Statistics for Business

Elements of Probability Theory

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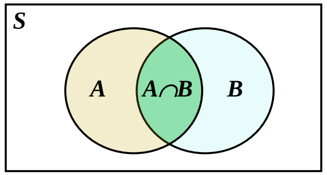
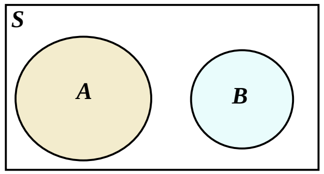
First Draft: July 15, 2015. This Draft: August 28, 2023.

Important Terms in Probability – I

- ***Random Experiment*** – it is a process leading to an uncertain outcome
- ***Basic Outcome*** (S_i) – a possible outcome (the most basic one) of a random experiment
- ***Sample Space*** (S) – the collection of all possible (basic) outcomes of a random experiment
- ***Event*** A – is any subset of basic outcomes from the sample space ($A \subseteq S$). This is our object of interest here – among other things.

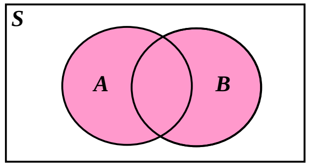
Important Terms in Probability – II

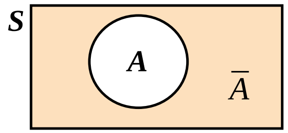




- ***Intersection of Events*** – If A and B are two events in a sample space S, then their intersection, $A \cap B$, is the set of all outcomes in S that belong to **both** A and B
- We say that A and B are ***Mutually Exclusive Events*** if they have no basic outcomes in common i.e., the set $A \cap B$ is empty ($\varnothing$)

Important Terms in Probability – III





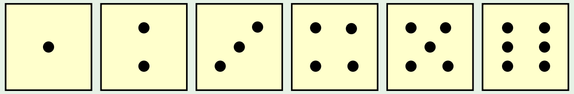
- ***Union of Events*** – If A and B are two events in a sample space S, then their union, $A \cup B$, is the set of all outcomes in S that belong to either A or B
- The ***Complement*** of an event A is the set of all basic outcomes in the sample space that do not belong to A. The complement is denoted $\bar{A}$ or A^c.

Important Terms in Probability – IV

- Events $E_1, E_2, \ldots, E_k$ are ***Collectively Exhaustive*** events if $E_1 \cup E_2 \cup \ldots \cup E_k = S$, i.e., the events completely cover the sample space.

Examples

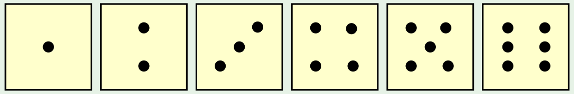
Let the ***Sample Space*** be the collection of all possible outcomes of rolling one die $S = \{1, 2, 3, 4, 5, 6\}$.



- Let A be the event "Number rolled is even": $A = \{2, 4, 6\}$
- Let B be the event "Number rolled is at least 4" : $B = \{4, 5, 6\}$
- ***Mutually exclusive***: A and B are **not** mutually exclusive. The outcomes 4 and 6 are common to both.

Important Terms in Probability – V

Examples (Continued)



$A = \{2, 4, 6\}$ $\quad B = \{4, 5, 6\}$

- ***Collectively exhaustive***: A and B are **not** collectively exhaustive. $A \cup B$ does not contain 1 or 3.
- ***Complements***: $\bar{A} = \{1, 3, 5\}$ and $\bar{B} = \{1, 2, 3\}$
- ***Intersections***: $A \cap B = \{4, 6\}$; $\bar{A} \cap B = \{5\}$; $A \cap \bar{B} = \{2\}$; $\bar{A} \cap \bar{B} = \{1, 3\}$.
- ***Unions***: $A \cup B = \{2, 4, 5, 6\}$; $A \cup \bar{A} = \{1, 2, 3, 4, 5, 6\} = S$.

Assessing Probability – I

- ***Probability*** **–** the chance that an uncertain event A will occur is always between 0 and 1.

$$\underbrace{0}_{\text{Impossible}} \leq \Pr(A) \leq \underbrace{1}_{\text{Certain}}$$

- There are three approaches to assessing the probability of an uncertain event:

Assessing Probability – II

1. ***Classical Definition of Probability***:

$$\text{Probability of an event } A = \frac{N_A}{N}$$
$$= \frac{\text{number of outcomes that satisfy the event } A}{\text{total number of outcomes in the sample space } S}$$

   - Assumes all outcomes in the sample space are equally likely to occur.
   - ***Example***: Consider the experiment of tossing 2 coins. The sample space is $S = \{HH, HT, TH, TT\}$.
   - Event $A = \{\text{one } T\} = \{TH, HT\}$. Hence $\Pr(A) = 0.5$ – assuming that all basic outcomes are equally likely.
   - Event $B = \{\text{at least one } T\} = \{TH, HT, TT\}$. So $\Pr(B) = 0.75$.

