Induction Course in Quantitative Methods for Finance

Probability, Discrete Random Variables and Probability Distributions

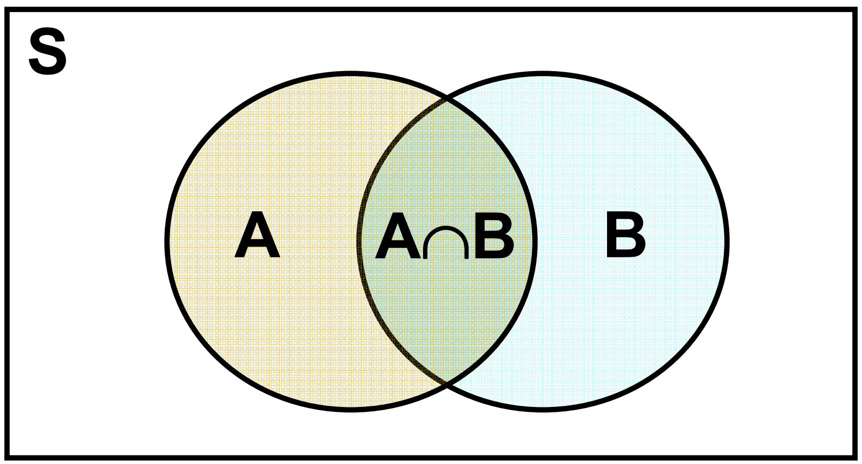
Important Terms

- **Random Experiment** – a process leading to an uncertain outcome
- **Basic Outcome** – a possible outcome of a random experiment
- **Sample Space** – the collection of all possible outcomes of a random experiment
- **Event** – any subset of basic outcomes from the sample space

Important Terms

(continued)

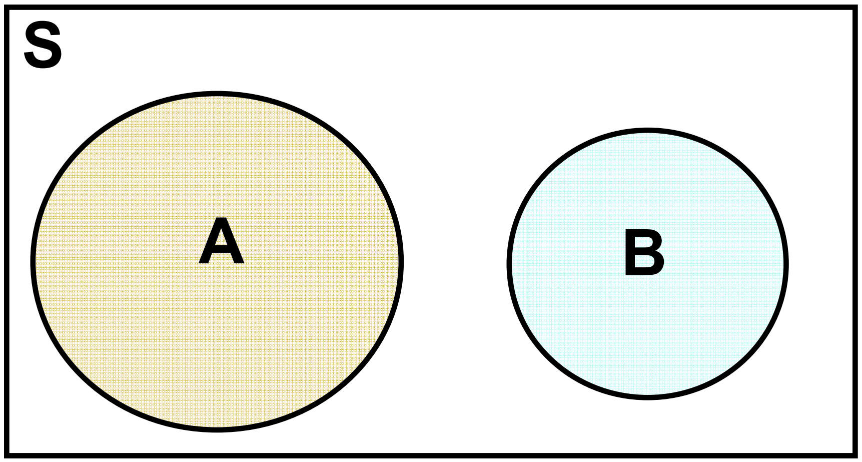
- **Intersection of Events** – If A and B are two events in a sample space S, then the intersection, A ∩ B, is the set of all outcomes in S that belong to both A and B



Important Terms

(continued)

- A and B are **Mutually Exclusive Events** if they have no basic outcomes in common
  - i.e., the set A ∩ B is empty

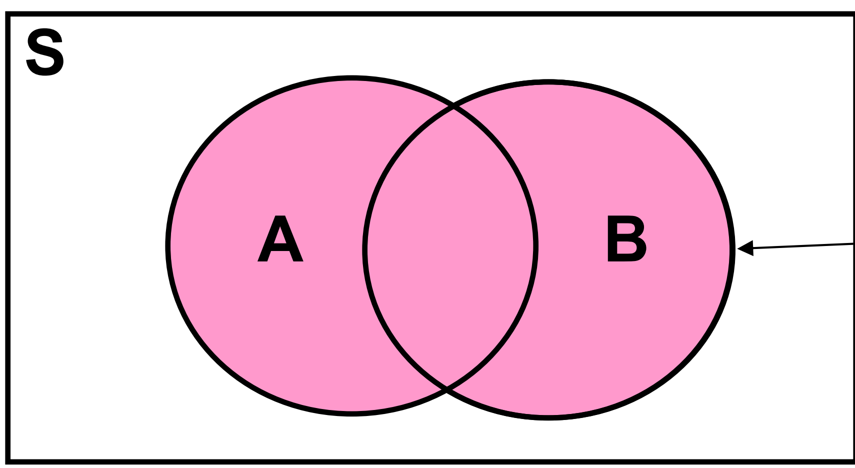


Important Terms

(continued)

- **Union of Events** – If A and B are two events in a sample space S, then the union, A U B, is the set of all outcomes in S that belong to either

  A or B



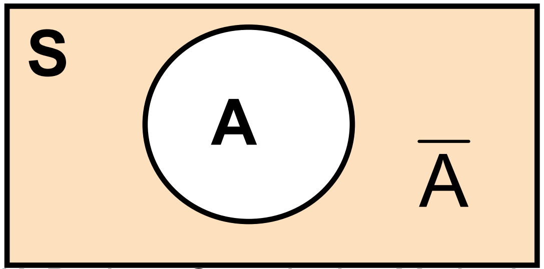
The entire shaded area represents **A** U **B**

Important Terms

(continued)

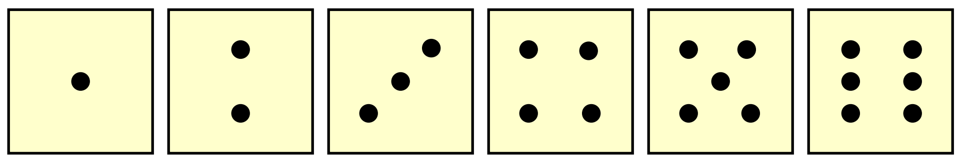
- Events E_1, E_2, … E_k are **Collectively Exhaustive** events if $E_1 \cup E_2 \cup \ldots \cup E_k = S$
  - i.e., the events completely cover the sample space
- The **Complement** of an event A is the set of all basic outcomes in the sample space that do not belong to A.

  The complement is denoted $\overline{A}$



Examples

Let the **Sample Space** be the collection of all possible outcomes of rolling one die:



S = [1, 2, 3, 4, 5, 6]

Let A be the event “Number rolled is even”

Let B be the event “Number rolled is at least 4”

Then

A = [2, 4, 6] and B = [4, 5, 6]

Examples

(continued)

S = [1, 2, 3, 4, 5, 6] | A = [2, 4, 6] | B = [4, 5, 6]

Complements:

$$\overline{A} = [1, 3, 5] \qquad \overline{B} = [1, 2, 3]$$

Intersections:

$$A \cap B = [4, 6] \qquad \overline{A} \cap B = [5]$$

Unions:

$$A \cup B = [2, 4, 5, 6]$$

$$A \cup \overline{A} = [1, 2, 3, 4, 5, 6] = S$$

Examples

(continued)

S = [1, 2, 3, 4, 5, 6]	A = [2, 4, 6]	B = [4, 5, 6]

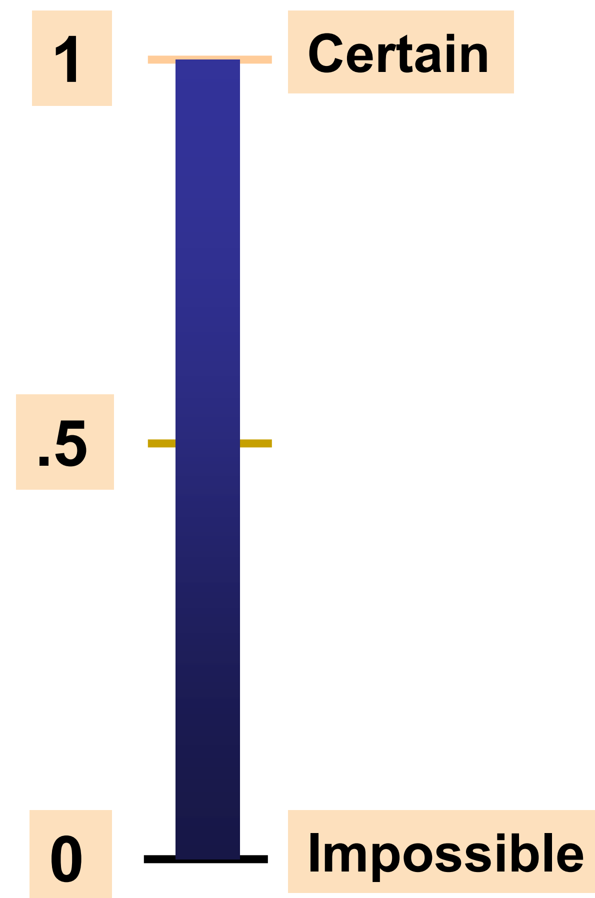
- **Mutually exclusive**:
  - A and B are **not** mutually exclusive
    - The outcomes 4 and 6 are common to both

- **Collectively exhaustive**:
  - A and B are **not** collectively exhaustive
    - $A \cup B$ does not contain 1 or 3

Probability

- **Probability** – the chance that an uncertain event will occur (always between 0 and 1)

$0 \leq P(A) \leq 1$ For any event A



Assessing Probability

- There are three approaches to assessing the probability of an uncertain event:

1. classical probability

$$\text{probability of event } A = \frac{N_A}{N} = \frac{\text{number of outcomes that satisfy the event}}{\text{total number of outcomes in the sample space}}$$

- Assumes all outcomes in the sample space are equally likely to occur

Counting the Possible Outcomes

- Use the **Combinations formula** to determine the number of combinations of n things taken k at a time

$$C_k^n = \frac{n!}{k!(n-k)!}$$

- where
  - $n! = n(n-1)(n-2)\ldots(1)$
  - $0! = 1$ by definition

Assessing Probability

Three approaches (continued)

2. relative frequency probability

$$\text{probability of event A} = \lim_{n\to\infty} \frac{n_A}{n} = \frac{\text{number of times that the event A has occured}}{\text{number of times that the experiment is performed}}$$

- the limit of the proportion of times that an event A occurs in a large number of trials, *n*

3. subjective probability

an individual opinion or belief about the probability of occurrence

Probability Postulates

1. If A is any event in the sample space S, then

$$\boxed{0 \le P(A) \le 1}$$

2. Let A be an event in S, and let O_i denote the basic outcomes. Then

(the notation means that the summation is over all the basic outcomes in A)

$$\boxed{P(A) = \sum_{A} P(O_i)}$$

3.

$$\boxed{P(S) = 1}$$

Probability Rules

- The Complement rule:

$$\boxed{P(\overline{A}) = 1 - P(A)} \qquad \text{i.e., } P(A) + P(\overline{A}) = 1$$

- The Addition rule:
  - The probability of the union of two events is

$$\boxed{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$$

A Probability Table

Probabilities and joint probabilities for two events A and B are summarized in this table:

	B	$\overline{B}$	
A	$P(A \cap B)$	$P(A \cap \overline{B})$	$P(A)$
$\overline{A}$	$P(\overline{A} \cap B)$	$P(\overline{A} \cap \overline{B})$	$P(\overline{A})$
	$P(B)$	$P(\overline{B})$	$P(S) = 1.0$

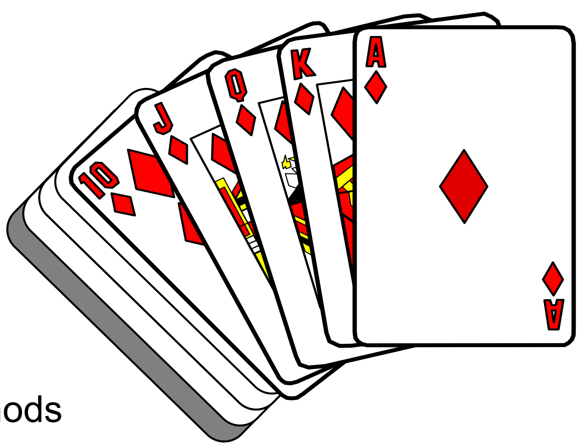
Addition Rule Example

Consider a standard deck of 52 cards, with four suits:

♥ ♣ ♦ ♠

Let event A = card is an Ace

Let event B = card is from a red suit



Addition Rule Example

(continued)

$$P(\text{Red} \cup \text{Ace}) = P(\text{Red}) + P(\text{Ace}) - P(\text{Red} \cap \text{Ace})$$

$$= 26/52 + 4/52 - \boxed{2/52} = 28/52$$

Don't count the two red aces twice!

