Financial Derivatives#13
The Black–Scholes–Merton Framework
and the Pricing of Options

Andrianos E. Tsekrekos

January–March, 2011

1 Preview

- An important result in stochastic calculus: Itô's lemma
- The Black–Scholes–Merton model and the hedging argument to pricing options
- Risk neutral valuation
- The effect of dividends
- Options on stock indices, currencies and futures contracts
- Formal analysis of the Greeks and the hedging of options

2 More on stochastic calculus

- In order to price option contracts, we assume that the underlying asset S follows a specific stochastic process, the parameters of which will determine whether the option will end up "in" or "out–of–the–money".
- The usual assumption in finance is that S follows a gBm (i.e. a Markov, Itô process that is always positive as long as $S_0 > 0$).
- However, regardless of the process of dS, an option holder/writer would like to know how the option value changes as S changes, i.e. what is the process for dc (for a call) or for dp (for a put).

- For any general Itô process

$$dx = \alpha(x,t)\,dt + b(x,t)\,dz \tag{1}$$

and any real function $f(x,t) : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$, Itô* [10] has shown that

$$df(x,t) = \frac{\partial f(x,t)}{\partial t}dt + \frac{\partial f(x,t)}{\partial x}dx + \frac{1}{2}\frac{\partial^2 f(x,t)}{\partial x^2}(dx)^2 \tag{2}$$

where $(dx)^2$ can be calculated using the "stochastic multiplication table":

	dt	dz
dt	0	0
dz	0	dt

- This important mathematical result, referred to as *Itô's lemma*, can be thought of as the differentiation rule for functions of stochastic variables.

*Kiyoshi Itô, Japanese mathematician (born 1915).

EXAMPLE 1: x follows a gBm ("popular" version of the lemma)
For x following

$$dx = \mu x dt + \sigma x dz \tag{3}$$

and for a general function $f(.)$ of x and time, t, apply (2) to get

$$\begin{aligned} df(x,t) &= \frac{\partial f(x,t)}{\partial t}dt + \frac{\partial f(x,t)}{\partial x}[\mu x dt + \sigma x dz] + \frac{1}{2}\frac{\partial^2 f(x,t)}{\partial x^2}[\mu x dt + \sigma x dz]^2 \\ &= \frac{\partial f(x,t)}{\partial t}dt + \mu x \frac{\partial f(x,t)}{\partial x}dt + \sigma x \frac{\partial f(x,t)}{\partial x}dz \\ &\quad + \frac{1}{2}\frac{\partial^2 f(x,t)}{\partial x^2}\left[\mu^2 x^2 (dt)^2 + \sigma^2 x^2 (dz)^2 + 2\mu\sigma x^2 (dz)^2\right] \\ &= \left[\frac{\partial f(x,t)}{\partial t} + \mu x \frac{\partial f(x,t)}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 f(x,t)}{\partial x^2}\right]dt + \sigma x \frac{\partial f(x,t)}{\partial x}dz \end{aligned}$$

EXAMPLE 2: The Forward Price

If the interest rates are constant, we know that the forward price F_t is

$$F_t = S_t e^{r(T-t)}$$

If the spot price S_t follows a gBm,

$$dS_t = \mu S_t dt + \sigma S_t dz \tag{4}$$

how does the forward price changes (dF), as time passes?

View the forward price as $F(t, S)$ and apply Itô's lemma (equation (2)):

$$\frac{\partial F}{\partial S} = e^{r(T-t)} \qquad \frac{\partial^2 F}{\partial S^2} = 0 \qquad \frac{\partial F}{\partial t} = -rSe^{r(T-t)}$$

$$dF = \left[e^{r(T-t)}\mu S - rSe^{r(T-t)}\right] dt + e^{r(T-t)}\sigma S dz$$

Substitute $F = Se^{r(T-t)}$ to get

$$dF = (\mu - r) F dt + \sigma F dz$$

EXAMPLE 3: The Lognormal Property

Let S follow a gBm (equation (4)) and define the function $G(S,t) = \ln(S)$. What is the process that G follows?

Because

$$\frac{\partial G}{\partial S} = e^{r(T-t)} \qquad \frac{\partial^2 G}{\partial S^2} = 0 \qquad \frac{\partial G}{\partial t} = -rSe^{r(T-t)}$$

it follows through Itô's lemma that

$$dG = \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma dz$$

thus $G = \ln(S)$ follows a generalised Wiener process (i.e. an aBm).

From the properties of the aBm we know that the change in $\ln(S)$ between time zero and time T is normally distributed with mean $\left(\mu - \frac{\sigma^2}{2}\right)T$ and variance $\sigma^2 T$

$$\ln(S_T) - \ln(S_0) \sim \mathcal{N}\left[\left(\mu - \frac{\sigma^2}{2}\right)T, \sigma T^{\frac{1}{2}}\right]$$

This is why if the stock price S follows (4), then $\ln\left(\frac{S_T}{S_0}\right)$ is normally distributed (i.e. S is log–normal).

3 The Black–Scholes–Merton model

- In 1973 Black and Scholes managed to solve a problem that had puzzled the most prominent finance scholars for at least 15 years (notable efforts by Boness [3] and Samuelson [13] are characteristic of the time period).
- The problem was the pricing of European–style options written on shares that pay no dividends: more specifically, it was not clear how future cash flows that the option writer might have to make are supposed to be discounted.

