Induction Course in Quantitative Methods for Finance

Estimation: Single Population

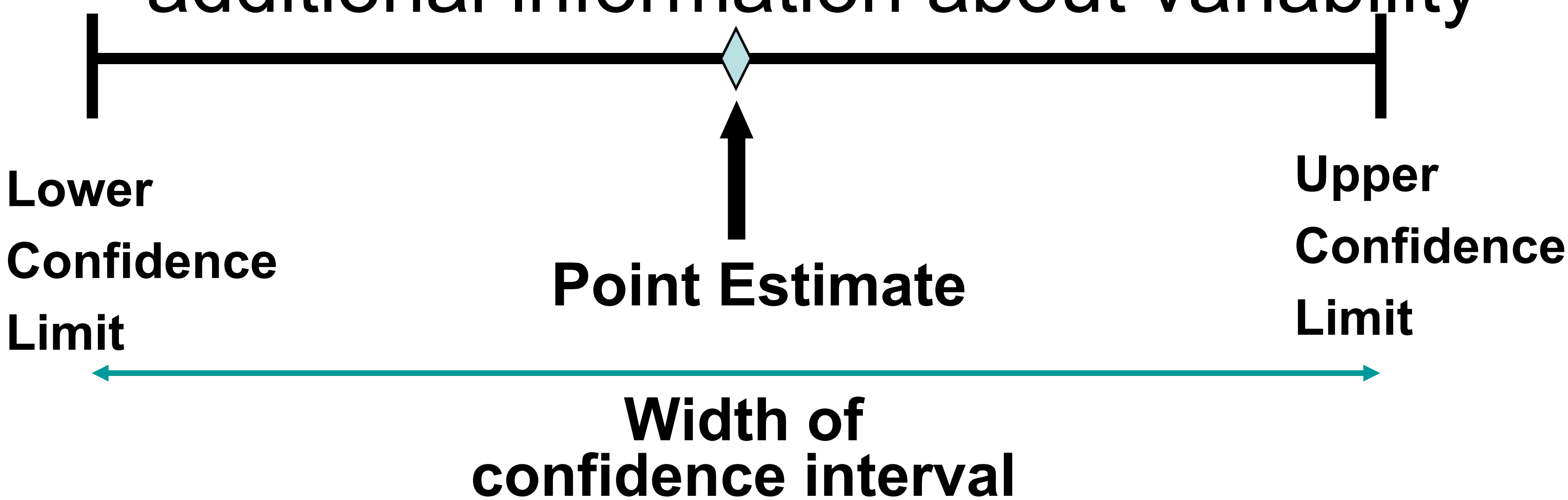
Definitions

- An estimator of a population parameter is
  - a random variable that depends on sample information . . .
  - whose value provides an approximation to this unknown parameter

- A specific value of that random variable is called an estimate

Point and Interval Estimates

- A point estimate is a single number,
- a confidence interval provides additional information about variability



Point Estimates

We can estimate a Population Parameter …		with a Sample Statistic (a Point Estimate)
Mean	μ	$\bar{x}$
Proportion	P	$\hat{p}$

Unbiasedness

- A point estimator $\hat{\theta}$ is said to be an unbiased estimator of the parameter θ if the expected value, or mean, of the sampling distribution of $\hat{\theta}$ is θ,