Type	Color		Total
	Red	Black	
Ace	2	2	4
Non-Ace	24	24	48
Total	26	26	52

Conditional Probability

- A conditional probability is the probability of one event, given that another event has occurred:

$$P(A\,|\,B) = \frac{P(A \cap B)}{P(B)}$$

→ The conditional probability of A given that B has occurred

$$P(B\,|\,A) = \frac{P(A \cap B)}{P(A)}$$

→ The conditional probability of B given that A has occurred

Conditional Probability Example

- Of the cars on a used car lot, 70% have air conditioning (AC) and 40% have a CD player (CD). 20% of the cars have both.

- What is the probability that a car has a CD player, given that it has AC ?

i.e., we want to find $P(CD \mid AC)$

Conditional Probability Example

(continued)

- Of the cars on a used car lot, **70%** have air conditioning (AC) and **40%** have a CD player (CD).
  20% of the cars have both.

	CD	No CD	Total
AC	.2	.5	.7
No AC	.2	.1	.3
Total	.4	.6	1.0

$$P(CD \mid AC) = \frac{P(CD \cap AC)}{P(AC)} = \frac{.2}{.7} = .2857$$

Conditional Probability Example

(continued)

- Given AC, we only consider the top row (70% of the cars). Of these, 20% have a CD player. 20% of 70% is 28.57%.

	CD	No CD	Total
AC	.2	.5	.7
No AC	.2	.1	.3
Total	.4	.6	1.0

$$P(CD \mid AC) = \frac{P(CD \cap AC)}{P(AC)} = \frac{.2}{.7} = .2857$$

Multiplication Rule

- Multiplication rule for two events A and B:

$$\boxed{P(A \cap B) = P(A \mid B)P(B)}$$

- also $$\boxed{P(A \cap B) = P(B \mid A)P(A)}$$

Multiplication Rule Example

P(Red ∩ Ace) = P(Red| Ace)P(Ace)

$$= \left(\frac{2}{4}\right)\left(\frac{4}{52}\right) = \frac{2}{52}$$

$$= \frac{\text{number of cards that are red and ace}}{\text{total number of cards}} = \frac{2}{52}$$

Type	Color		Total
	Red	Black	
Ace	2	2	4
Non-Ace	24	24	48
Total	26	26	52

Statistical Independence

- Two events are statistically independent if and only if:

$$\boxed{P(A \cap B) = P(A)P(B)}$$

  - Events A and B are independent when the probability of one event is not affected by the other event

- If A and B are independent, then

$$\boxed{P(A \mid B) = P(A)} \qquad \text{if } P(B)>0$$

$$\boxed{P(B \mid A) = P(B)} \qquad \text{if } P(A)>0$$

Statistical Independence Example

- Of the cars on a used car lot, **70%** have air conditioning (AC) and **40%** have a CD player (CD). **20%** of the cars have both.

	CD	No CD	Total
AC	.2	.5	.7
No AC	.2	.1	.3
Total	.4	.6	1.0

- Are the events AC and CD statistically independent?

Statistical Independence Example

(continued)

	CD	No CD	Total
AC	.2	.5	.7
No AC	.2	.1	.3
Total	.4	.6	1.0

$P(AC \cap CD) = 0.2$

$$\left.\begin{array}{l} P(AC) = 0.7 \\ P(CD) = 0.4 \end{array}\right\} P(AC)P(CD) = (0.7)(0.4) = 0.28$$

$P(AC \cap CD) = 0.2 \quad \neq \quad P(AC)P(CD) = 0.28$

So the two events are not statistically independent

Bivariate Probabilities

Outcomes for bivariate events:

	B_1	B_2	. . .	B_k
A_1	$P(A_1 \cap B_1)$	$P(A_1 \cap B_2)$	. . .	$P(A_1 \cap B_k)$
A_2	$P(A_2 \cap B_1)$	$P(A_2 \cap B_2)$	. . .	$P(A_2 \cap B_k)$
. . .	. . .	. . .	. . .	. . .
A_h	$P(A_h \cap B_1)$	$P(A_h \cap B_2)$	. . .	$P(A_h \cap B_k)$

Joint and Marginal Probabilities

- The probability of a joint event, A ∩ B:

$$P(A \cap B) = \frac{\text{number of outcomes satisfying A and B}}{\text{total number of elementary outcomes}}$$

- Computing a marginal probability:

$$P(A) = P(A \cap B_1) + P(A \cap B_2) + \cdots + P(A \cap B_k)$$

- Where B_1, B_2, …, B_k are k mutually exclusive and collectively exhaustive events

Marginal Probability Example

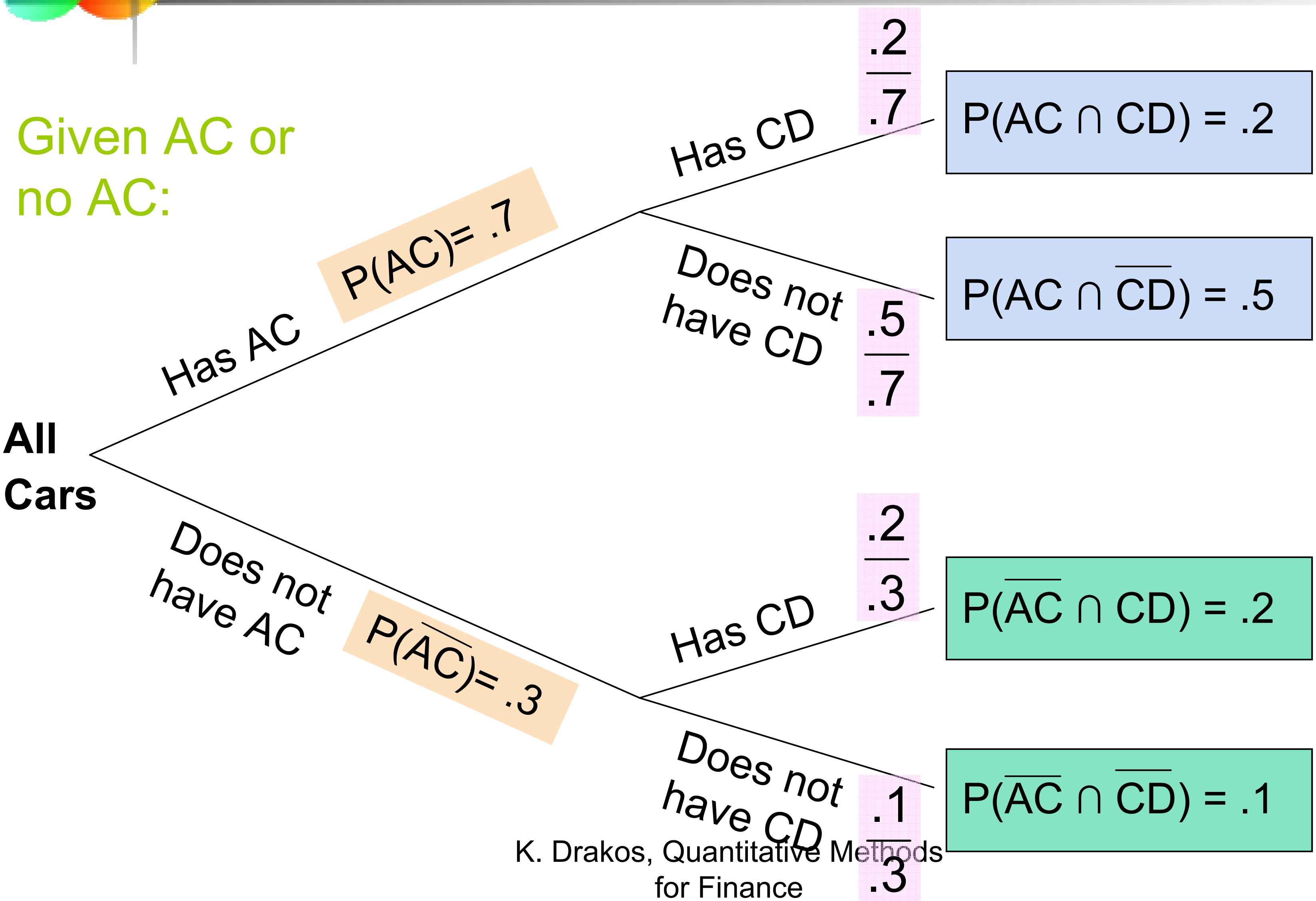
P(Ace)

$$= P(\text{Ace} \cap \text{Red}) + P(\text{Ace} \cap \text{Black}) = \frac{2}{52} + \frac{2}{52} = \frac{4}{52}$$

Type	Color		Total
	Red	Black	
Ace	2	2	4
Non-Ace	24	24	48
Total	26	26	52

Using a Tree Diagram

Given AC or no AC:



Bayes' Theorem

$$P(E_i \mid A) = \frac{P(A|E_i)P(E_i)}{P(A)}$$

$$= \frac{P(A|E_i)P(E_i)}{P(A|E_1)P(E_1) + P(A|E_2)P(E_2) + \ldots + P(A|E_k)P(E_k)}$$

- where:

  E_i = i^{th} event of k mutually exclusive and collectively exhaustive events

  A = new event that might impact $P(E_i)$

Bayes' Theorem Example

- A drilling company has estimated a 40% chance of striking oil for their new well.
- A detailed test has been scheduled for more information. Historically, 60% of successful wells have had detailed tests, and 20% of unsuccessful wells have had detailed tests.
- Given that this well has been scheduled for a detailed test, what is the probability that the well will be successful?