- If the option ends up in–the–money, the option writer faces a risk that is 1 : 1 with the riskiness of the underlying asset; if the option ends up out–of–the–money, the writer faces no risk. However, an option writer trying to determine a fair price for the option cannot know in advance which of the two cases is going to materialise at maturity.
- The key idea behind the breakthrough by Black and Scholes is to construct a "*no–arbitrage*" portfolio, whose theoretically–correct discount factor can be determined.

4 Model Assumptions

- The share price follows the process

$$dS = \mu S dt + \sigma S dz$$

  with μ and σ constants.
- Short–selling of securities (with full use of the proceeds) is permitted.
- There are no transaction costs or taxes
- All securities are perfectly divisible.
- There are no riskless arbitrage opportunities.
- Security trading is continuous.
- The risk–free rate of interest, r, is constant and the same for all maturities.

5 Derivation of the model

- The stock price (S) process is

$$dS = \mu S dt + \sigma S dz \tag{5}$$

and let $c(S,t)$ denote the price of a European call option. Apply Itô's lemma for $c(S,t)$ to get

$$dc = \left(\frac{\partial c}{\partial S}\mu S + \frac{\partial c}{\partial t} + \frac{1}{2}\frac{\partial^2 c}{\partial S^2}\sigma^2 S^2 \right) dt + \frac{\partial c}{\partial S}\sigma S dz \tag{6}$$

- Consider the following *"no–arbitrage" portfolio*

$$\begin{array}{cl} -1 & \text{call option, } c \\ \Delta \equiv +\frac{\partial c}{\partial S} & \text{shares, } S \end{array} \tag{7}$$

i.e. go short (sell) one call option and buy (go long) an amount $\frac{\partial c}{\partial S}$ of shares. Define Π as the value of this portfolio; clearly

$$\Pi = -c + \frac{\partial c}{\partial S} S \tag{8}$$

and the change in the portfolio value over an infinitesimal time interval dt is

$$d\Pi = -dc + \frac{\partial c}{\partial S} dS \tag{9}$$

- Substituting (5) and (6) into equation (9) yields

$$\begin{aligned} d\Pi &= \underbrace{-\frac{\partial c}{\partial S}\mu S dt - \frac{\partial c}{\partial t}dt - \frac{1}{2}\frac{\partial^2 c}{\partial S^2}\sigma^2 S^2 dt - \frac{\partial c}{\partial S}\sigma S dz}_{-dc} + \frac{\partial c}{\partial S}\underbrace{(\mu S dt + \sigma S dz)}_{dS} \\ &= \left(-\frac{\partial c}{\partial t} - \frac{1}{2}\frac{\partial^2 c}{\partial S^2}\sigma^2 S^2\right) dt \end{aligned} \tag{10}$$

- The change in the portfolio value over dt is *riskless!!!* (It does not involve any dz term). To exclude the possibility of riskless arbitrage, the portfolio must instantaneously earn the same rate of return as other short–term risk–free securities. Thus

$$d\Pi = r\Pi dt$$

with r the risk–free interest rate.

- Substituting from equations (8) and (10) we get

$$d\Pi = r\Pi dt$$

$$\left(-\frac{\partial c}{\partial t} - \frac{1}{2}\frac{\partial^2 c}{\partial S^2}\sigma^2 S^2\right) dt = r\left(-c + \frac{\partial c}{\partial S}S\right) dt$$

$$-\frac{\partial c}{\partial t} - \frac{1}{2}\frac{\partial^2 c}{\partial S^2}\sigma^2 S^2 = -rc + \frac{\partial c}{\partial S}rS$$

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 c}{\partial S^2} + rS\frac{\partial c}{\partial S} + \frac{\partial c}{\partial t} - rc = 0 \tag{11}$$

- This is the Black–Scholes *partial differential equation* (hereafter PDE). It has many solutions; the particular solution for the European call is obtained by applying boundary conditions:

$$c(S,T) = (S_T - K)^+ \equiv \max(S_T - K, 0) \qquad (12)$$
$$c(0,t) = 0 \qquad \forall t \in [0,T]$$

The solution of (11) subject to (12) comes from physics (the "heat equation") and is the Black–Scholes formula for a European call.

$$c = S_0 \Phi(d_1) - K e^{-rT} \Phi(d_2) \qquad (13)$$

$$d_{1,2} = \frac{\ln S - \ln K + \left(r \pm \frac{1}{2}\sigma^2\right) T}{\sigma T^{1/2}} \qquad \Phi(x) = (2\pi)^{-1/2} \int_{-\infty}^{x} e^{-\frac{1}{2}s^2} ds$$

where the function $\Phi(.)$ is the cumulative probability distribution function for a standardised normal distribution. Equivalently, the European put option is given by ($d_{1,2}$ as above)

$$p = K e^{-rT} \Phi(-d_2) - S_0 \Phi(-d_1) \qquad (14)$$

6 What is a PDE?

- Any equation that involves a function $f(x)$, and its derivatives $f'(x)$, $f''(x)$, $f'''(x)$, etc. is called a *differential equation* (DE). For example,

$$\frac{1}{2}f''(x) - 5f'(x) + f(x) = 5$$

$$f'(x) - [f(x)]^2 = 0$$

  are differential equations.