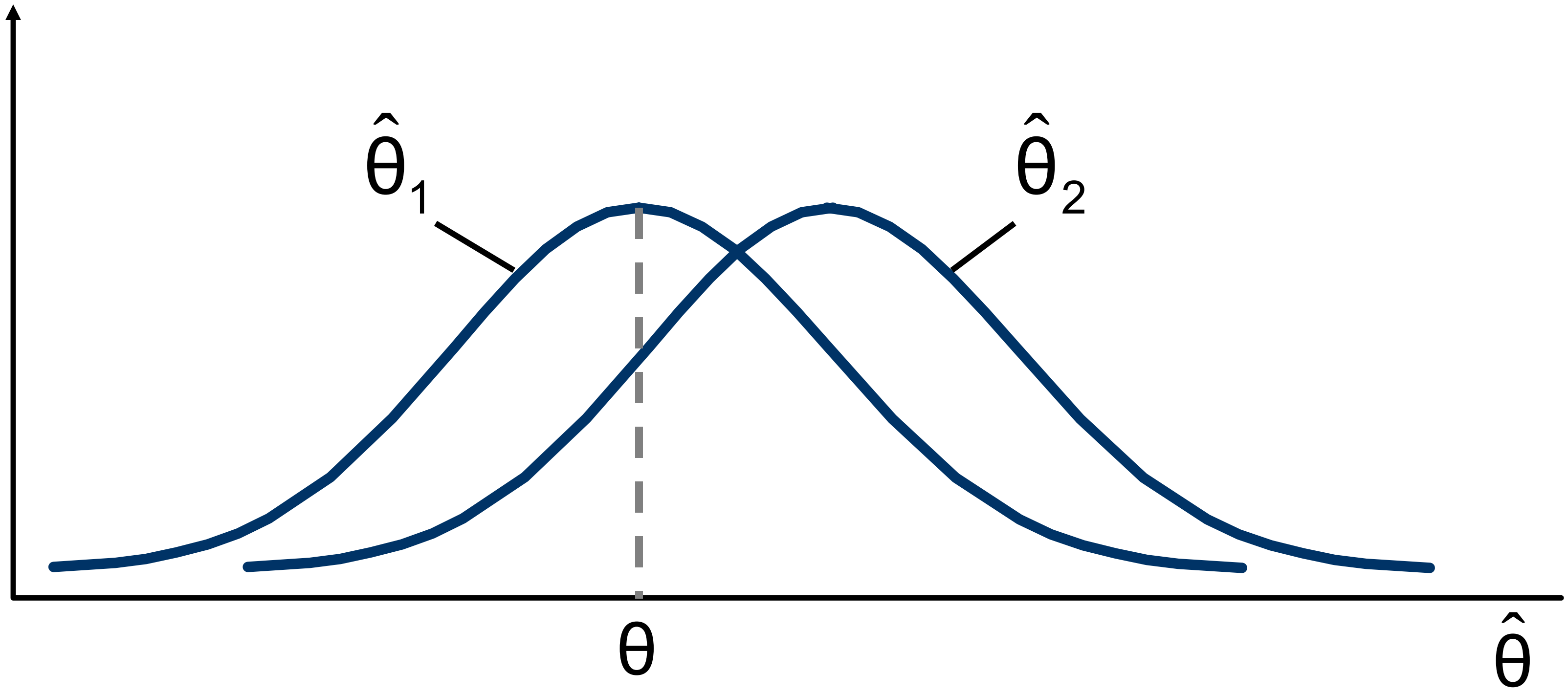
$$E(\hat{\theta}) = \theta$$

- Examples:
  - The sample mean is an unbiased estimator of μ
  - The sample variance is an unbiased estimator of σ^2
  - The sample proportion is an unbiased estimator of P

Unbiasedness

(continued)

- $\hat{\theta}_1$ is an unbiased estimator, $\hat{\theta}_2$ is biased:



Bias

- Let $\hat{\theta}$ be an estimator of θ
- The bias in $\hat{\theta}$ is defined as the difference between its mean and θ

$$\text{Bias}(\hat{\theta}) = E(\hat{\theta}) - \theta$$

- The bias of an unbiased estimator is 0

Consistency

- Let $\hat{\theta}$ be an estimator of θ
- $\hat{\theta}$ is a consistent estimator of θ if the difference between the expected value of and θ decreases as the sample size increases

- Consistency is desired when unbiased estimators cannot be obtained

Most Efficient Estimator

- Suppose there are several unbiased estimators of θ
- The most efficient estimator or the minimum variance unbiased estimator of θ is the unbiased estimator with the smallest variance

- Let $\hat{\theta}_1$ and $\hat{\theta}_2$ be two unbiased estimators of θ, based on the same number of sample observations. Then,
  - $\hat{\theta}_1$ is said to be more efficient than $\hat{\theta}_2$ if $\text{Var}(\hat{\theta}_1) < \text{Var}(\hat{\theta}_2)$
  - The relative efficiency of $\hat{\theta}_1$ with respect to $\hat{\theta}_2$ is the ratio of their variances:

$$\text{Relative Efficiency} = \frac{\text{Var}(\hat{\theta}_2)}{\text{Var}(\hat{\theta}_1)}$$

Confidence Intervals

- How much uncertainty is associated with a point estimate of a population parameter?
- An interval estimate provides more information about a population characteristic than does a point estimate
- Such interval estimates are called confidence intervals

Confidence Interval Estimate

- An interval gives a range of values:
  - Takes into consideration variation in sample statistics from sample to sample
  - Based on observation from 1 sample
  - Gives information about closeness to unknown population parameters
  - Stated in terms of level of confidence
    - Can never be 100% confident

Confidence Interval and Confidence Level

- If $P(a < \theta < b) = 1 - \alpha$ then the interval from a to b is called a $100(1 - \alpha)\%$ confidence interval of θ.