Bayes' Theorem Example

(continued)

- Let S = successful well

  U = unsuccessful well

- P(S) = .4 , P(U) = .6 (prior probabilities)
- Define the detailed test event as D
- Conditional probabilities:

  P(D|S) = .6 P(D|U) = .2

- Goal is to find P(S|D)

Bayes’ Theorem Example

(continued)

Apply Bayes’ Theorem:

$$P(S|D) = \frac{P(D|S)P(S)}{P(D|S)P(S) + P(D|U)P(U)}$$

$$= \frac{(.6)(.4)}{(.6)(.4) + (.2)(.6)}$$

$$= \frac{.24}{.24 + .12} = .667$$

So the revised probability of success (from the original estimate of .4), given that this well has been scheduled for a detailed test, is .667

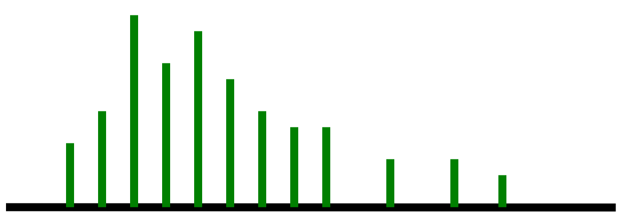
Discrete Random Variables and Probability Distributions

Introduction to Probability Distributions

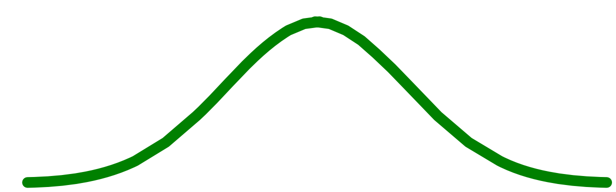
- **Random Variable**
  - Represents a possible numerical value from a random experiment

Random Variables

Discrete Random Variable



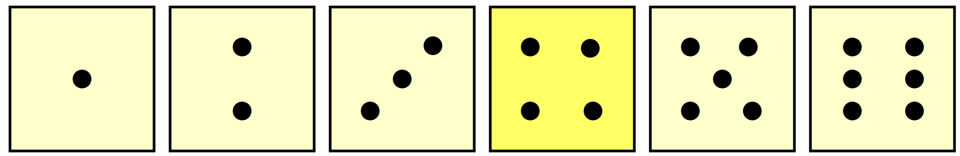
Continuous Random Variable



Discrete Random Variables

- Can only take on a countable number of values

Examples:



- Roll a die twice
  Let X be the number of times 4 comes up
  (then X could be 0, 1, or 2 times)

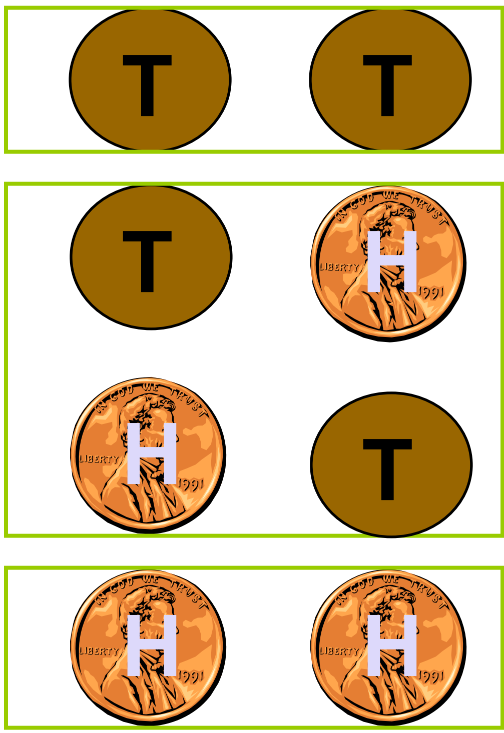
- Toss a coin 5 times.
  Let X be the number of heads
  (then X = 0, 1, 2, 3, 4, or 5)

Discrete Probability Distribution

Experiment: Toss 2 Coins. Let X = # heads.

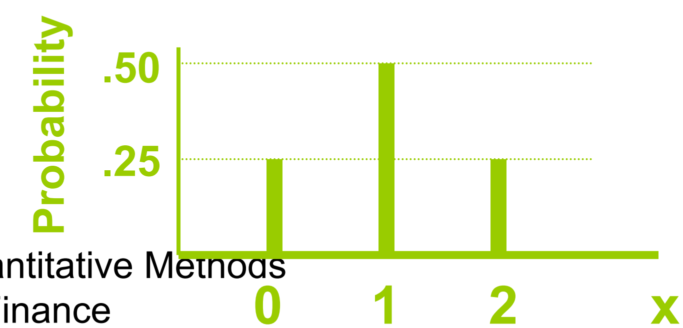
Show P(x) , i.e., P(X = x) , for all values of x:

4 possible outcomes



Probability Distribution

x Value	Probability
0	1/4 = .25
1	2/4 = .50
2	1/4 = .25



Probability Distribution Required Properties

- $\boxed{P(x) \geq 0}$ for any value of x

- The individual probabilities sum to 1;

$$\boxed{\sum_x P(x) = 1}$$

(The notation indicates summation over all possible x values)

Cumulative Probability Function

- The cumulative probability function, denoted $F(x_0)$, shows the probability that X is less than or equal to x_0

$$F(x_0) = P(X \leq x_0)$$

- In other words,

$$F(x_0) = \sum_{x \leq x_0} P(x)$$

Expected Value

- Expected Value (or mean) of a discrete distribution (Weighted Average)

$$\mu = E(x) = \sum_x xP(x)$$

x	P(x)
0	.25
1	.50
2	.25

– Example: Toss 2 coins, x = # of heads, compute expected value of x:

E(x) = (0 x .25) + (1 x .50) + (2 x .25)

= 1.0

Variance and Standard Deviation

- Variance of a discrete random variable X

$$\sigma^2 = E(X-\mu)^2 = \sum_x (x-\mu)^2 P(x)$$

- Standard Deviation of a discrete random variable X

$$\sigma = \sqrt{\sigma^2} = \sqrt{\sum_x (x-\mu)^2 P(x)}$$

Standard Deviation Example

– Example: Toss 2 coins, X = # heads, compute standard deviation (recall E(x) = 1)

$$\sigma = \sqrt{\sum_x (x-\mu)^2 P(x)}$$

$$\sigma = \sqrt{(0-1)^2(.25) + (1-1)^2(.50) + (2-1)^2(.25)} = \sqrt{.50} = .707$$

Possible number of heads = 0, 1, or 2

Functions of Random Variables

- If $P(x)$ is the probability function of a discrete random variable X, and $g(X)$ is some function of X, then the expected value of function g is

$$\boxed{E[g(X)] = \sum_{x} g(x)P(x)}$$

Linear Functions of Random Variables

- Let a and b be any constants.
- a) $\boxed{E(a) = a \qquad \text{and} \qquad Var(a) = 0}$

  i.e., if a random variable always takes the value a, it will have mean a and variance 0
- b) $\boxed{E(bX) = b\mu_X \qquad \text{and} \qquad Var(bX) = b^2\sigma_X^2}$

  i.e., the expected value of b·X is b·E(x)

Linear Functions of Random Variables

(continued)

- Let random variable X have mean μ_x and variance σ^2_x
- Let a and b be any constants.
- Let Y = a + bX
- Then the mean and variance of Y are

$$\boxed{\mu_Y = E(a + bX) = a + b\mu_X}$$

$$\boxed{\sigma^2_Y = Var(a + bX) = b^2\sigma^2_X}$$

- so that the standard deviation of Y is $\boxed{\sigma_Y = |b|\sigma_X}$

Probability Distributions

- **Probability Distributions**
  - **Discrete** Probability Distributions
    - Binomial
    - Hypergeometric
    - Poisson
  - **Continuous** Probability Distributions
    - Uniform
    - Normal
    - Exponential

Bernoulli Distribution

- Consider only two outcomes: "success" or "failure"
- Let P denote the probability of success
- Let $1 - P$ be the probability of failure
- Define random variable X:

  $x = 1$ if success, $x = 0$ if failure

- Then the Bernoulli probability function is

$$\boxed{P(0) = (1 - P) \quad \text{and} \quad P(1) = P}$$

Bernoulli Distribution Mean and Variance

- The mean is $\boxed{\mu = P}$

$$\mu = E(X) = \sum_{X} xP(x) = (0)(1-P) + (1)P = P$$

- The variance is $\boxed{\sigma^2 = P(1 - P)}$

$$\sigma^2 = E[(X-\mu)^2] = \sum_{X} (x-\mu)^2 P(x)$$
$$= (0-P)^2(1-P) + (1-P)^2 P = P(1-P)$$

Sequences of x Successes in n Trials

- The number of sequences with x successes in n independent trials is:

$$\boxed{C_x^n = \frac{n!}{x!(n-x)!}}$$

Where n! = n·(n – 1)·(n – 2)· . . . ·1 and 0! = 1

- These sequences are mutually exclusive, since no two can occur at the same time

Binomial Probability Distribution

- A fixed number of observations, n
  - e.g., 15 tosses of a coin; ten light bulbs taken from a warehouse
- Two mutually exclusive and collectively exhaustive categories
  - e.g., head or tail in each toss of a coin; defective or not defective light bulb
  - Generally called “success” and “failure”
  - Probability of success is P , probability of failure is 1 – P
- Constant probability for each observation
  - e.g., Probability of getting a tail is the same each time we toss the coin
- Observations are independent
  - The outcome of one observation does not affect the outcome of the other

Possible Binomial Distribution Settings

- A manufacturing plant labels items as either defective or acceptable
- A firm bidding for contracts will either get a contract or not
- A marketing research firm receives survey responses of “yes I will buy” or “no I will not”
- New job applicants either accept the offer or reject it

Binomial Distribution Formula

$$P(x) = \frac{n!}{x!(n-x)!} P^{x}(1-P)^{n-x}$$

P(x) = probability of **x** successes in **n** trials,
with probability of success **P** on each trial

x = number of 'successes' in sample,
(x = 0, 1, 2, ..., n)

n = sample size (number of trials or observations)

P = probability of "success"

Example: Flip a coin four times, let x = # heads:

n = 4

P = 0.5

1 - P = (1 - 0.5) = 0.5

x = 0, 1, 2, 3, 4

Example: Calculating a Binomial Probability

What is the probability of one success in five observations if the probability of success is 0.1?

