \tilde{\beta} - \beta) &= \mathbb{E}[\tilde{\beta}(\tilde{\beta} - \hat{\beta})'] - \mathbb{E}\tilde{\beta} \cdot \mathbb{E}\hat{\beta} = \\ &= \mathbb{E}[\tilde{\beta}\tilde{\beta}'] - \mathbb{E}[\tilde{\beta}\hat{\beta}'] - \mathbb{E}(\tilde{\beta})[\beta\mathbb{E}\tilde{\beta} - \beta\mathbb{E}\hat{\beta}]' \\ &= \mathbb{E}[\tilde{\beta}\tilde{\beta}'] - \mathbb{E}[\tilde{\beta}\hat{\beta}']\end{aligned}$$

$$\begin{aligned}\mathbb{E}[\tilde{\beta}\tilde{\beta}'] &= \mathbb{E}[(\beta + (\tilde{X}'\tilde{X})^{-1}\tilde{X}\tilde{\varepsilon})(\beta + (\tilde{X}'\tilde{X})^{-1}\tilde{X}\tilde{\varepsilon})'] = \\ &= \mathbb{E}[\beta\beta'] + \mathbb{E}[\beta\tilde{\varepsilon}'\tilde{X}(\tilde{X}'\tilde{X})^{-1}] + \mathbb{E}[\beta(\tilde{X}'\tilde{X})^{-1}\tilde{X}'\tilde{\varepsilon}] + \mathbb{E}[(\tilde{X}'\tilde{X})^{-1}\tilde{X}'\tilde{\varepsilon}\tilde{\varepsilon}'\tilde{X}(\tilde{X}'\tilde{X})^{-1}] \\ &= \beta\beta' + \beta 0\mathbb{E}[\tilde{\varepsilon}]\tilde{X}(\tilde{X}'\tilde{X})^{-1} + \beta(\tilde{X}'\tilde{X})0\tilde{X}'E[\tilde{\varepsilon}] + (\tilde{X}'\tilde{X})^{-1}\tilde{X}E[\tilde{\varepsilon}\tilde{\varepsilon}']\tilde{X}(\tilde{X}'\tilde{X})^{-1} \\ &= \beta\beta' + (\tilde{X}'\tilde{X})^{-1}\tilde{X}I_nC\Omega C'\tilde{X}(\tilde{X}'\tilde{X})^{-1}\sigma^2 = \beta\beta' + (\tilde{X}'\tilde{X})^{-1}\sigma^2\end{aligned}$$

$$\begin{aligned}\mathbb{E}[\tilde{\beta}\hat{\beta}'] &= \mathbb{E}[\beta\beta'] + \beta\mathbb{E}[\varepsilon']X(X'X)^{-1} + \beta(\tilde{X}'\tilde{X})^{-1}\mathbb{E}[\tilde{\varepsilon}] + (\tilde{X}'\tilde{X})^{-1}\tilde{X}\tilde{\varepsilon}\varepsilon'X(X'X)^{-1} = \\ &= \mathbb{E}[\beta\beta'] + (\tilde{X}'\tilde{X})^{-1}X'C'C\mathbb{E}[\varepsilon\varepsilon']X(X'X)^{-1} = \\ &\overset{(*)}{=} \beta\beta' + (\tilde{X}'\tilde{X})^{-1}X'\Omega^{-1}\Omega\sigma^2X(X'X)^{-1} \\ &= \beta\beta' + \sigma^2(\tilde{X}'\tilde{X})^{-1}\end{aligned}$$

(*) $(C')^{-1}V^{-1}C^{-1} = I_n, (C')^{-1}V^{-1} = C$

Then:

$$\mathrm{Cov}(\tilde{\beta}, \tilde{\beta} - \beta) = \beta\beta' + (\tilde{X}'\tilde{X})^{-1}\sigma^2 - \beta\beta' + \sigma^2(\tilde{X}'\tilde{X})^{-1} = 0$$

2. What is the covariance matrix of the OLS and GLS estimators of β?

$$\text{Var}(\hat\beta) = \mathbb{E}[(\hat\beta-\beta)(\hat\beta-\beta)'] = \mathbb{E}[((X'X)^{-1}X'\varepsilon)((X'X)^{-1}X'\varepsilon)']$$
$$= \mathbb{E}[(X'X)^{-1}X'\varepsilon\varepsilon'X(X'X)^{-1}] = \sigma^2(X'X)^{-1}X'\Omega X(X'X)^{-1}$$
$$\text{Var}(\tilde\beta) = \mathbb{E}[(\tilde\beta-\beta)(\tilde\beta-\beta)'] = \sigma^2 I_n$$

(Since $\tilde\beta$ was derived by the linear map C, to get spherical errors.)