- The quantity $(1-\alpha)$ is called the confidence level of the interval (α between 0 and 1)

  – In repeated samples of the population, the true value of the parameter θ would be contained in $100(1 - \alpha)\%$ of intervals calculated this way.
  – The confidence interval calculated in this manner is written as $a < \theta < b$ with $100(1 - \alpha)\%$ confidence

Confidence Level, $(1-\alpha)$

(continued)

- Suppose confidence level = 95%
- Also written $(1 - \alpha) = 0.95$
- A relative frequency interpretation:
  – From repeated samples, 95% of all the confidence intervals that can be constructed will contain the unknown true parameter
- A specific interval either will contain or will not contain the true parameter
  – No probability involved in a specific interval

General Formula

- The general formula for all confidence intervals is:

Point Estimate $\pm$ (Reliability Factor)(Standard Error)

- The value of the reliability factor depends on the desired level of confidence

Confidence Interval for μ (σ^2 Known)

- Assumptions
  - Population variance σ^2 is known
  - Population is normally distributed
  - If population is not normal, use large sample
- Confidence interval estimate:

$$\bar{x} - z_{\alpha/2} \frac{\sigma}{\sqrt{n}} < \mu < \bar{x} + z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$$

(where $z_{\alpha/2}$ is the normal distribution value for a probability of $\alpha/2$ in each tail)

Margin of Error

- The confidence interval,

$$\bar{x} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} < \mu < \bar{x} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$$

- Can also be written as $\bar{x} \pm ME$
  where ME is called the margin of error

$$ME = z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$$

- The interval width, w, is equal to twice the margin of error

Reducing the Margin of Error

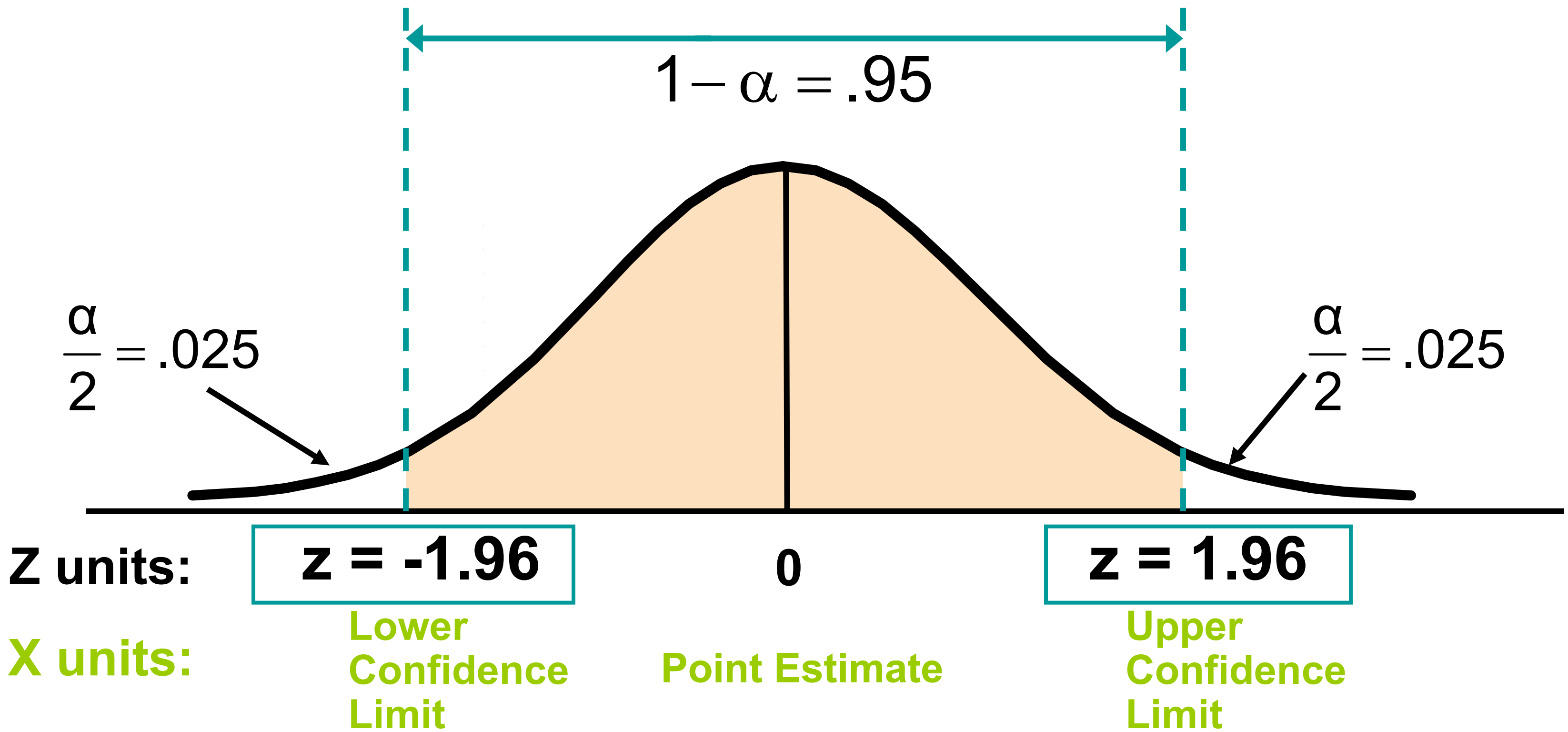
$$ME = z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$$