x = 1, n = 5, and P = 0.1

$$P(x=1) = \frac{n!}{x!(n-x)!} P^X (1-P)^{n-X}$$

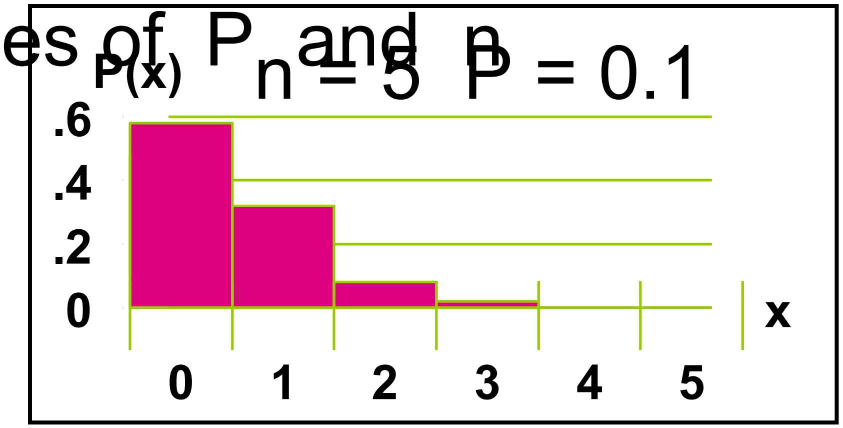
$$= \frac{5!}{1!(5-1)!} (0.1)^1 (1-0.1)^{5-1}$$

$$= (5)(0.1)(0.9)^4$$

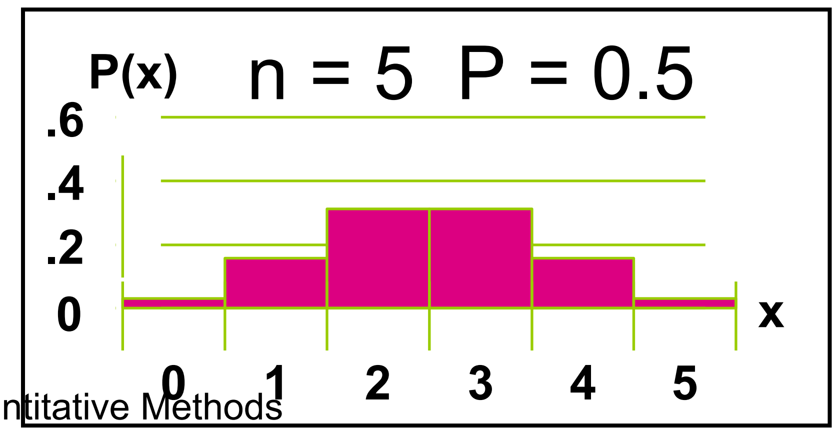
$$= .32805$$

Binomial Distribution

- The shape of the binomial distribution depends on the values of P and n
  - Here, n = 5 and P = 0.1



  - Here, n = 5 and P = 0.5



Binomial Distribution Mean and Variance

- Mean

$$\mu = E(x) = nP$$

- Variance and Standard Deviation

$$\sigma^2 = nP(1-P)$$

$$\sigma = \sqrt{nP(1-P)}$$

Where n = sample size

P = probability of success

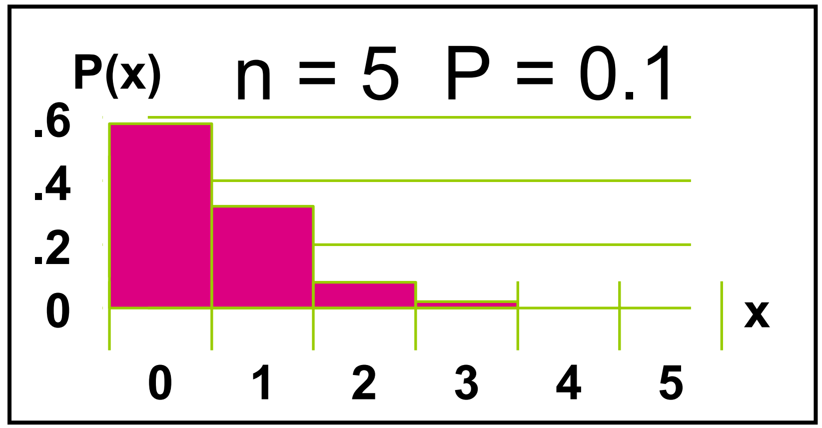
(1 – P) = probability of failure

Binomial Characteristics

Examples

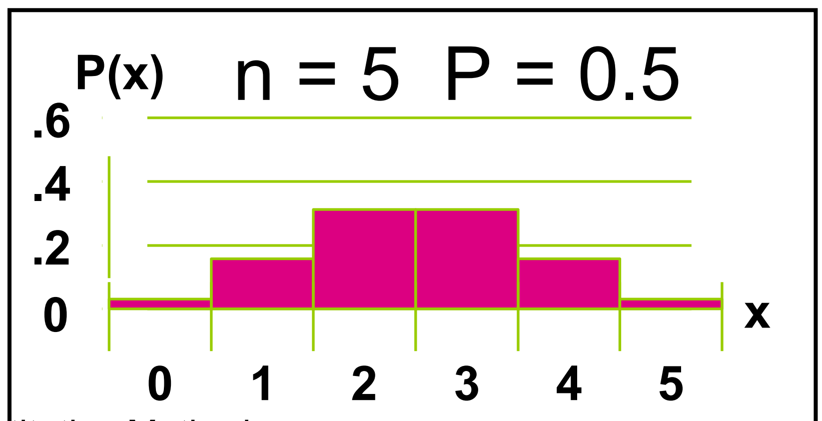
$$\mu = nP = (5)(0.1) = 0.5$$

$$\sigma = \sqrt{nP(1-P)} = \sqrt{(5)(0.1)(1-0.1)} = 0.6708$$



$$\mu = nP = (5)(0.5) = 2.5$$

$$\sigma = \sqrt{nP(1-P)} = \sqrt{(5)(0.5)(1-0.5)} = 1.118$$



The Poisson Distribution

- Apply the Poisson Distribution when:
  - You wish to count the number of times an event occurs in a given continuous interval
  - The probability that an event occurs in one subinterval is very small and is the same for all subintervals
  - The number of events that occur in one subinterval is independent of the number of events that occur in the other subintervals
  - There can be no more than one occurrence in each subinterval
  - The average number of events per unit is λ (lambda)

Poisson Distribution Formula

$$P(x) = \frac{e^{-\lambda}\lambda^{x}}{x!}$$

where:

x = number of successes per unit

λ = expected number of successes per unit

e = base of the natural logarithm system (2.71828...)

Poisson Distribution Characteristics

- Mean

$$\boxed{\mu = E(x) = \lambda}$$

- Variance and Standard Deviation

$$\boxed{\sigma^2 = E[(X-\mu)^2] = \lambda}$$

$$\boxed{\sigma = \sqrt{\lambda}}$$

where λ = expected number of successes per unit

Using Poisson Tables

X	λ								
	0.10	0.20	0.30	0.40	**0.50**	0.60	0.70	0.80	0.90
0	0.904	0.818	0.740	0.670	0.606	0.548	0.496	0.449	0.4066
1	0.090	0.163	0.222	0.268	0.303	0.329	0.347	0.359	0.3659
2	0.004	0.016	0.033	0.053	**0.075**	0.098	0.121	0.143	0.1647
3	0.000	0.001	0.003	0.007	0.012	0.019	0.028	0.038	0.0494
4	0.000	0.000	0.000	0.000	0.001	0.003	0.005	0.007	0.0111
5	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.0020
6	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.0003
7	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.0000

Example: Find P(X = 2) if λ = .50

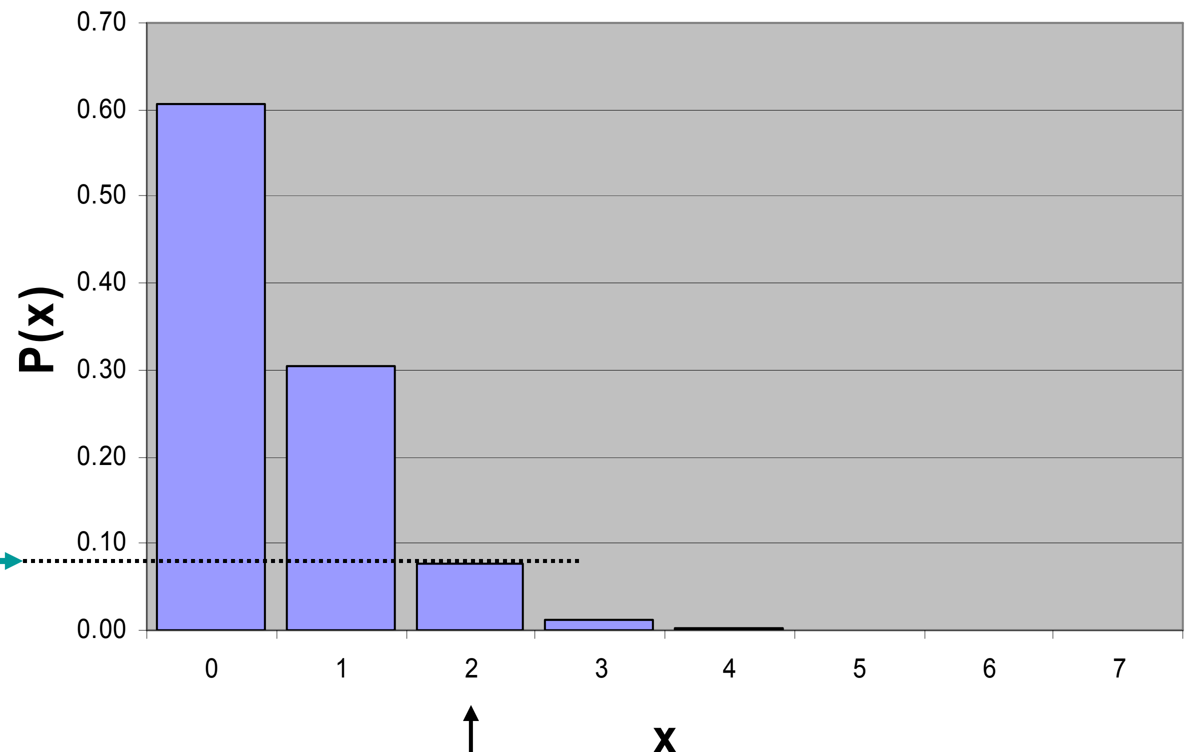
$$P(X = 2) = \frac{e^{-\lambda}\lambda^{X}}{X!} = \frac{e^{-0.50}(0.50)^2}{2!} = .0758$$

Graph of Poisson Probabilities

Graphically:

λ = .50

X	**λ = 0.50**
0	0.6065
1	0.3033
2	0.0758
3	0.0126
4	0.0016
5	0.0002
6	0.0000
7	0.0000

