QUESTION 1

1

i) There is a typo so this question is cancelled.

ii) We have that

$$L(y|\theta) = (1-\theta)^{\sum_{i=1}^{n} y_i - n} \theta^n$$

So $\ln L = \left(\sum_{i=1}^{n} x_i - n\right) \ln(1-\theta) + n \ln \theta$

$$\frac{\partial \ln L}{\partial \theta} = \frac{n}{\theta^2} - \frac{\sum_{i=1}^{n} x_i - n}{1-\theta}$$

$$\frac{\partial^2 \ln L}{\partial \theta^2} = -\frac{n}{\theta^2} - \frac{\sum x_i - n}{(1-\theta)^2}$$

Therefore, $E\left(-\frac{\partial^2 \ln L}{\partial \theta^2}\right) =$

$$= E\left(\frac{n}{\theta^2} + \frac{\sum x_i - n}{(1-\theta)^2}\right)$$

$$= \frac{n}{\theta^2} + \frac{1}{(1-\theta)^2}\left(E(\sum y_i) - n\right)$$

$E y_i = \frac{1}{\theta}$

$$= \frac{n}{\theta^2} + \frac{1}{(1-\theta)^2}\left(\frac{n}{\theta} - n\right) =$$

$$= n\left(\frac{1}{\theta^2} + \frac{1}{(1-\theta)\theta}\right) = \frac{n}{\theta^2(1-\theta)}$$

iii) Working Similarly to ii) we have that

$$L(y|\theta) = \theta^{\sum_{i=1}^{n} y_i} (1-\theta)^{n-\sum_{i=1}^{n} y_i}$$

and thus

$$\ln L(y|\theta) = \sum_{i=1}^{n} y_i \ln\theta + \left(n - \sum_{i=1}^{n} y_i\right) \ln(1-\theta)$$

Then you need to calculate

$$\frac{\partial \ln L}{\partial \theta} \quad \text{and} \quad \frac{\partial^2 \ln L}{\partial \theta^2}$$

and you will see that

$$E\left(-\frac{\partial^2 \ln L}{\partial \theta^2}\right) = \frac{n}{\theta(1-\theta)}$$

iv) Let I_G and I_B the information matrices for geometric and Bernoulli models respectively. We have that for $\theta \in (0,1)$

$$I_G > I_B . (*)$$

Moreover, from the Cramer-Rao lower Bound we have that $Var(\hat{\theta}) \geq \frac{1}{I(\hat{\theta})}$ for any estimator θ. Therefore $(*)$ implies that the estimator of the gemetric model has lower variance and thus is more accurate.

(B)

QUESTION 2

The log-likelihood of the model is (1)

$$\ell(\beta,\sigma^2) - \frac{n}{2}\ln(2\pi) - \frac{n}{2}\ln(\sigma^2) - \frac{1}{2\sigma^2}(y-X\beta)'(y-X\beta)$$

and we have that by FOCs

$$\hat{\beta}_{MLE} = (X'X)^{-1}X'y$$

and $\hat{\sigma}^2_{MLE} = \dfrac{\hat{e}'\hat{e}}{n}$ where $\hat{e} = y - X\hat{\beta}_{MLE}$

(1)

If we replace β and σ^2 by $\hat{\beta}_{MLE}$ and $\hat{\sigma}_{MLE}$ we get

$$\ell(\beta_{MLE}, \sigma^2_{MLE}) = -\frac{n}{2}\ln(2\pi\sigma^2_{MLE}) - \frac{n}{2}$$

The likelihood ratio for the test we perform is calculated as

$$-2\ln\left(\frac{L_R}{L_u}\right)$$ where L_R and L_u are the likelihood functions for the model with and without the restrictions respectively

In the restricted model we have that $\beta_2 = 0$ which implies that $\hat{\varepsilon}_i = y_i - \beta_1 X_{si}$ and $\tilde{\sigma}^2 = \frac{1}{n}\sum \hat{\varepsilon}_i^2$

If we denote by $\hat{\sigma}^2$ the MLE calculated in the previous page the

$$-2\ln L_R + 2\ln L_u =$$

$$= n\ln(2\pi\tilde{\sigma}^2) \cancel{+n} - n\ln(2\pi\hat{\sigma}^2) \cancel{-n}$$

$$= n\ln\left(\frac{\cancel{2\pi}\tilde{\sigma}^2}{\cancel{2\pi}\hat{\sigma}^2}\right) = n\ln\left(\frac{\tilde{\sigma}^2}{\hat{\sigma}^2}\right)$$

QUESTION 3

i) Let $\theta = (\beta_0, \beta_1, v)$ the likelihood function is

$$L(y|\theta) = \prod_{i=1}^{n} \frac{\Gamma\left(\frac{v+1}{2}\right)}{\sqrt{v\pi}\,\Gamma\left(\frac{v}{2}\right)} \left(1 + \frac{(y_i - x_i\beta_1 - \beta_0)^2}{v\sigma^2}\right)^{-\frac{v}{2}}$$

$$= (v\pi)^{-n/2}\,\Gamma\left(\frac{v}{2}\right)^{-n}\,\Gamma\left(\frac{v+1}{2}\right)\prod_{i=1}^{n}\left(1 + \frac{y_i - x_i\beta_1 - \beta_0}{v\sigma^2}\right)^{-v}$$

Therefore,

$$\ln L(y|\theta) = -\frac{n}{2}\ln(v\pi) - n\ln\Gamma\left(\frac{v}{2}\right) + n\ln\Gamma\left(\frac{v+1}{2}\right) +$$

$$+ \sum_{i=1}^{n} \log\left(1 + \frac{y_i - x_i\beta_1 - \beta_0}{v\sigma^2}\right)^2 \qquad (1)$$

ii) To find $\frac{\partial \ln L(y|\theta)}{\partial \beta_0}$, $\frac{\partial \ln L(y|\theta)}{\partial \beta_1}$, $\frac{\partial \ln L(y|\theta)}{\partial v}$ and $\frac{\partial \ln L}{\partial \sigma^2}$ you need to differentiate (1) appropriately. Notice that the derivatives of $\beta_0, \beta_1, \sigma^2$ do not depend on the Gamma function since β_0, β_1 they are not appearing there in whereas the derivative of v does and your answer should include terms of the form $\frac{1}{\Gamma(v)}\cdot\frac{\partial \Gamma}{\partial v}$.

6

iii) You should notice that FOCs correspond to highly non-linear system with four equations and four unknowns. Thus, you can solve this system only numerically.

You may also notice that if you fix some parameters then you can solve the system and obtain a solution similar to the OLS for the Gaussian errors.

QUESTION 5

a) See book: "Bayesian Data Analysis" by A. Gelman, J. Carlin, H. Stern and D. Rubin, Second edition p. 48 and p. 49

b) See p. 50 in the same book as above.

QUESTION 6: See the same book as in QUESTION 5, p. 75-76