- A *partial differential equation* (PDE) is an equation that involves a (multivariate) function $f(x, y, \ldots)$ and its partial derivatives $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, etc. Examples are

$$
\begin{aligned}
\frac{\partial f(x,y)}{\partial x} + \frac{\partial f(x,y)}{\partial y} - 3\frac{\partial^2 f(x,y)}{\partial y^2} &= \pi \\
x\frac{\partial f(x,y,z)}{\partial x} + y\frac{\partial f(x,y,z)}{\partial y} + z\frac{\partial f(x,y,z)}{\partial z} &= \\
= xyz\left[\frac{\partial^2 f(x,y,z)}{\partial x^2} - \frac{\partial^2 f(x,y,z)}{\partial y^2}\right]
\end{aligned}
$$

- Solving a DE or a PDE involves determining what is the function for which the equation is true. For example, what is the solution of

$$f'''(x) - f''(x) + f'(x) - f(x) = 0$$

A simple substitution would convince you that

$$f(x) = e^x + c \quad c \in \mathbb{R}$$

What about

$$2f'(x) - xf''(x) = 0$$

Maybe you can see that

$$f(x) = \frac{1}{x} + c \quad c \in \mathbb{R}$$

- To determine one of the possible (infinite) solutions, a *boundary condition* is required. In the first example, if we know that

$$f(0) = 5 \Rightarrow f(x) = e^{x} + 4$$

In the second example, if we are given the extra condition

$$\lim_{x \to +\infty} f(x) = 1 \Rightarrow f(x) = \frac{1}{x} + 1$$

7 Risk–neutral valuation

- One key property of the Black–Scholes–Merton PDE and the resulting option price is that they do not involve any variable affected by the *risk preferences* of investors.
- The expected return of the underlying asset, μ, which depends on risk preferences of investors (the higher the risk aversion of the investor, the higher the required return), does not appear in the equation.
- If risk preferences do not enter the equation, they cannot affect its solution. Thus, any set of preferences can be used when evaluating the option price. In particular, the very simple assumption that all investors are *risk neutral* can be made.
- In a risk–neutral world, the expected return of all securities is r. Moreover, the present value of any future cash flow can be obtained by discounting

with the risk free rate. This takes care of the "discount–factor" problem we discussed earlier.

- For a European call, risk–neutral valuation implies:
  - Assume the expected rate of return of S is the risk–free rate of return (i.e. assume $\mu = r$)
  - Calculate the expected payoff from the option at maturity
  - Discount the expected payoff at the risk–free interest rate
- Namely,

$$c = e^{-rT} E^{\mathbb{Q}} \left[(S_T - K)^+ \right] \tag{15}$$

where $E^{\mathbb{Q}}[.]$ denotes the expected value in a risk–neutral world. It can be shown† that the expectation is equal to

$$E^{\mathbb{Q}} \left[(S_T - K)^+ \right] = S_0 e^{rT} \Phi(d_1) - K \Phi(d_2) \tag{16}$$

†Proof available at Hull [9, pp. 262-264].

Substitute this in (15) to get the Black–Scholes equation

$$c = S_0 \Phi(d_1) - K e^{-rT} \Phi(d_2)$$

- Equation (16) brings out a very important insight of the formula: $\Phi(d_2)$ is the probability that the call option will be exercised, $K\Phi(d_2)$ is the strike price times the probability that the strike price will be paid and $S_0 e^{rT} \Phi(d_1)$ is the expected value of a variable that equals S_T if $S_T > K$ and zero otherwise in a risk–neutral world.

8 Extensions & alternative option pricing models

- Cox and Ross [5] *pure jump model*

$$\begin{aligned} \frac{dS}{S} &= \mu dt + (k-1)\,dq \\ &= \mu dt + \begin{cases} k-1 & \text{with probability } \lambda dt \\ 0 & \text{with probability } 1-\lambda dt \end{cases} \end{aligned}$$

where q is a continuous–time Poisson process, λ is the intensity of the process, and $k-1$ is the jump amplitude. This model says that the percentage change on asset price S is composed of a drift term, μdt and a term dq that with probability λdt will jump the percentage stock change to $k-1$ and with probability $1-\lambda dt$ will do nothing.

- Merton [12] *mixed diffusion–jump model*

$$\frac{dS}{S} = \mu dt + \sigma dz + (k-1)\, dq$$

where dz a Brownian motion and dq, k as before. The model's plausibility comes from the intuition that stock prices seem to have small, almost continuous movements most of the time but sometimes experience large discrete jumps when important new information arrives.

- Cox [4] and Cox and Ross [6] *constant elasticity of variance model*

$$dS = \mu S dt + \sigma S^{\frac{\alpha}{2}} dz$$

where α is the elasticity of variance factor ($0 \leq \alpha < 2$). If $\alpha = 2$ the model collapses to the Black–Scholes formula. For $\alpha < 2$ the standard deviation of the return distribution moves inversely with the level of the asset price. The intuitive appeal of such an argument is that if the price of a company's share decreases significantly, this should make the company more risky and that the variance/standard deviation of its returns' distribution should increase.