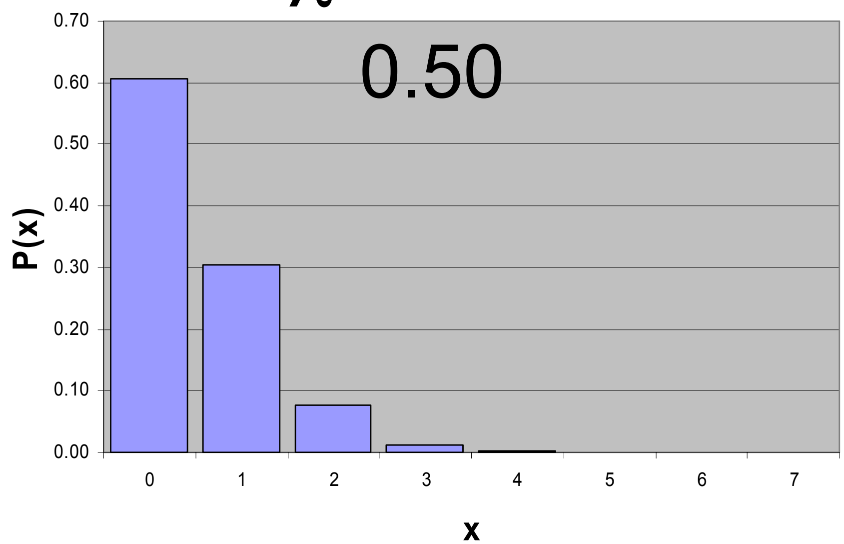


P(X = 2) = .0758

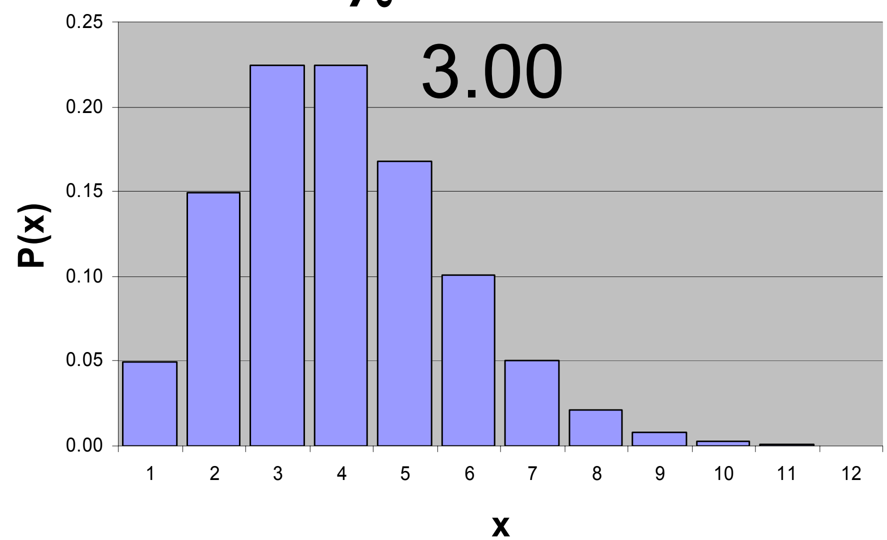
Poisson Distribution Shape

- The shape of the Poisson Distribution depends on the parameter λ :

$\lambda = 0.50$



$\lambda = 3.00$



Joint Probability Functions

- A joint probability function is used to express the probability that X takes the specific value x and simultaneously Y takes the value y, as a function of x and y

$$\boxed{P(x,y) = P(X = x \cap Y = y)}$$

- The marginal probabilities are

$$\boxed{P(x) = \sum_{y} P(x,y)} \qquad \boxed{P(y) = \sum_{x} P(x,y)}$$

Conditional Probability Functions

- The conditional probability function of the random variable Y expresses the probability that Y takes the value y when the value x is specified for X.

$$\boxed{P(y \mid x) = \frac{P(x,y)}{P(x)}}$$

- Similarly, the conditional probability function of X, given $Y = y$ is:

$$\boxed{P(x \mid y) = \frac{P(x,y)}{P(y)}}$$

Independence

- The jointly distributed random variables X and Y are said to be independent if and only if their joint probability function is the product of their marginal probability functions:

$$\boxed{P(x,y) = P(x)P(y)}$$

for all possible pairs of values x and y

- A set of k random variables are independent if and only if

$$\boxed{P(x_1, x_2, \cdots, x_k) = P(x_1)P(x_2)\cdots P(x_k)}$$

Covariance

- Let X and Y be discrete random variables with means μ_X and μ_Y
- The expected value of $(X - \mu_X)(Y - \mu_Y)$ is called the covariance between X and Y
- For discrete random variables

$$\mathrm{Cov}(X,Y) = E[(X-\mu_X)(Y-\mu_Y)] = \sum_x \sum_y (x-\mu_x)(y-\mu_y)P(x,y)$$

- An equivalent expression is

$$\mathrm{Cov}(X,Y) = E(XY) - \mu_x\mu_y = \sum_x \sum_y xyP(x,y) - \mu_x\mu_y$$

Covariance and Independence

- The covariance measures the strength of the linear relationship between two variables

- If two random variables are statistically independent, the covariance between them is 0
  - The converse is not necessarily true

Correlation

- The correlation between X and Y is:

$$\rho = \mathrm{Corr}(X, Y) = \frac{\mathrm{Cov}(X, Y)}{\sigma_X \sigma_Y}$$

- $\rho = 0 \Rightarrow$ no linear relationship between X and Y
- $\rho > 0 \Rightarrow$ positive linear relationship between X and Y
  - » when X is high (low) then Y is likely to be high (low)
  - » $\rho = +1 \Rightarrow$ perfect positive linear dependency
- $\rho < 0 \Rightarrow$ negative linear relationship between X and Y
  - » $\rho = -1 \Rightarrow$ perfect negative linear dependency
  - » when X is high (low) then Y is likely to be low (high)

Portfolio Analysis

- Let random variable X be the price for stock A
- Let random variable Y be the price for stock B
- The market value, W, for the portfolio is given by the linear function

$$\boxed{W = aX + bY}$$

(a is the number of shares of stock A,
b is the number of shares of stock B)

Portfolio Analysis

(continued)

- The mean value for W is

$$\boxed{\begin{aligned}\mu_W &= E[W] = E[aX + bY] \\ &= a\mu_X + b\mu_Y\end{aligned}}$$

- The variance for W is

$$\boxed{\sigma_W^2 = a^2\sigma_X^2 + b^2\sigma_Y^2 + 2ab\,\mathrm{Cov}(X,Y)}$$

or using the correlation formula

$$\boxed{\sigma_W^2 = a^2\sigma_X^2 + b^2\sigma_Y^2 + 2ab\,\mathrm{Corr}(X,Y)\sigma_X\sigma_Y}$$

Example: Investment Returns

Return per $1,000 for two types of investments

$P(x_i y_i)$	Economic condition	Investment: Passive Fund X	Investment: Aggressive Fund Y
.2	Recession	- $ 25	- $200
.5	Stable Economy	+ 50	+ 60
.3	Expanding Economy	+ 100	+ 350

$$E(x) = \mu_x = (-25)(.2) + (50)(.5) + (100)(.3) = 50$$

$$E(y) = \mu_y = (-200)(.2) + (60)(.5) + (350)(.3) = 95$$

Computing the Standard Deviation for Investment Returns

$P(x_i y_i)$	Economic condition	Investment	
		Passive Fund X	Aggressive Fund Y
0.2	Recession	- $ 25	- $200
0.5	Stable Economy	+ 50	+ 60
0.3	Expanding Economy	+ 100	+ 350

$$\sigma_X = \sqrt{(-25-50)^2(0.2)+(50-50)^2(0.5)+(100-50)^2(0.3)}$$
$$= 43.30$$

$$\sigma_y = \sqrt{(-200-95)^2(0.2)+(60-95)^2(0.5)+(350-95)^2(0.3)}$$
$$= 193.71$$

Covariance for Investment Returns

$P(x_i y_i)$	Economic condition	Investment	
		Passive Fund X	Aggressive Fund Y
.2	Recession	- $ 25	- $200
.5	Stable Economy	+ 50	+ 60
.3	Expanding Economy	+ 100	+ 350

$$
\begin{aligned}
\mathrm{Cov}(X,Y) &= (-25-50)(-200-95)(.2)+(50-50)(60-95)(.5) \\
&\quad +(100-50)(350-95)(.3) \\
&= 8250
\end{aligned}
$$

Portfolio Example

Investment X: $\mu_x = 50$ $\sigma_x = 43.30$

Investment Y: $\mu_y = 95$ $\sigma_y = 193.21$

$\sigma_{xy} = 8250$

Suppose 40% of the portfolio (P) is in Investment X and 60% is in Investment Y:

$$E(P) = .4(50) + (.6)(95) = 77$$

$$\sigma_P = \sqrt{(.4)^2(43.30)^2 + (.6)^2(193.21)^2 + 2(.4)(.6)(8250)}$$
$$= 133.04$$

The portfolio return and portfolio variability are between the values for investments X and Y considered individually

Interpreting the Results for Investment Returns

- The aggressive fund has a higher expected return, but much more risk

$$\mu_y = 95 \; > \; \mu_x = 50$$

but

$$\sigma_y = 193.21 \; > \; \sigma_x = 43.30$$

- The Covariance of 8250 indicates that the two investments are positively related and will vary in the same direction

Continuous Random Variables and Probability Distributions

Continuous Probability Distributions

- A continuous random variable is a variable that can assume any value in an interval
  - thickness of an item
  - time required to complete a task
  - temperature
  - height, in inches
- These can potentially take on any value, depending only on the ability to measure accurately.

Cumulative Distribution Function

- The cumulative distribution function, F(x), for a continuous random variable X expresses the probability that X does not exceed the value of x

$$\boxed{F(x) = P(X \leq x)}$$

- Let a and b be two possible values of X, with a < b. The probability that X lies between a and b is

$$\boxed{P(a < X < b) = F(b) - F(a)}$$

Probability Density Function

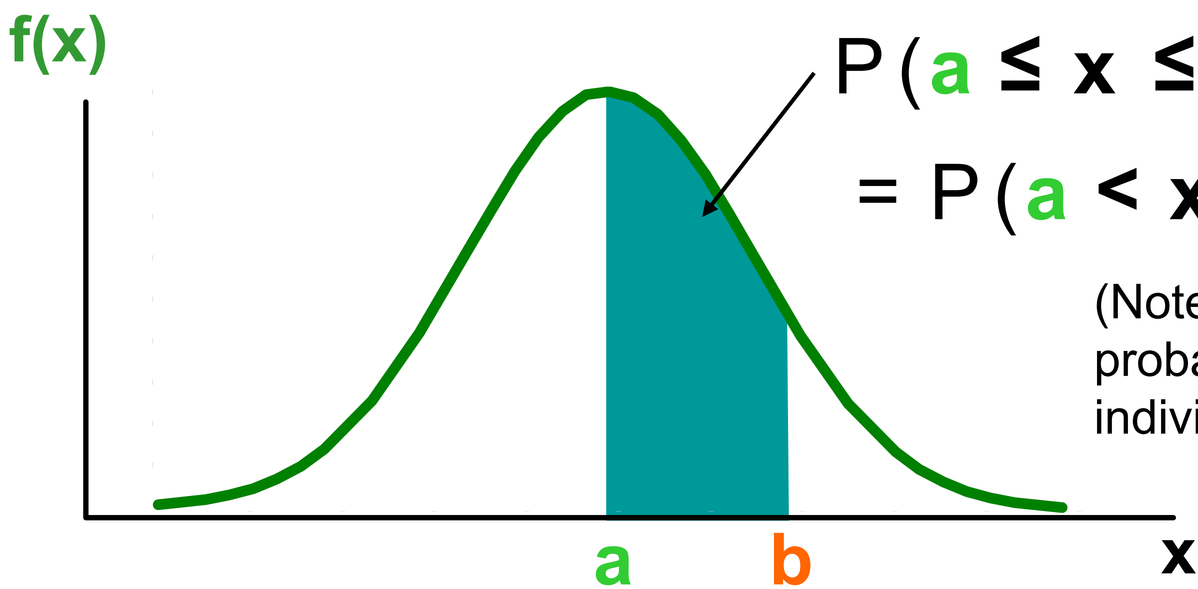
The probability density function, f(x), of random variable X has the following properties:

1. f(x) > 0 for all values of x
2. The area under the probability density function f(x) over all values of the random variable X is equal to 1.0
3. The probability that X lies between two values is the area under the density function graph between the two values
4. The cumulative density function $F(x_0)$ is the area under the probability density function f(x) from the minimum x_m value up to x_0

$$F(x_0) = \int_{x_m}^{x_0} f(x)dx$$

Probability as an Area

Shaded area under the curve is the probability that X is between a and b



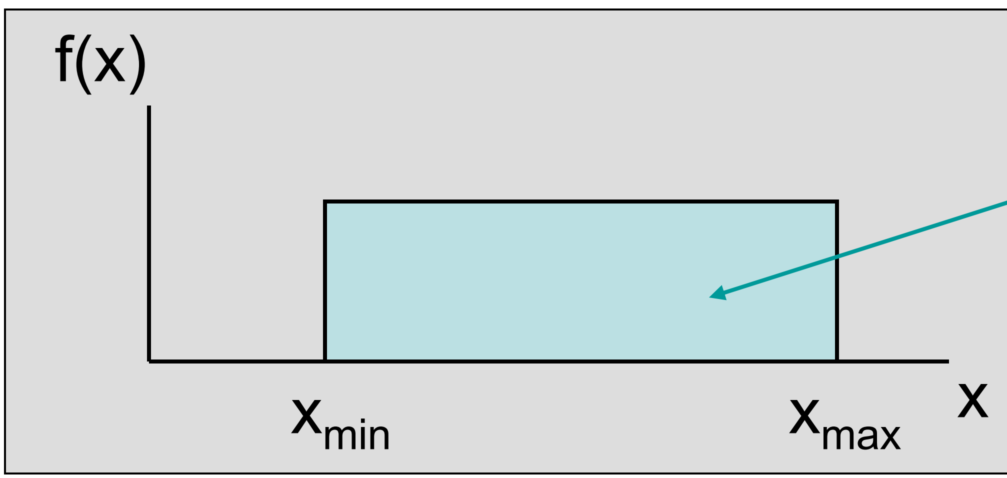
$$P(a \le x \le b)$$

$$= P(a < x < b)$$

(Note that the probability of any individual value is zero)

The Uniform Distribution

- The uniform distribution is a probability distribution that has equal probabilities for all possible outcomes of the random variable



Total area under the uniform probability density function is 1.0

The Uniform Distribution

(continued)

The Continuous Uniform Distribution:

$$f(x) = \begin{cases} \dfrac{1}{b-a} & \text{if } a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$

where

f(x) = value of the density function at any x value

a = minimum value of x

b = maximum value of x

Properties of the Uniform Distribution

- The mean of a uniform distribution is

$$\mu = \frac{a + b}{2}$$

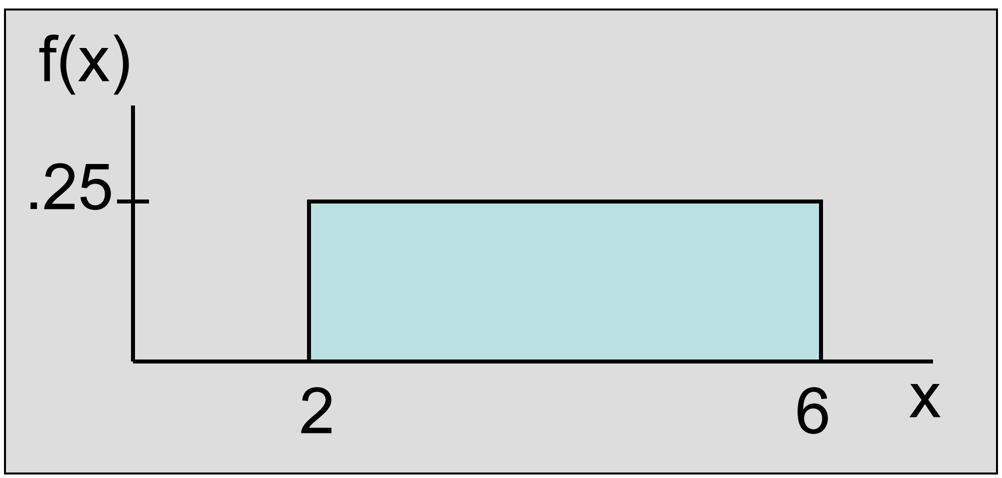
- The variance is

$$\sigma^2 = \frac{(b - a)^2}{12}$$

Uniform Distribution Example

Example: Uniform probability distribution over the range $2 \le x \le 6$:

$$f(x) = \frac{1}{6-2} = .25 \quad \text{for } 2 \le x \le 6$$



$$\mu = \frac{a+b}{2} = \frac{2+6}{2} = 4$$

$$\sigma^2 = \frac{(b-a)^2}{12} = \frac{(6-2)^2}{12} = 1.333$$

Expectations for Continuous Random Variables

- The mean of X, denoted μ_X, is defined as the expected value of X

$$\boxed{\mu_X = E(X)}$$

- The variance of X, denoted σ_X^2, is defined as the expectation of the squared deviation, $(X - \mu_X)^2$, of a random variable from its mean

$$\boxed{\sigma_X^2 = E[(X - \mu_X)^2]}$$

Linear Functions of Variables

- Let $\boxed{W = a + bX}$, where X has mean μ_X and variance σ_X^2, and a and b are constants
- Then the mean of W is

$$\boxed{\mu_W = E(a+bX) = a + b\mu_X}$$

- the variance is

$$\boxed{\sigma_W^2 = \mathrm{Var}(a+bX) = b^2\sigma_X^2}$$

- the standard deviation of W is

$$\boxed{\sigma_W = |b|\sigma_X}$$

Linear Functions of Variables

(continued)

- An important special case of the previous results is the standardized random variable

$$\boxed{Z = \frac{X - \mu_X}{\sigma_X}}$$

- which has a mean 0 and variance 1

The Normal Distribution

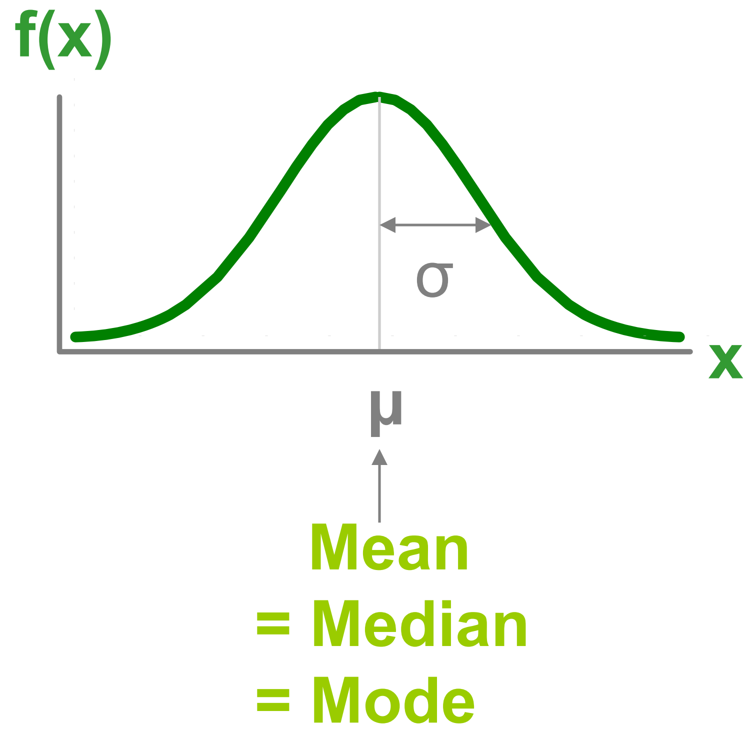
(continued)

- Bell Shaped’
- Symmetrical
- Mean, Median and Mode are Equal

Location is determined by the mean, μ

Spread is determined by the standard deviation, σ

The random variable has an infinite theoretical range: $+\infty$ to $-\infty$

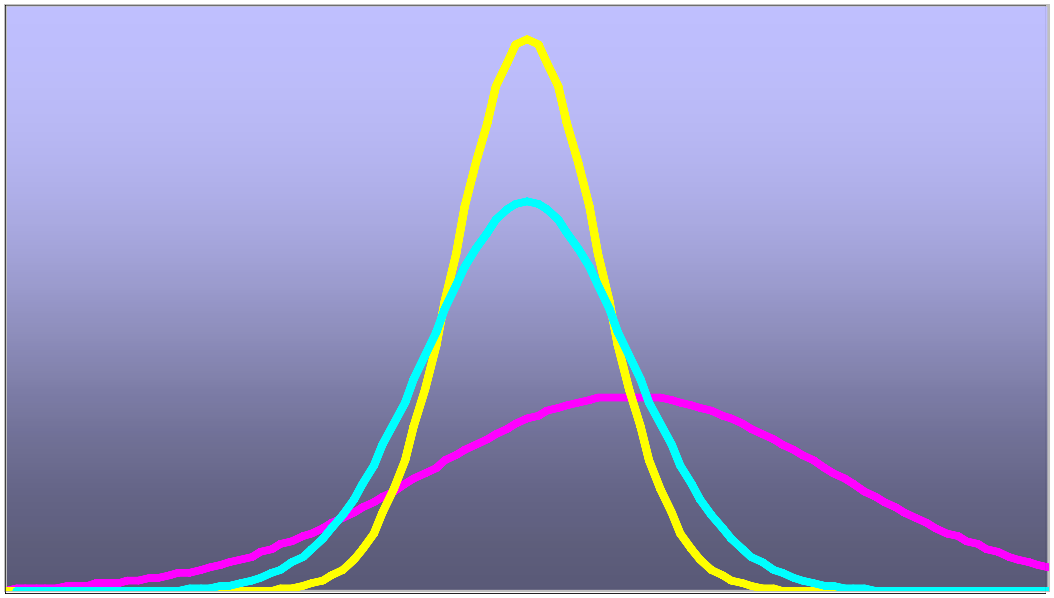


The Normal Distribution

(continued)