The margin of error can be reduced if

- the population standard deviation can be reduced ($\sigma\downarrow$)
- The sample size is increased ($n\uparrow$)
- The confidence level is decreased, $(1-\alpha)\downarrow$

Finding the Reliability Factor, $z_{\alpha/2}$

- Consider a 95% confidence interval:



- Find $z_{.025} = \pm 1.96$ from the standard normal distribution table

Common Levels of Confidence

- Commonly used confidence levels are 90%, 95%, and 99%

Confidence Level	***Confidence Coefficient,*** $1-\alpha$	$Z_{\alpha/2}$ ***value***
80%	**.80**	**1.28**
90%	**.90**	**1.645**
95%	**.95**	**1.96**
98%	**.98**	**2.33**
99%	**.99**	**2.58**
99.8%	**.998**	**3.08**
99.9%	**.999**	**3.27**

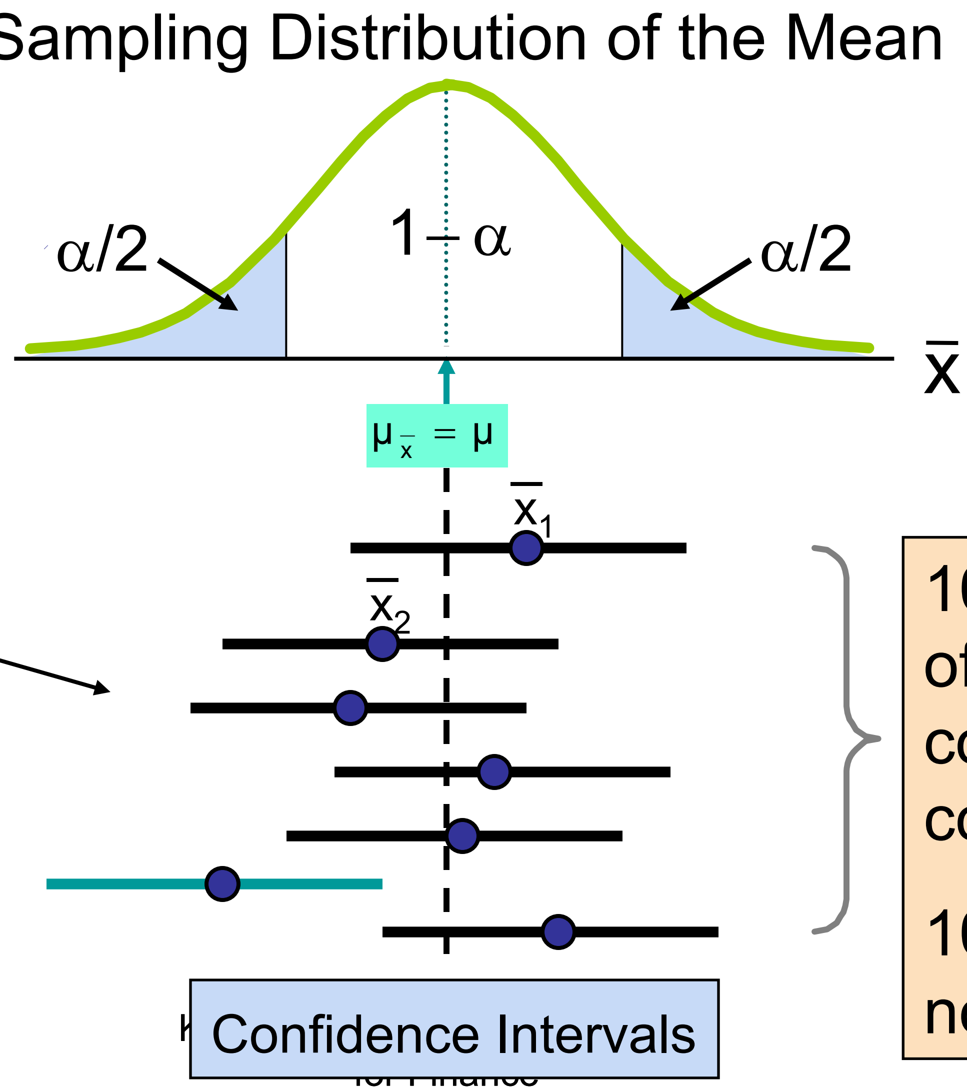
Intervals and Level of Confidence

Intervals extend from

$$\bar{x} - z\frac{\sigma}{\sqrt{n}}$$

to

$$\bar{x} + z\frac{\sigma}{\sqrt{n}}$$



100(1-α)% of intervals constructed contain μ;

100(α)% do not.

Example

- A sample of 11 firms from a large normal population has a mean monthly return of 2.20%. We know from past testing that the population standard deviation is 0.35%.
- Determine a 95% confidence interval for the true mean return of the population.

Example

(continued)

$$\bar{x} \pm z \frac{\sigma}{\sqrt{n}}$$

$$= 2.20 \pm 1.96\,(.35/\sqrt{11})$$

$$= 2.20 \pm .2068$$

$$1.9932 < \mu < 2.4068$$

Interpretation

- We are 95% confident that the true mean return is between 1.9932 and 2.4068 %
- Although the true mean may or may not be in this interval, 95% of intervals formed in this manner will contain the true mean

Student's t Distribution

- Consider a random sample of n observations
  - with mean $\bar{x}$ and standard deviation s
  - from a normally distributed population with mean μ
- Then the variable $$t = \frac{\bar{x} - \mu}{s/\sqrt{n}}$$
- follows the Student's t distribution with (n - 1) degrees of freedom

Confidence Interval for μ (σ^2 Unknown)

- If the population standard deviation σ is unknown, we can substitute the sample standard deviation, s
- This introduces extra uncertainty, since s is variable from sample to sample
- So we use the t distribution instead of the normal distribution

Confidence Interval for μ (σ Unknown)

(continued)

- Assumptions
  - Population standard deviation is unknown
  - Population is normally distributed
  - If population is not normal, use large sample
- Use Student's t Distribution
- Confidence Interval Estimate:

$$\bar{x} - t_{n-1,\alpha/2}\frac{S}{\sqrt{n}} < \mu < \bar{x} + t_{n-1,\alpha/2}\frac{S}{\sqrt{n}}$$

where $t_{n-1,\alpha/2}$ is the critical value of the t distribution with n-1 d.f. and an area of $\alpha/2$ in each tail:

$$P(t_{n-1} > t_{n-1,\alpha/2}) = \alpha/2$$

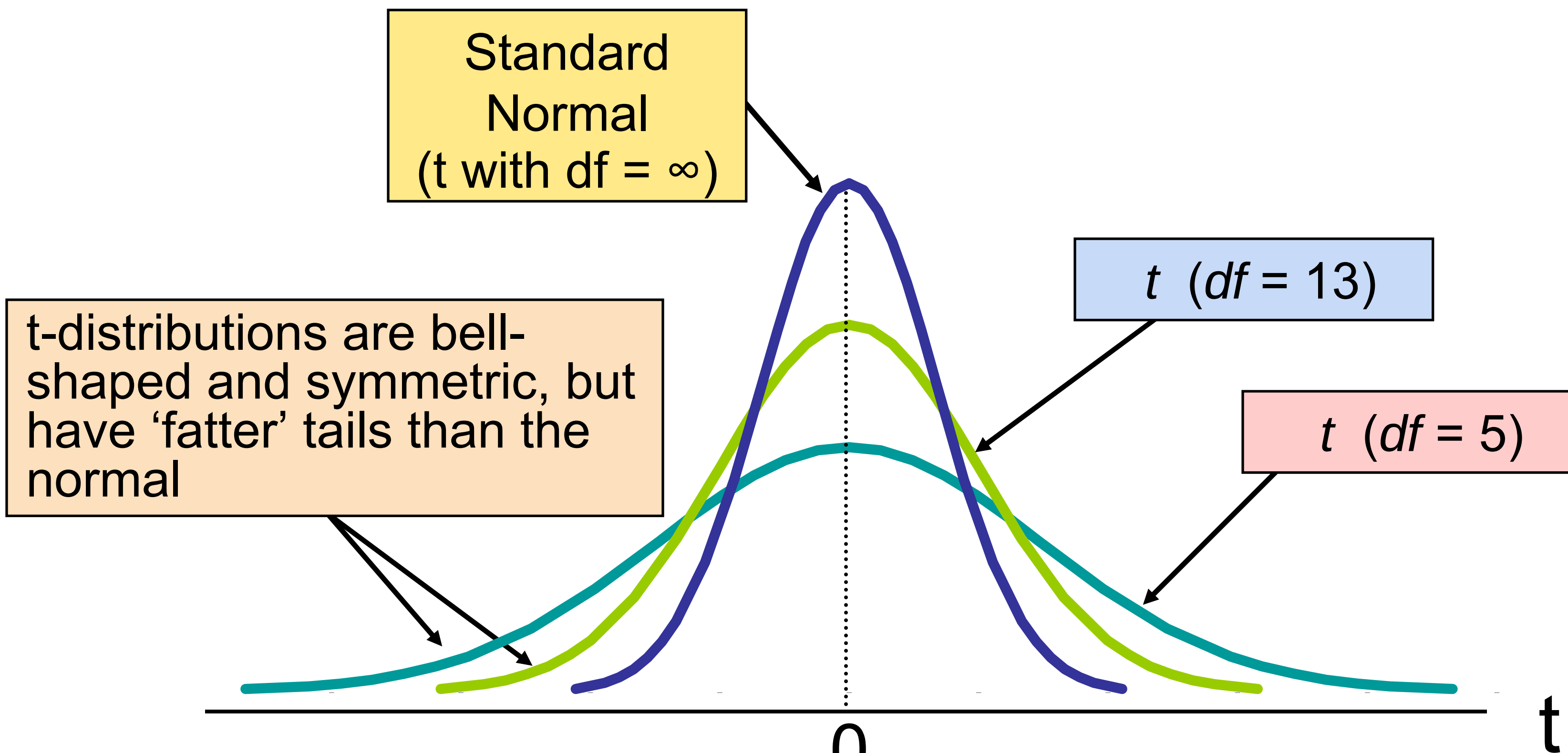
Student's t Distribution

- The t is a family of distributions
- The t value depends on degrees of freedom (d.f.)
  - Number of observations that are free to vary after sample mean has been calculated