9 The effect of dividends

- Merton has modified the Black and Scholes [2] European call option formula, so as to accommodate the possibility that the underlying asset pays a continuous dividend yield, δ, i.e.

$$dS = (\mu - \delta)\, S dt + \sigma S dz$$

- The no–arbitrage portfolio is the same in this case as well. However observe that the change in the portfolio $d\Pi$ is not given by (10). At any instant dt, the holder of the portfolio earns $-\frac{\partial c}{\partial t} - \frac{1}{2}\frac{\partial^2 c}{\partial S^2}\sigma^2 S^2$ plus the dividends received by being "long" Δ number of shares, i.e. $\delta S \frac{\partial c}{\partial S} dt$

$$d\Pi = \left(-\frac{\partial c}{\partial t} - \frac{1}{2}\frac{\partial^2 c}{\partial S^2}\sigma^2 S^2 + \delta S \frac{\partial c}{\partial S} \right) dt \qquad (17)$$

- This portfolio is again riskless, thus the equilibrium condition is again $d\Pi = r\Pi dt$ but with $d\Pi$ now given by (17). The resulting PDE is

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 c}{\partial S^2} + (r - \delta) S \frac{\partial c}{\partial S} + \frac{\partial c}{\partial t} - rc = 0$$

which when solved subject to (12) yields

$$c = S_0 e^{-\delta T} \Phi(d_1) - K e^{-rT} \Phi(d_2) \tag{18}$$

$$p = K e^{-rT} \Phi(-d_2) - S_0 e^{-\delta T} \Phi(-d_1) \tag{19}$$

$$d_{1,2} = \frac{\ln S - \ln K + \left(r - \delta \pm \frac{1}{2}\sigma^2\right) T}{\sigma T^{1/2}}$$

- Merton has also demonstrated that the option bounds we derived in lecture #8 need to be modified in instances when the underlying asset provides a dividend yield

1. *[Call lower bound]* For a European call written on an asset that pays a continuous dividend yield δ over $[0, T]$, the following inequality holds
$$c \geq \left(S_0 e^{-\delta T} - K e^{-rT}\right)^+ \tag{20}$$
2. *[Put lower bound]* For a European put written on an asset that pays that pays a continuous dividend yield δ over $[0, T]$, the following inequality holds
$$p \geq \left(K e^{-rT} - S_0 e^{-\delta T}\right)^+ \tag{21}$$
3. *[Put–Call parity]* The price of a European call option can be deduced from the price of the corresponding put, and vice versa.
$$c - p = S_0 e^{-\delta T} - K e^{-rT} \tag{22}$$

4. *[American options & early exercise]* It is always optimal to early–exercise American options (calls and puts) written on a flow–paying underlying assets. Thus

$$
\begin{aligned}
C &\geq K - S \\
P &\geq K - S_0 \\
P &\geq p \\
C &\geq c
\end{aligned}
$$

10 Options on stock indices (stock index options)

- The underlying asset of such options is an index tracking the movement of a stock market as a whole (FTSE, S&P, NASDAQ).
- Traded both in OTC and exchange markets, can either be European (e.g. S&P 500) or American (e.g. S&P 100) in nature. When exercised, they are settled in cash (no delivery of the "underlying asset" is involved).
- Under the assumption that the index follows a gBm, European index options can be priced by the dividend–adjusted formula of Black–Scholes–Merton (equations (18)–(19)). In this instance, the dividend yield δ should be set equal to the average, continuously–compounded and annualised dividend yield of all the shares comprising the index (not a trivial task).

11 Options on foreign currencies (currency options)

- These are options to buy or sell a foreign currency up until or at a prespecified date, at a predetermined exchange rate.
- Traded in both OTC and exchange markets, can either be European or American style.
- Very useful contracts for hedging FX risk.[‡]
- Garman and Kohlhagen [7] and Grabbe [8] have shown that if the spot exchange rate follows a gBm, the dividend–adjusted formulae (18)–(19) can be used to price European–style currency options by setting the dividend yield δ equal to r_f, the foreign country risk–free rate of interest.

[‡]Homework: what is the difference between hedging a future currency exposure with a currency option and with a foreign exchange forward contract?

12 Options on futures contracts (futures options)

- Written on both financial and commodities futures, traded in exchange markets, usually American in style.
- When exercised the holder acquires a long position (for a call option) or a short position (for a put) in the underlying futures contract
- Their popularity is due to the fact that it is easier to deliver a futures contract than the underlying commodity or financial asset. Moreover, seldom are the underlying futures contracts held until their maturity.
- Black [1] has shown that under the assumption of a gBm futures price, the dividend–adjusted Black–Scholes–Merton framework (equations (18)–(19)) can be applied to European futures options by setting $\delta = r$ and $S_0 e^{-\delta T} = f_0$.