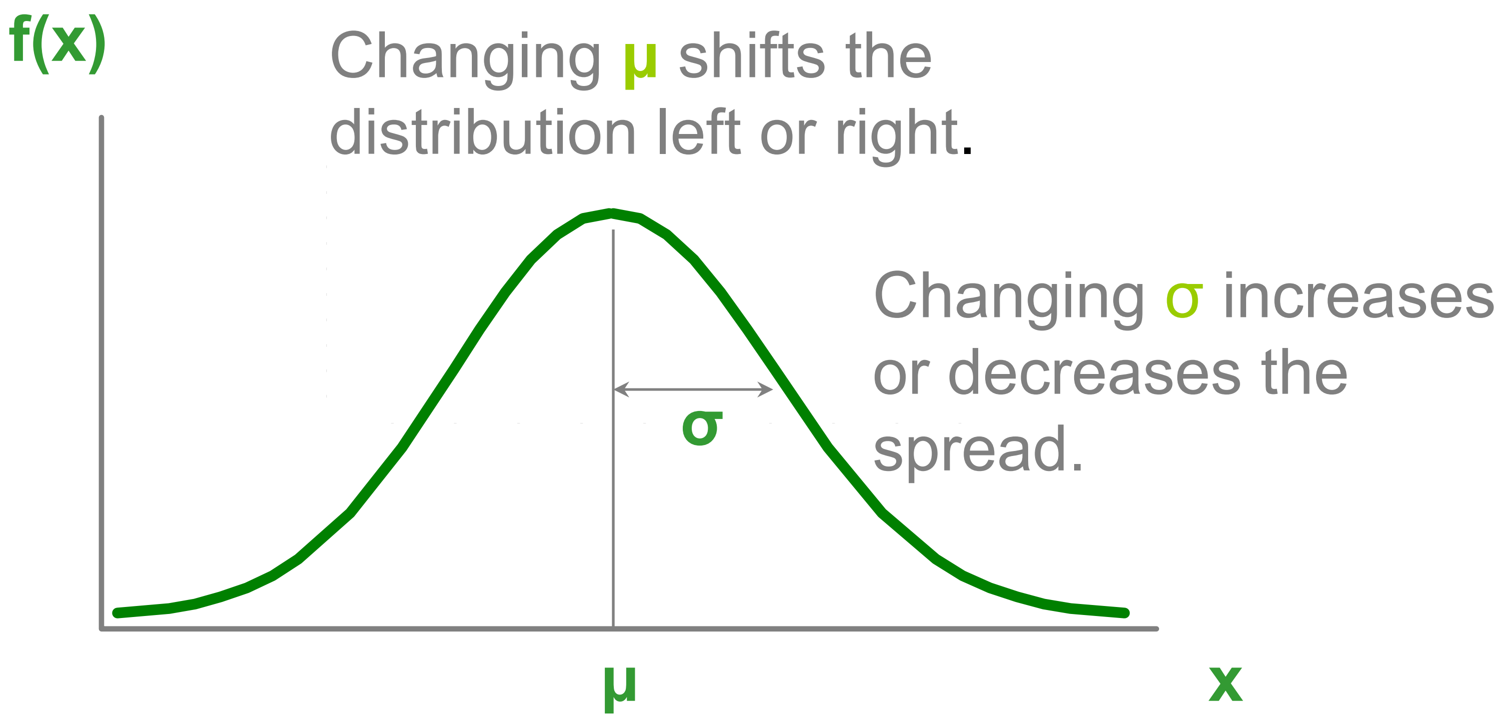
- The normal distribution closely approximates the probability distributions of a wide range of random variables
- Distributions of sample means approach a normal distribution given a “large” sample size
- Computations of probabilities are direct and elegant
- The normal probability distribution has led to good business decisions for a number of applications

Many Normal Distributions



By varying the parameters μ and σ, we obtain different normal distributions

The Normal Distribution Shape



Given the mean μ and variance σ we define the normal distribution using the notation

$$X \sim N(\mu,\sigma^2)$$

The Normal Probability Density Function

- The formula for the normal probability density function is

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}$$

Where e = the mathematical constant approximated by 2.71828

π = the mathematical constant approximated by 3.14159

μ = the population mean

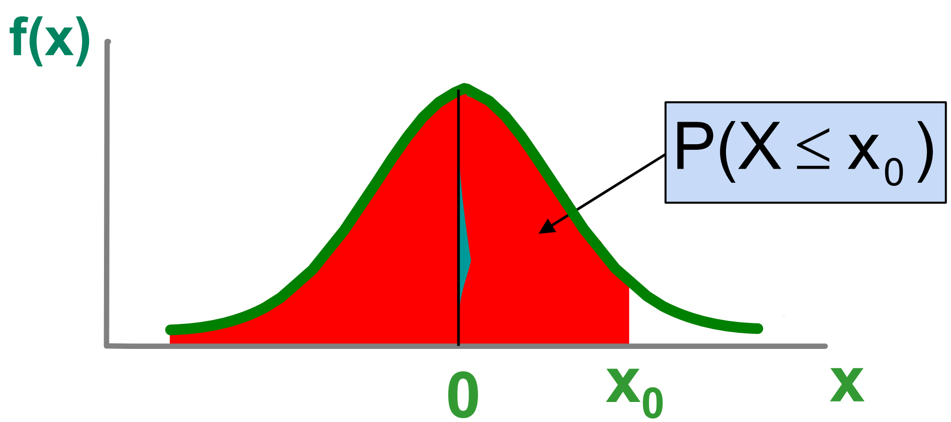
σ = the population standard deviation

x = any value of the continuous variable, $-\infty < x < \infty$

Cumulative Normal Distribution

- For a normal random variable X with mean μ and variance σ^2 , i.e., $X \sim N(\mu, \sigma^2)$, the cumulative distribution function is

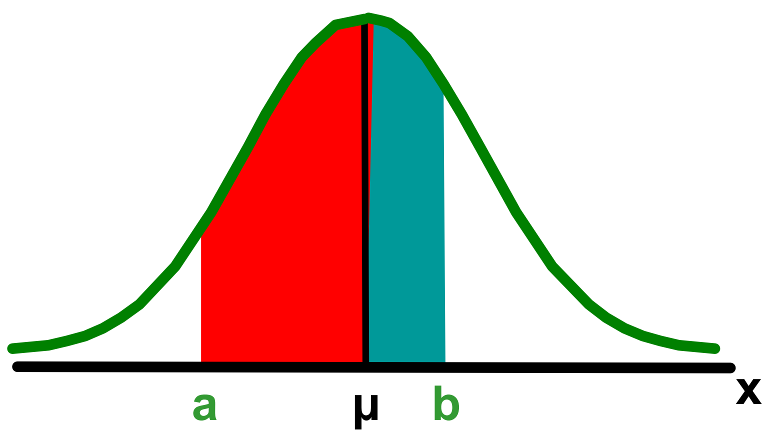
$$\boxed{F(x_0) = P(X \le x_0)}$$



Finding Normal Probabilities

The probability for a range of values is measured by the area under the curve

$$P(a < X < b) = F(b) - F(a)$$



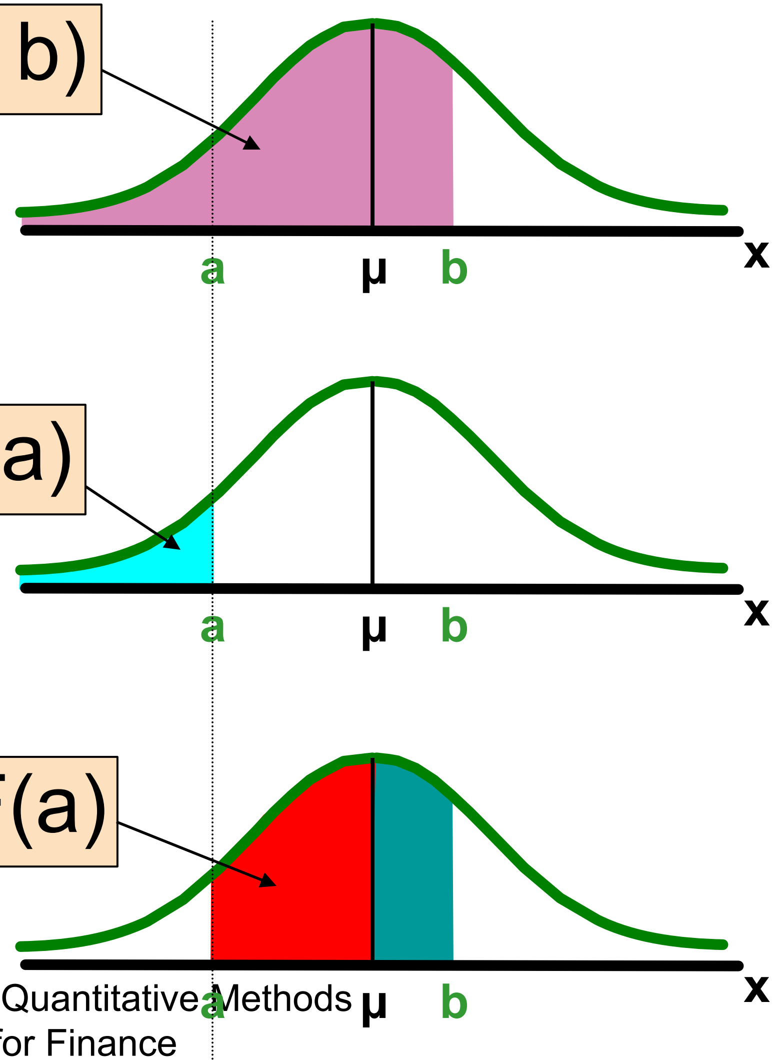
Finding Normal Probabilities

(continued)

$$F(b) = P(X < b)$$

$$F(a) = P(X < a)$$

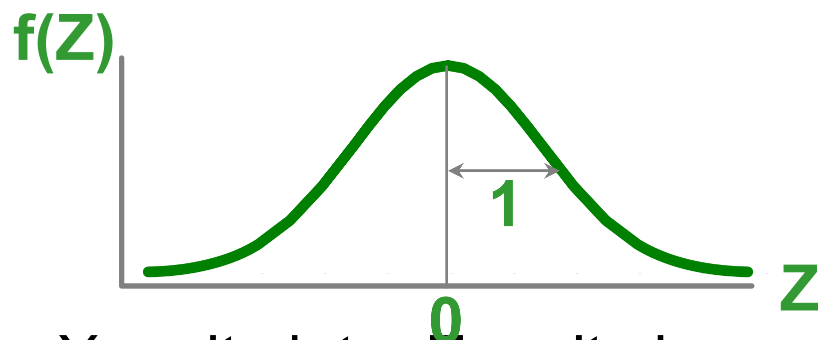
$$P(a < X < b) = F(b) - F(a)$$



The Standardized Normal

- Any normal distribution (with any mean and variance combination) can be transformed into the standardized normal distribution (Z), with mean 0 and variance 1

$$\boxed{Z \sim N(0,1)}$$



- Need to transform X units into Z units by subtracting the mean of X and dividing by its standard deviation

$$\boxed{Z = \frac{X - \mu}{\sigma}}$$

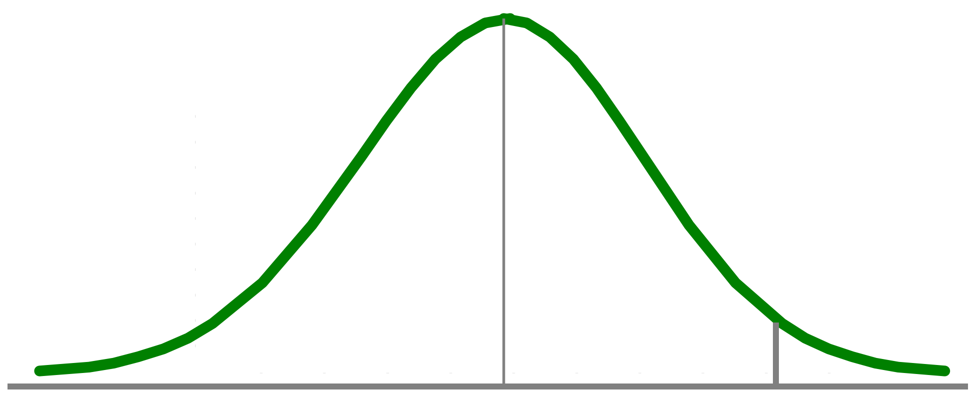
Example

- If X is distributed normally with mean of 100 and standard deviation of 50, the Z value for X = 200 is

$$Z = \frac{X - \mu}{\sigma} = \frac{200 - 100}{50} = 2.0$$

- This says that X = 200 is two standard deviations (2 increments of 50 units) above the mean of 100.