$$d.f. = n - 1$$

Student's t Distribution

Note: t ⟶ Z as n increases

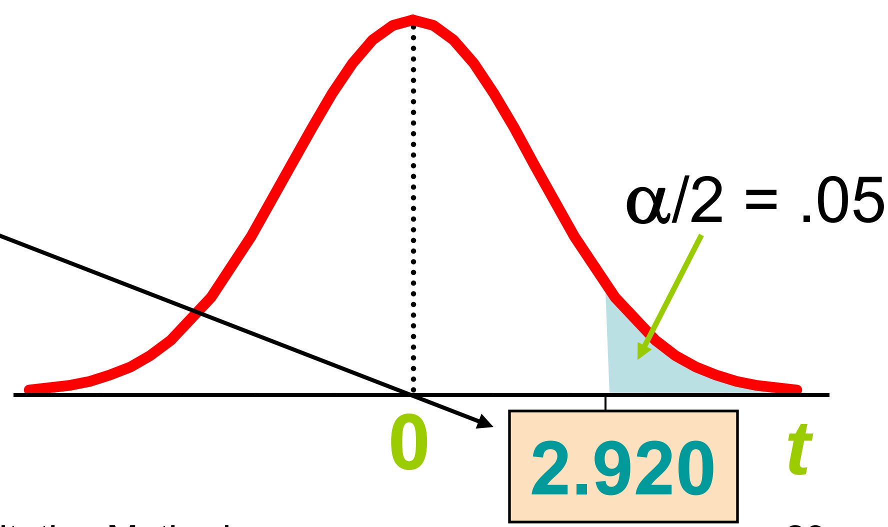


Student's t Table

	Upper Tail Area		
df	.10	**.05**	.025
1	3.078	6.314	12.706
2	1.886	**2.920**	4.303
3	1.638	2.353	3.182