EXAMPLE 4: Price of a stock index option

A European call option on S&P 500 is two months away from maturity. The current value of the index is 930, the strike price 900, the risk–free rate is 8% per annum, the volatility of the index is 20% per annum and the index is expected to pay a dividend yield of $\delta = 3\%$ in the next year. The call price is

$$\begin{aligned} d_1 &= \frac{\ln S - \ln K + \left(r - \delta + \frac{1}{2}\sigma^2\right) T}{\sigma T^{1/2}} \\ &= \frac{\ln 930 - \ln 900 + \left(0.08 - 0.03 + \frac{1}{2}0.2^2\right) \times \frac{2}{12}}{0.2 \times \left(\frac{2}{12}\right)^{1/2}} \\ &= 0.5444 \end{aligned}$$

$$
\begin{aligned}
d_2 &= d_1 - \sigma T^{\frac{1}{2}} \\
&= 0.5444 - 0.2 \times \left(\frac{2}{12}\right)^{1/2} \\
&= 0.4628
\end{aligned}
$$

$$
\Phi(d_1) = \Phi(0.5444) = 0.7069 \qquad \Phi(d_2) = \Phi(0.4628) = 0.6782
$$

$$
\begin{aligned}
c &= S_0 e^{-\delta T}\Phi(d_1) - K e^{-rT}\Phi(d_2) \\
&= 930 \times e^{-0.03\times\frac{2}{12}} \times 0.7069 - 900 \times e^{-0.08\times\frac{2}{12}} \times 0.4628 = 51.83
\end{aligned}
$$

Thus, one option contract would cost $5,183 (since one index option contract is on 100 times the index)

13 Hedging and the Greeks

- Consider a financial institution or a market maker that has written (sold) a number of options. How can this position be hedged?
  - Naked position: One possible strategy is to do nothing. This strategy works well if the options sold end up out–of–the–money and are not exercised. However, there is a danger that if the options end up deep in–the–money, this strategy involves buying (expensively) the underlying assets necessary to cover the options exercise.
  - Covered position: This strategy involves buying—immediately after the options are written—the necessary assets to cover possible options exercise. This strategy is ok if the options are indeed exercised, but otherwise we are left with a position in the underlying asset that might depreciate in value.

- Stop–Loss strategy: This strategy involves buying (selling) the underlying asset the minute the option becomes in–the–money (out–of–the–money). It will work fine for deep in or out–of–the–money options, but for options that are written at–the–money, it will involve large transaction costs. Even in the absence of transaction costs, this strategy will involve substantial costs: this is because when the underlying asset price is equal to the exercise price $(S = K)$, it is impossible to know whether in the next instant it will move in a direction that will make the options written in or out–of–the–money.

- Most traders use more sophisticated hedging schemes than those mentioned above, based on calculations of delta, gamma, vega, etc.

14 Delta hedging

- Recall that the *delta* of an option, Δ, was defined in lecture #3 as the rate of change of the option price with respect to the price of the underlying asset; for a call
$$\Delta = \frac{\partial c}{\partial S}$$
- Delta is very closely related to the Black–Scholes–Merton analysis. See in equation (7) how the *"no–arbitrage" portfolio* is constructed in the derivation.
- Delta summarises the number of underlying assets an option writer must hold in order to hedge the exposure associated with future exercise. If the writer holds Δ shares for every option written, the position is said to be *delta neutral*.

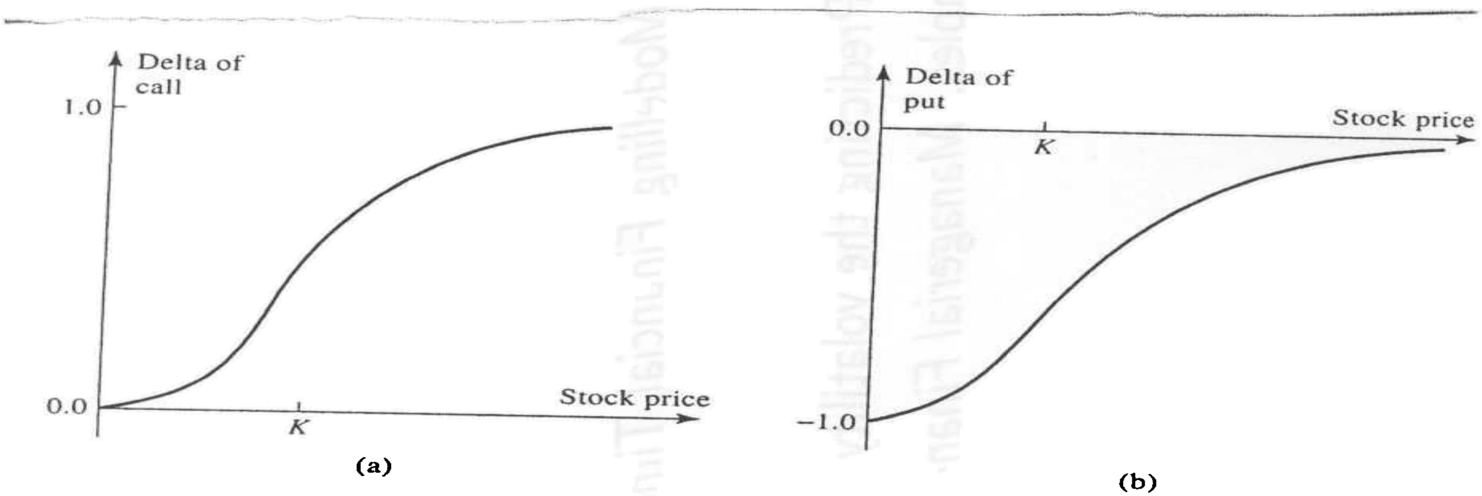
- It is important to realise that, because delta changes as S changes, the investor's position remains delta–hedged (delta neutral) only for a relatively short period of time. The hedge has to be adjusted (*rebalanced*) periodically.
- In the Black–Scholes–Merton framework, delta can be calculated by differentiating equations (13), (14) or (18), (19) with respect to S (not trivial!). The results are:

	No dividends	Dividends
Call Δ	$\Phi(d_1)$	$e^{-\delta T}\Phi(d_1)$
Put Δ	$\Phi(d_1) - 1$	$e^{-\delta T}[\Phi(d_1) - 1]$

- The delta of a portfolio of options is a weighted average of the individual deltas of the options in the portfolio

$$\Delta_\pi = \sum_{i=1}^{n} w_i \Delta_i$$

where w_i the quantity of option i in portfolio π.


Delta of call
1.0
0.0
K
Stock price
(a)
Delta of put
0.0
K
Stock price
−1.0
(b)

15 Theta

- Recall that the *theta* of an option, Θ, was defined in lecture #3 as the rate of change of the option price with respect to the passage of time; for a call

$$\Theta = \frac{\partial c}{\partial t}$$

- Theta summarises the change in the price of an option when time decreases as we reach the option maturity. It is also known as the *time decay* of the option
- In the Black–Scholes–Merton framework, theta can be calculated by differentiating equations (13), (14) or (18), (19) with respect to T (not trivial!).

The results are:

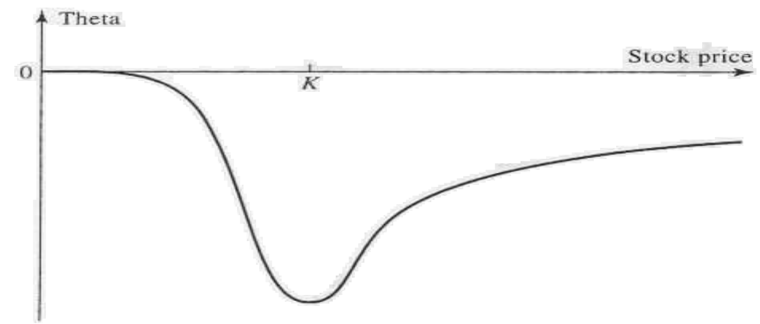
	No dividends	Dividends
Call Θ	$-\frac{S_0\Phi'(d_1)\sigma}{2T^{\frac{1}{2}}}$ $-rKe^{-rT}\Phi\left(d_2\right)$	$-\frac{S_0\Phi'(d_1)\sigma e^{-\delta T}}{2T^{\frac{1}{2}}} + \delta S_0 \Phi\left(d_1\right) e^{-\delta T}$ $-rKe^{-rT}\Phi\left(d_2\right)$
Put Θ	$-\frac{S_0\Phi'(d_1)\sigma}{2T^{\frac{1}{2}}}$ $+rKe^{-rT}\Phi\left(-d_2\right)$	$-\frac{S_0\Phi'(d_1)\sigma e^{-\delta T}}{2T^{\frac{1}{2}}} - \delta S_0 \Phi\left(-d_1\right) e^{-\delta T}$ $+rKe^{-rT}\Phi\left(-d_2\right)$

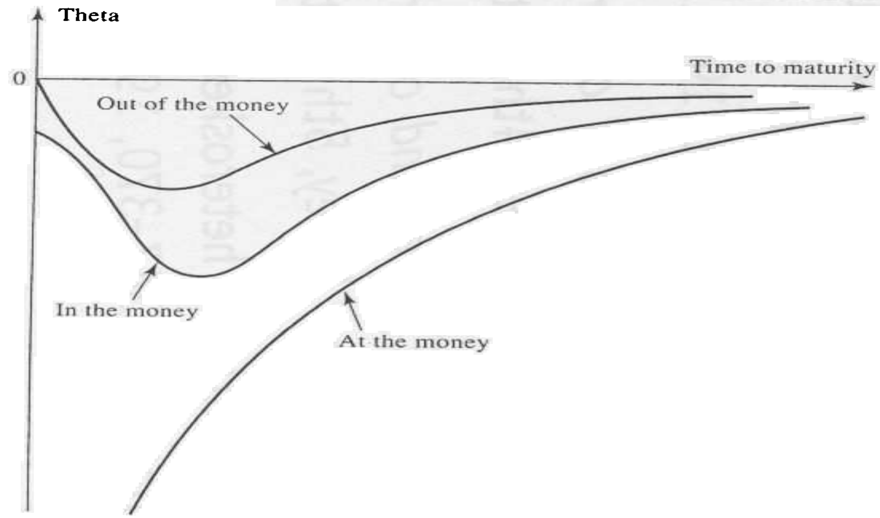
where

$$\Phi'\left(x\right) = \frac{1}{\left(2\pi\right)^{1/2}} e^{-\frac{x^2}{2}}$$

- In the formulae above, time is measured in years. Divide Θ with 365 or 252 to get the theta of an option per calendar or trading day respectively.