Comparing X and Z units



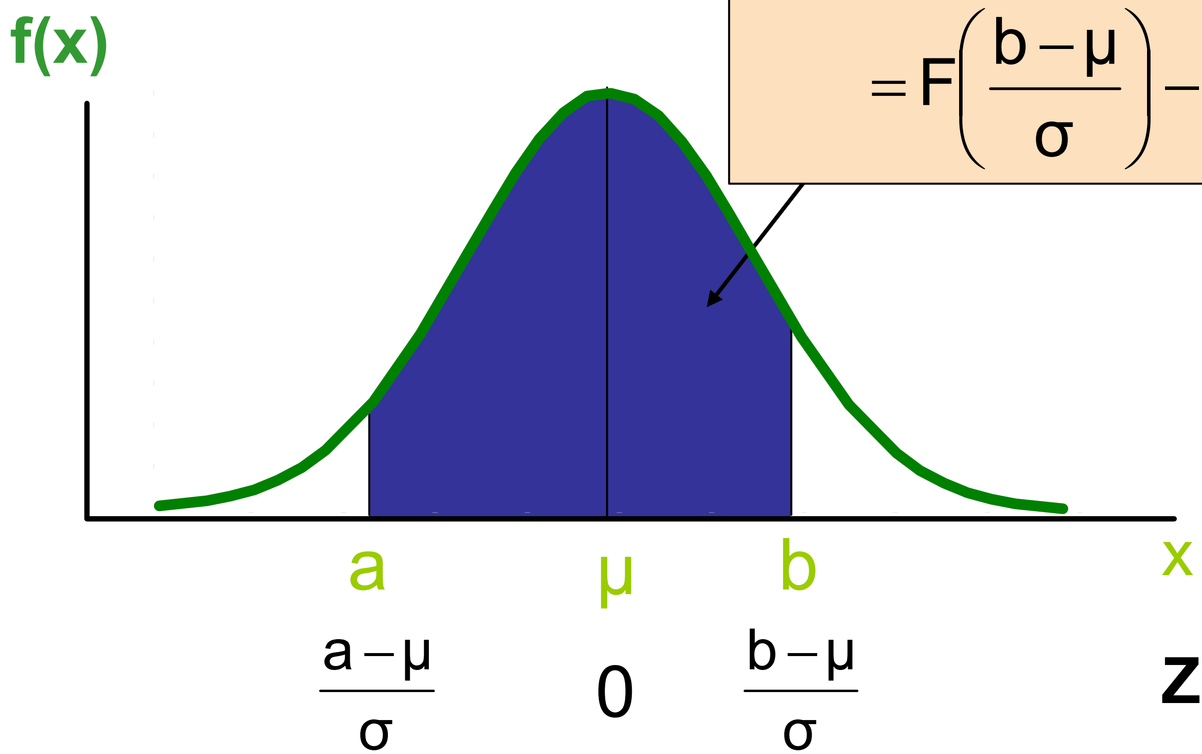
100	200	X	($\mu = 100$, $\sigma = 50$)
0	2.0	Z	($\mu = 0$, $\sigma = 1$)

Note that the distribution is the same, only the scale has changed. We can express the problem in original units (X) or in standardized units (Z)

Finding Normal Probabilities

$$P(a < X < b) = P\left(\frac{a-\mu}{\sigma} < Z < \frac{b-\mu}{\sigma}\right)$$

$$= F\left(\frac{b-\mu}{\sigma}\right) - F\left(\frac{a-\mu}{\sigma}\right)$$



Example

In the Czech Republic in 2002, the average region had an unemployment rate of 9.94% with a standard deviation of 4.15%. Assume that unemployment rates are Normally distributed. What fraction of regions would you expect to have an unemployment rate of 5 to 15%?

Here we want to know

$$P(5 < X < 15) = P\left(\frac{5 - 9.94}{4.15} < Z < \frac{15 - 9.94}{4.15}\right) = P(-1.19 < Z < 1.22) =$$
$$= F(1.22) - F(-1.19)$$

Appendix D

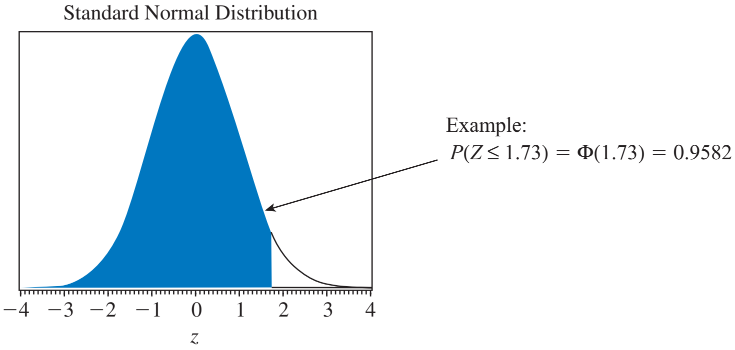


Table 1 **Cumulative Probabilities for the Standard Normal Distribution**
$\Phi(z) = P(Z \leq z)$

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990

Source: This table was generated using the SAS® function PROBNORM.

Example

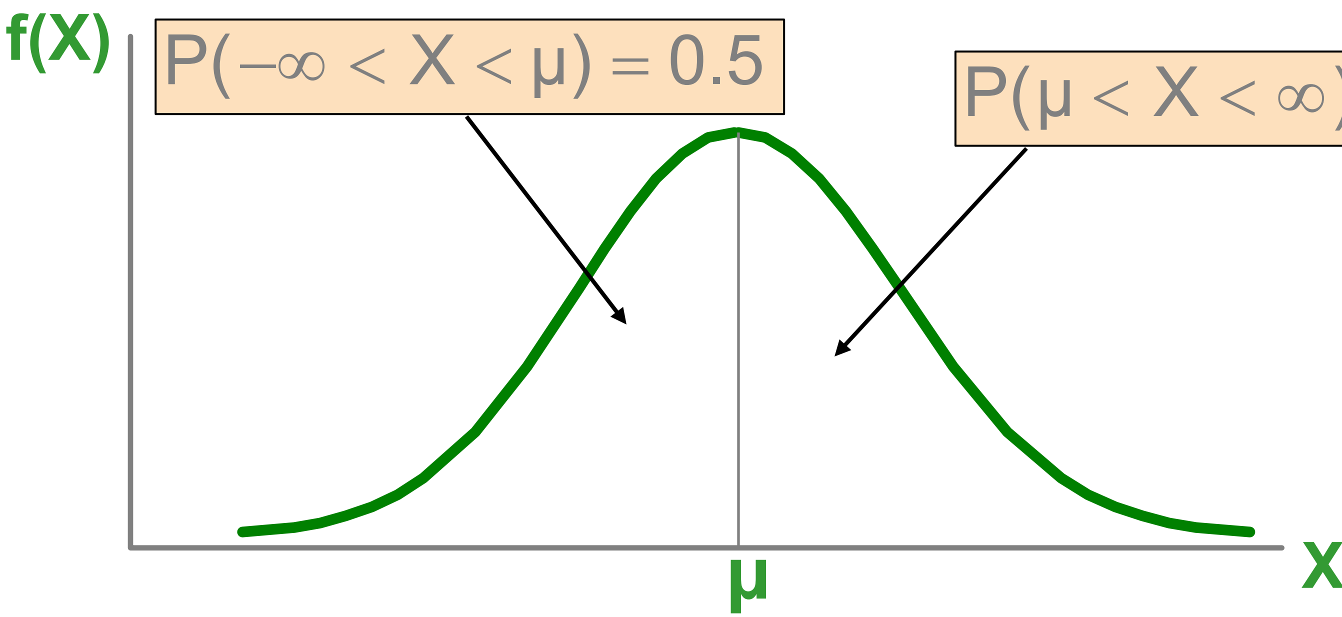
From the table we have that $F(1.22) = 0.888$

and $F(-1.19) = 1 – F(1.19) = 1 – 0.8830 = 0.117$.

Thus, $P(5 < X < 15) = 0.888 – 0.117 = 0.771$

Probability as Area Under the Curve

The total area under the curve is 1.0, and the curve is symmetric, so half is above the mean, half is below



$$P(-\infty < X < \infty) = 1.0$$

The Exponential Distribution

- Used to model the length of time between two occurrences of an event (the time between arrivals)
  - Examples:
    - Time between trucks arriving at an unloading dock
    - Time between transactions at an ATM Machine
    - Time between phone calls to the main operator

The Exponential Distribution

(continued)

- The exponential random variable T (t>0) has a probability density function

$$\boxed{f(t) = \lambda e^{-\lambda t} \quad \text{for } t > 0}$$

- Where
  - λ is the mean number of occurrences per unit time
  - t is the number of time units until the next occurrence
  - e = 2.71828
- T is said to follow an exponential probability distribution

The Exponential Distribution

- Defined by a single parameter, its mean λ (lambda)
- The cumulative distribution function (the probability that an arrival time is less than some specified time t) is

$$F(t) = 1 - e^{-\lambda t}$$

where e = mathematical constant approximated by 2.71828

λ = the population mean number of arrivals per unit

t = any value of the continuous variable where $t > 0$

Exponential Distribution Example

Example: Customers arrive at the service counter at the rate of 15 per hour. What is the probability that the arrival time between consecutive customers is less than three minutes?

- The mean number of arrivals per hour is 15, so $\lambda = 15$
- Three minutes is .05 hours
- P(arrival time < .05) $= 1 – e^{-\lambda X} = 1 – e^{-(15)(.05)} = \boxed{0.5276}$
- So there is a 52.76% probability that the arrival time between successive customers is less than three minutes