Assessing Probability – III

2. ***Probability as Relative Frequency***:

$$\text{Probability of an event } A = \frac{n_A}{n}$$
$$= \frac{\text{number of events in the population that satisfy event } A}{\text{total number of events in the population}}$$

- The limit of the proportion of times that an event A occurs in a large number of trials, n.

Assessing Probability – IV

3. ***Subjective Probability***: an individual has opinion or belief about the probability of occurrence of A.
   - When economic conditions or a company's circumstances change rapidly, it might be inappropriate to assign probabilities based solely on historical data
   - We can use any data available as well as our experience and intuition, but ultimately a probability value should express our degree of belief that the experimental outcome will occur.

Measuring Outcomes – I

Classical Definition of Probability

- ***Basic Rule of Counting***: If an experiment consists of a sequence of k steps in which there are n_1 possible results for the first step, n_2 possible results for the second step, and so on, then the total number of experimental outcomes is given by $(n_1)(n_2)\ldots(n_k)$ – tree diagram...

Measuring Outcomes – II

Classical Definition of Probability

- ***Counting Rule for Combinations*** (Number of Combinations of n Objects taken k at a time): A second useful counting rule enables us to count the number of experimental outcomes when k objects are to be selected from a set of n objects (the ordering does not matter)

$$C_k^n = \binom{n}{k} = \frac{n!}{k!(n-k)!},$$

where $n! = n(n-1)(n-2)\ldots(2)(1)$ and $0! = 1$.

Measuring Outcomes – III

Classical Definition of Probability

- ***Example***: Suppose we flip three coins. How many are the possible combinations with (exactly) 1 T?

$$C_1^3 = \binom{3}{1} = \frac{3!}{1!(3-1)!} = 3.$$

- ***Example***: Suppose we flip three coins. How many are the possible combinations with *at least* $1T$?
- ***Example***: Suppose that there are two groups of questions. Group A with 6 questions and group B with 4 questions. How many are the possible half-a-dozens we can put together?

$$n = 6 + 4 = 10; \; C_6^{10} = \binom{10}{6} = \frac{10!}{6!(10-6)!} = 210.$$

Measuring Outcomes – IV

Classical Definition of Probability

- ***Example***: How many possible half-a-dozens we can put together, preserving the ratio 4 : 2?

$$\binom{6}{4} \times \binom{4}{2} = 15 \times 6 = 90.$$

- **Probability**: What is the probability of selecting a particular half-a-dozen (with ratio 4 : 2), when we choose at random? Using the classical definition of probability

$$\frac{90}{210} = 0.4286$$

Measuring Outcomes – V

Classical Definition of Probability

- ***Counting Rule for Permutations*** (Number of Permutations of n Objects taken k at a time): A third useful counting rule enables us to count the number of experimental outcomes when k objects are to be selected from a set of n objects, **where the order of selection is important**

$$P_k^n = \frac{n!}{(n-k)!}.$$

Measuring Outcomes – VI

Classical Definition of Probability

- ***Example***: How many 3-digit lock combinations can we make from the numbers 1, 2, 3, and 4?
  The order of the choice is important! So

$$P_3^4 = \frac{4!}{1!} = 4! = 4(3)(2)(1) = 24.$$

- ***Example***: Let the characters A, B, Γ. In how many ways can we combine them in making triads?

$$P_3^3 = \frac{3!}{0!} = 3! = 3(2)(1) = 6.$$

These are: $AB\Gamma, A\Gamma B, BA\Gamma, B\Gamma A, \Gamma AB$,and ΓBA.

Measuring Outcomes – VII

Classical Definition of Probability

- ***Example***: Let the characters A, B, Γ, Δ, E. In how many ways is it possible to combine them into pairs?

* If the order matters, we may have

$$P_2^5 = \frac{5!}{3!} = (5)(4) = 20.$$

* If the order does not matters, we may choose pairs

$$C_2^5 = \binom{5}{2} = \frac{5!}{2!(5-2)!} = \frac{5!}{2!3!} = 10$$

Probability Axioms

- The following ***Axioms*** hold

1. If A is any event in the sample space S, then

$$0 \leq \Pr(A) \leq 1.$$

2. Let A be an event in S, and let S_i denote the basic outcomes. Then

$$\Pr(A) = \sum_{\text{all } S_i \text{ in } A} \Pr(S_i).$$

3. $\Pr(S) = 1.$

Probability Rules – I

- The ***Complement Rule***:

$$\Pr(\bar{A}) = 1 - \Pr(A) \text{ [i.e., } \Pr(A) + \Pr(\bar{A}) = 1].$$

- The ***Addition Rule***: The probability of the union of two events is

$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$$

- Probabilities and joint probabilities for two events A and B are summarized in the following table:

	B	$\bar{B}$	
A	$\Pr(A \cap B)$	$\Pr(A \cap \bar{B})$	$\Pr(A)$
$\bar{A}$	$\Pr(\bar{A} \cap B)$	$\Pr(\bar{A} \cap \bar{B})$	$\Pr(\bar{A})$
	$\Pr(B)$	$\Pr(\bar{B})$	$\Pr(S) = 1$

Probability Rules – II

Example (Addition Rule)

Consider a standard deck of 52 cards, with four suits ♡♣♢♠. Let event A = card is an Ace and event B = card is from a red suit.