- Theta is negative usually because as time to maturity decreases (with all else equal) the option tends to become less valuable.


Theta
0
Stock price
K


Theta
0
Time to maturity
Out of the money
In the money
At the money

16 Gamma

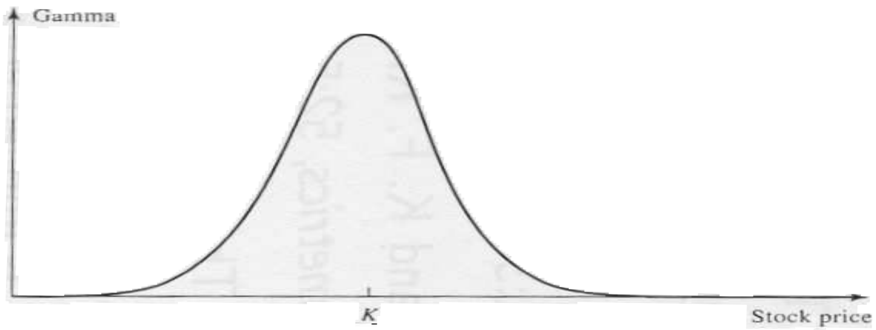
- Gamma (Γ) is the rate of change of delta with respect to the price of the underlying asset. It is the second partial derivative of the option price with respect to the underlying asset price; for a call

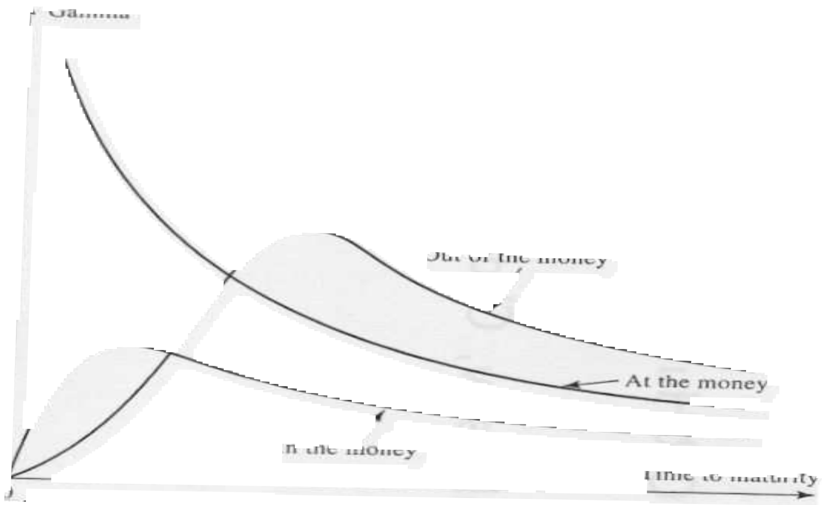
$$\Gamma = \frac{\partial \Delta}{\partial S} = \frac{\partial^2 c}{\partial S^2}$$

- If gamma is small, delta changes slowly, and adjustments to keep a position delta neutral need to be made only relatively infrequently.

- In the Black–Scholes–Merton framework, gamma can be calculated by differentiating equations (13), (14) or (18), (19) twice with respect to S (not trivial!). The results are:

	No dividends	Dividends
Call Γ	$\frac{\Phi'(d_1)}{S_0\sigma T^{\frac{1}{2}}}$	$\frac{\Phi'(d_1)e^{-\delta T}}{S_0\sigma T^{\frac{1}{2}}}$
Put Γ	$\frac{\Phi'(d_1)}{S_0\sigma T^{\frac{1}{2}}}$	$\frac{\Phi'(d_1)e^{-\delta T}}{S_0\sigma T^{\frac{1}{2}}}$


Gamma
K
Stock price


At the money

17 Vega

- Recall that the *vega* of an option, $\mathcal{V}$, was defined in lecture #3 as the rate of change of the option price with respect to the volatility of the underlying asset; for a call

$$\mathcal{V} = \frac{\partial c}{\partial \sigma}$$

- Despite the fact that volatility is assumed constant in the Black–Scholes–Merton framework, in reality it is not! Thus we expect the price of an option to change if the arrival of news alters the volatility of the underlying asset.
- If vega is high (low) in absolute value, the position is very (only slightly) sensitive to small changes in volatility. Vega neutrality protects against large changes in the price of the underlying asset between hedge rebalancing.

- In the Black–Scholes–Merton framework, vega can be calculated by differentiating equations (13), (14) or (18), (19) with respect to σ (not trivial!). The results are:

	No dividends	Dividends
Call $\mathcal{V}$	$S_0 T^{\frac{1}{2}} \Phi'(d_1)$	$S_0 T^{\frac{1}{2}} \Phi'(d_1)\, e^{-\delta T}$
Put $\mathcal{V}$	$S_0 T^{\frac{1}{2}} \Phi'(d_1)$	$S_0 T^{\frac{1}{2}} \Phi'(d_1)\, e^{-\delta T}$

18 Rho

- Recall that the *rho* of an option, ρ, was defined in lecture #3 as the rate of change of the option price with respect to the interest rate; for a call

$$\rho = \frac{\partial c}{\partial r}$$

- It measures the sensitivity of the option price to interest rate changes, despite the fact that in the Black–Scholes–Merton framework r is assumed constant.