The body of the table contains t values, not probabilities

Let: n = 3
df = n - 1 = 2
α = .10
$\alpha/2$ =.05



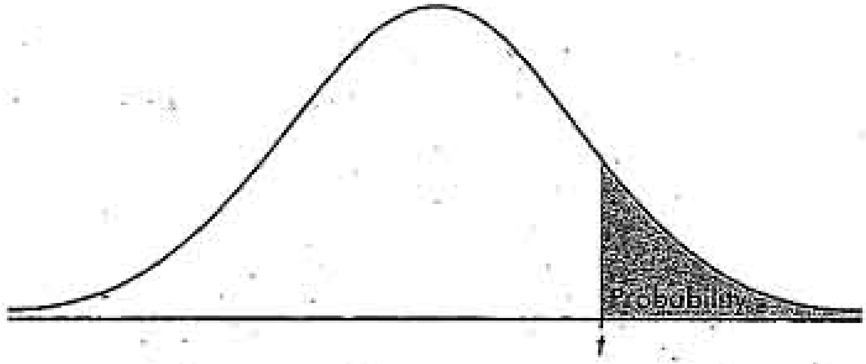


TABLE B: t-DISTRIBUTION CRITICAL VALUES

	Tail probability p											
df	.25	.20	.15	.10	.05	.025	.02	.01	.005	.0025	.001	.0005
1	1.000	1.376	1.963	3.078	6.314	12.71	15.89	31.82	63.66	127.3	318.3	636.6
2	.816	1.061	1.386	1.886	2.920	4.303	4.849	6.965	9.925	14.09	22.33	31.60
3	.765	.978	1.250	1.638	2.353	3.182	3.482	4.541	5.841	7.453	10.21	12.92
4	.741	.941	1.190	1.533	2.132	2.776	2.999	3.747	4.604	5.598	7.173	8.610
5	.727	.920	1.156	1.476	2.015	2.571	2.757	3.365	4.032	4.773	5.893	6.869
6	.718	.906	1.134	1.440	1.943	2.447	2.612	3.143	3.707	4.317	5.208	5.959
7	.711	.896	1.119	1.415	1.895	2.365	2.517	2.998	3.499	4.029	4.785	5.408
8	.706	.889	1.108	1.397	1.860	2.306	2.449	2.896	3.355	3.833	4.501	5.041
9	.703	.883	1.100	1.383	1.833	2.262	2.398	2.821	3.250	3.690	4.297	4.781
10	.700	.879	1.093	1.372	1.812	2.228	2.359	2.764	3.169	3.581	4.144	4.587
11	.697	.876	1.088	1.363	1.796	2.201	2.328	2.718	3.106	3.497	4.025	4.437
12	.695	.873	1.083	1.356	1.782	2.179	2.303	2.681	3.055	3.428	3.930	4.318
13	.694	.870	1.079	1.350	1.771	2.160	2.282	2.650	3.012	3.372	3.852	4.221
14	.692	.868	1.076	1.345	1.761	2.145	2.264	2.624	2.977	3.326	3.787	4.140
15	.691	.866	1.074	1.341	1.753	2.131	2.249	2.602	2.947	3.286	3.733	4.073
16	.690	.865	1.071	1.337	1.746	2.120	2.235	2.583	2.921	3.252	3.686	4.015
17	.689	.863	1.069	1.333	1.740	2.110	2.224	2.567	2.898	3.222	3.646	3.965
18	.688	.862	1.067	1.330	1.734	2.101	2.214	2.552	2.878	3.197	3.611	3.922
19	.688	.861	1.066	1.328	1.729	2.093	2.205	2.539	2.861	3.174	3.579	3.883
20	.687	.860	1.064	1.325	1.725	2.086	2.197	2.528	2.845	3.153	3.552	3.850
21	.686	.859	1.063	1.323	1.721	2.080	2.189	2.518	2.831	3.135	3.527	3.819
22	.686	.858	1.061	1.321	1.717	2.074	2.183	2.508	2.819	3.119	3.505	3.792
23	.685	.858	1.060	1.319	1.714	2.069	2.177	2.500	2.807	3.104	3.485	3.768
24	.685	.857	1.059	1.318	1.711	2.064	2.172	2.492	2.797	3.091	3.467	3.745