Pr(Red ∪ Ace) = Pr(Red) + Pr(Ace) - Pr(Red ∩ Ace)

= 26/52 + 4/52 - 2/52 = 28/52

Don't count the two red aces twice!

Type	Color		Total
	Red	Black	
Ace	2	2	4
Non-Ace	24	24	48
Total	26	26	52

Conditional Probability – I

- A ***conditional probability*** is the probability of one event, given that another event has occurred:

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)} \text{ (if } \Pr(B) > 0);$$

$$\Pr(B|A) = \frac{\Pr(A \cap B)}{\Pr(A)} \text{ (if } \Pr(A) > 0)$$

Conditional Probability – II

Example (Conditional Probability)

Of the cars on a used car lot, 70% have air conditioning (AC) and 40% have a CD player (CD). 20% of the cars have both. What is the probability that a car has a CD player, given that it has AC? [$\Pr(CD|AC) = ?$]

	CD	No CD	Total
AC	.2	.5	.7
No AC	.2	.1	.3
Total	.4	.6	1.0

$$\Pr(\text{CD}|\text{AC}) = \frac{\Pr(\text{CD} \cap \text{AC})}{\Pr(\text{AC})} = \frac{.2}{.7} = .2857$$

Multiplication Rule

- The ***Multiplication Rule*** for two events A and B:

$$\Pr(A \cap B) = \Pr(A|B)\Pr(B) = \Pr(B|A)\Pr(A)$$

Example (Multiplication Rule)

Pr(Red ∩ Ace) = Pr(Red| Ace)Pr(Ace)

$$= \left(\frac{2}{4}\right)\left(\frac{4}{52}\right) = \frac{2}{52}$$

$$= \frac{\text{number of cards that are red and ace}}{\text{total number of cards}} = \frac{2}{52}$$

Type	Color: Red	Color: Black	Total
Ace	2	2	4
Non-Ace	24	24	48
Total	26	26	52

Statistical Independence – I

- Two events are ***statistically independent*** if and only if:

$$\Pr(A \cap B) = \Pr(A)\Pr(B).$$

  - Events A and B are independent when the probability of one event is not affected by the other event.
  - If A and B are independent, then

$$\begin{aligned} \Pr(A|B) &= \Pr(A), \text{ if } \Pr(B) > 0; \\ \Pr(B|A) &= \Pr(B), \text{ if } \Pr(A) > 0. \end{aligned}$$

Statistical Independence – II

Example (Statistical Independence)

Of the cars on a used car lot, 70% have air conditioning (AC) and 40% have a CD player (CD). 20% of the cars have both. Are the events AC and CD statistically independent?

	CD	No CD	Total
AC	.2	.5	.7
No AC	.2	.1	.3
Total	.4	.6	1.0

P(AC ∩ CD) = 0.2

$\left.\begin{array}{l}\text{P(AC)} = 0.7 \\ \text{P(CD)} = 0.4\end{array}\right\}$ P(AC)P(CD) = (0.7)(0.4) = 0.28

P(AC ∩ CD) = 0.2 ≠ P(AC)P(CD) = 0.28
So the two events are not statistically independent

Statistical Independence – III

Remark (Exclussive Events and Statistical Independence)

Let two events A and B with $\Pr(A) > 0$ *and* $\Pr(B) > 0$ *which are mutually exclusive. Are A and B independent?* ***NO!***

To see this use a Venn diagram and the formula of conditional probability (or the multiplication rule).

- If one mutually exclusive event is known to occur, the other cannot occur; thus, the probability of the other event occurring is reduced to zero (and they are therefore dependent).

Examples – I

- **Example 1.** In a certain population, 10% of the people can be classified as being high risk for a heart attack. Three people are randomly selected from this population. What is the probability that exactly one of the three are high risk?
  - Define H: high risk, and N: not high risk. Then

$$\Pr(\text{exactly one high risk}) = \Pr(HNN) + \Pr(NHN) + \Pr(NNH) =$$
$$= \Pr(H)\Pr(N)\Pr(N) + \Pr(N)\Pr(H)\Pr(N) + \Pr(N)\Pr(N)\Pr(H)$$
$$= (.1)(.9)(.9) + (.9)(.1)(.9) + (.9)(.9)(.1) = 3(.1)(.9)^2 = .243$$

Examples – II

- **Example 2.** Suppose we have additional information in the previous example. We know that only 49% of the population are female. Also, of the female patients, 8% are high risk. A single person is selected at random. What is the probability that it is a high risk female?
  - Define H: high risk, and F: female. From the example, $\Pr(F) = .49$ and $\Pr(H|F) = .08$. Using the Multiplication Rule:

$$\Pr(\text{high risk female}) = \Pr(H \cap F)$$
$$= \Pr(F)\Pr(H|F) = .49(.08) = .0392$$