	No dividends	Dividends
Call ρ	$KTe^{-rT}\Phi(d_2)$	$KTe^{-rT}\Phi(d_2)$
Put ρ	$-KTe^{-rT}\Phi(-d_2)$	$-KTe^{-rT}\Phi(-d_2)$

19 Hedging option positions in practice

- Ideally, an option writers would like to make their positions delta, gamma, vega, etc. neutral by frequently rebalancing their positions. This however is impossible most of the times.
- Traders would usually zero out delta sensitivity at least once every day and monitor the other sensitivities; in case any of those become too large in a positive or negative direction, either corrective action is taken or trading is curtailed.
- Moreover, traders benefit from the economies in scale present in hedging derivatives "books" with large volumes of both calls and puts and of different maturities and strike prices.

20 Using the Greeks for hedging funds/portfolios

- Consider a manager in charge of a $\$X$ million fund tracking the S&P 100, that wishes to hedge against the possibility that the portfolio will decline below $\$\,(X - Y)$ million.
- Alternatives:
  - Buy put options on the stocks that comprise the portfolio (expensive)
  - Buy put options on the S&P 100 index (satisfactory if option market makers have the liquidity to absorb the trades that large fund managers wish to carry out, and if required strike prices and exercise dates are available on exchanges)
  - Create the put option *synthetically*: Adjust the holdings of the fund so as to maintain a position with a delta (Δ) equal to the delta of the desired option

EXAMPLE 5: Synthetic Put

A portfolio that mimics the S&P 500 is worth \$90 million. To protect against market downturns, the managers require a six–month European put on the portfolio with a strike price of \$87 million. The S&P 500 currently stands at 900, the risk–free rate is 9% p.a., dividend yield 3% p.a. and the volatility of the portfolio is 25%

One alternative is to buy $1,000$ puts on the S&P 500 with a strike price of 870 (if available). The other alternative is to create the option synthetically:

- Calculate the Δ of the option required

$$\Delta = e^{-\delta T} \left[\Phi \left(d_1 \right) - 1 \right]$$

$$
\begin{aligned}
d_1 &= \frac{\ln S - \ln K + \left(r - \delta + \frac{1}{2}\sigma^2\right) T}{\sigma T^{1/2}} \\
&= \frac{\ln 900 - \ln 870 + \left(0.09 - 0.03 + \frac{1}{2}0.25^2\right) \times \frac{6}{12}}{0.25 \times \left(\frac{6}{12}\right)^{1/2}} = 0.4499
\end{aligned}
$$

and thus

$$
\begin{aligned}
\Delta &= e^{-0.03 \times \frac{6}{12}} \times \left(\Phi\left(0.4499\right) - 1\right) \\
&= -0.3215
\end{aligned}
$$

This shows that 32.15% of the portfolio should be sold, to match the sensitivity (delta) of the position required.

21 Reading

- Hull [9], Chapters 12, 13, 14
- Jarrow and Turnbull [11], Chapters 8, 9
- Stulz [14], Chapter 12
- Black and Scholes [2]

References

[1] F. Black. The pricing of commodity contracts. *Journal of Financial Economics*, 3:167–179, 1976.

[2] F. Black and M. Scholes. The pricing of options and corporate liabilities. *Journal of Political Economy*, 81(3):637–654, 1973.

[3] A. J. Boness. Elements of a theory of stock option value. *Journal of Political Economy*, 72(2):163–175, 1964.

[4] J. C. Cox. Notes on option pricing I: Constant elasticity of diffusions. Unpublished, Stanford University, 1975.

[5] J. C. Cox and S. A. Ross. The pricing of options for jump processes. Working Paper, University of Pennsylvania, 1975.

[6] J. C. Cox and S. A. Ross. The valuation of options for alternative stochastic processes. *Journal of Financial Economics*, 3(1):145–166, 1976.

[7] M. B. Garman and S. W. Kohlhagen. Foreign currency option values. *Journal of International Money and Finance*, 2(3):231–237, 1983.

[8] J. O. Grabbe. The pricing of call and put options on foreign exchange. *Journal of International Money and Finance*, 2(3):239–253, 1983.

[9] J. C. Hull. *Options, Futures and other Derivatives*. Prentice Hall Inc., Upper Saddle River, New Jersey, 5th edition, 2003.

[10] K. Itô. On stochastic differential equations. *Memoirs, American Mathematical Society*, 4:1–51, 1951.

[11] R. A. Jarrow and S. Turnbull. *Derivative Securities*. South–Western College Publishing, Thomson Learning, 2nd edition, 2000.

[12] R. C. Merton. Option pricing when underlying stock returns are discontinuous. *Journal of Financial Economics*, 3(1):125–144, 1976.

[13] P. A. Samuelson. Rational theory of warrant pricing. *Industrial Management Review*, 6(Spring):13–31, 1965.

[14] R. M. Stulz. *Risk Management and Derivatives*. South–Western College Publishing, Thomson Learning, 2003.