$$\text{Cov}(\tilde\beta,\hat\beta) = \sigma^2(\tilde X'\tilde X)^{-1}$$

3. What is the covariance matrix of the OLS residual vector $\hat\varepsilon = y - X\hat\beta$?

$$\text{Var}(\hat\varepsilon) = \mathbb{E}[(\hat\varepsilon-\mathbb{E}[\hat\varepsilon])(\hat\varepsilon-\mathbb{E}[\hat\varepsilon])'] = \mathbb{E}[\hat\varepsilon\hat\varepsilon'] = \mathbb{E}[M\varepsilon\varepsilon'M'] = M\mathbb{E}(\varepsilon\varepsilon')M' \implies$$
$$\text{Var}(\hat\varepsilon) = \sigma^2 M\Omega M'$$

4. What is the covariance matrix of the GLS residual vector $\tilde\varepsilon = y - X\tilde\beta$?

$$\text{Var}(\tilde\varepsilon) = \mathbb{E}(\tilde\varepsilon\tilde\varepsilon') = \mathbb{E}[(y-X\tilde\beta)(y-X\tilde\beta)'] =$$
$$= \mathbb{E}[(Y - X(\tilde X'\tilde X)^{-1}\tilde X\tilde Y)(Y - X(\tilde X'\tilde X)^{-1}\tilde X\tilde Y)'] =$$
$$= \mathbb{E}[(Y - X(X'\Omega^{-1}X)^{-1}X'\Omega^{-1}Y)(Y - X(X'\Omega^{-1}X)^{-1}X'\Omega^{-1}Y)'']$$
$$= \mathbb{E}[(I_n - X(X'\Omega^{-1}X)^{-1}X'\Omega^{-1})YY'(I_n - X(X'\Omega^{-1}X)^{-1}X'\Omega^{-1})']$$

$$(I_n - X(X'\Omega^{-1}X)^{-1}X'\Omega^{-1})Y = (I_n - X(X'\Omega^{-1}X)^{-1}X'\Omega^{-1})(X\beta+\varepsilon)$$
$$(X - X\underbrace{(X'\Omega^{-1}X)^{-1}X'\Omega^{-1}X}_{=1})\beta + (I_n - X(X'\Omega^{-1}X)^{-1}X'\Omega^{-1})\varepsilon = (I_n - X(X'\Omega^{-1}X)^{-1}X'\Omega^{-1})\varepsilon$$

Therefore,

$$\text{Var}(\tilde\varepsilon) = \mathbb{E}[(I_n - X(X'\Omega^{-1}X)^{-1}X'\Omega^{-1})\varepsilon\varepsilon'(I_n - X(X'\Omega^{-1}X)^{-1}X'\Omega^{-1})'] =$$
$$= \sigma^2[(I_n - X(X'\Omega^{-1}X)^{-1}X'\Omega^{-1})\Omega(I_n - X(X'\Omega^{-1}X)^{-1}X'\Omega^{-1})']$$
$$= \sigma^2[(I_n - X(X'\Omega^{-1}X)^{-1}X'\Omega^{-1})\Omega(I - \Omega^{-1}X(X'\Omega^{-1}X)^{-1}X')]$$
$$= \sigma^2[\Omega - X(X'\Omega^{-1}X)^{-1}X'](I_n - \Omega^{-1}X(X'\Omega^{-1}X)^{-1}X')$$
$$= \sigma^2[\Omega - X(X'\Omega^{-1}X')^{-1}X' - X(X'\Omega^{-1}X)^{-1}X' + X(X'\Omega^{-1}X)^{-1}X'\Omega^{-1}X(X'\Omega^{-1})X']$$
$$= \sigma^2[\Omega + X(X'\Omega^{-1}X)^{-1}\ X'\Omega^{-1}X(X'\Omega^{-1})\ X']$$
$$\implies \text{Var}(\tilde\varepsilon) = \sigma^2[\Omega - X(X'\Omega^{-1}X)^{-1}X']$$

4. What is the covariance matrix of the OLS and GLS residual vectors, i.e, $\text{Cov}(\hat\varepsilon,\tilde\varepsilon \mid X)$

$$\text{Cov}(\hat\varepsilon,\tilde\varepsilon) = \mathbb{E}[\hat\varepsilon\tilde\varepsilon'] - \mathbb{E}\hat\varepsilon\mathbb{E}\tilde\varepsilon = \mathbb{E}[\hat\varepsilon\tilde\varepsilon] =$$
$$= \mathbb{E}[(Y - X\hat\beta)(Y - X\tilde\beta)'] = \mathbb{E}[(Y - X(X'X)^{-1}XY)(Y - X(\tilde X'\tilde X)^{-1}\tilde X'Y)']$$
$$\mathbb{E}[(I_n - X(X'X)^{-1}X')YY'(I_n - \Omega^{-1}X(X'\Omega X)X')]$$
$$= (\dots) = \sigma^2[(\Omega - X(X'X)^{-1}X'\Omega)(I_n - \Omega^{-1}X(X'\Omega^{-1}X)X')] =$$
$$= (\dots) = \sigma^2[\Omega - X(X'X)^{-1}X'\Omega] = \sigma^2[I_n - X(X'X)X'] = \sigma^2 M\Omega$$