25	.684	.856	1.058	1.316	1.708	2.060	2.167	2.485	2.787	3.078	3.450	3.725
26	.684	.856	1.058	1.315	1.706	2.056	2.162	2.479	2.779	3.067	3.435	3.707
27	.684	.855	1.057	1.314	1.703	2.052	2.158	2.473	2.771	3.057	3.421	3.690
28	.683	.855	1.056	1.313	1.701	2.048	2.154	2.467	2.763	3.047	3.408	3.674
29	.683	.854	1.055	1.311	1.699	2.045	2.150	2.462	2.756	3.038	3.396	3.659
30	.683	.854	1.055	1.310	1.697	2.042	2.147	2.457	2.750	3.030	3.385	3.646
40	.681	.851	1.050	1.303	1.684	2.021	2.123	2.423	2.704	2.971	3.307	3.551
50	.679	.849	1.047	1.299	1.676	2.009	2.109	2.403	2.678	2.937	3.261	3.496
60	.679	.848	1.045	1.296	1.671	2.000	2.099	2.390	2.660	2.915	3.232	3.460
80	.678	.846	1.043	1.292	1.664	1.990	2.088	2.374	2.639	2.887	3.195	3.416
100	.677	.845	1.042	1.290	1.660	1.984	2.081	2.364	2.626	2.871	3.174	3.390
1000	.675	.842	1.037	1.282	1.646	1.962	2.056	2.330	2.581	2.813	3.098	3.300
∞	.674	.841	1.036	1.282	1.645	1.960	2.054	2.326	2.576	2.807	3.091	3.291
	50%	60%	70%	80%	90%	95%	96%	98%	99%	99.5%	99.8%	99.9%
	Confidence level C											

t distribution values

With comparison to the Z value

Confidence Level	t (10 d.f.)	t (20 d.f.)	t (30 d.f.)	Z
.80	1.372	1.325	1.310	1.282
.90	1.812	1.725	1.697	1.645
.95	2.228	2.086	2.042	1.960
.99	3.169	2.845	2.750	2.576

Note: $t \longrightarrow Z$ as n increases

Example

A random sample of n = 25 has $\bar{x}$ = 50 and s = 8. Form a 95% confidence interval for μ

– d.f. = n – 1 = 24, so $t_{n-1,\alpha/2} = t_{24,.025} = 2.0639$

The confidence interval is

$$\bar{x} - t_{n\text{-}1,\alpha/2}\frac{S}{\sqrt{n}} < \mu < \bar{x} + t_{n\text{-}1,\alpha/2}\frac{S}{\sqrt{n}}$$

$$50 - (2.0639)\frac{8}{\sqrt{25}} < \mu < 50 + (2.0639)\frac{8}{\sqrt{25}}$$

$$46.698 < \mu < 53.302$$

Confidence Intervals for the Population Proportion, p

- An interval estimate for the population proportion (P) can be calculated by adding an allowance for uncertainty to the sample proportion ($\hat{p}$)

Confidence Intervals for the Population Proportion, p

(continued)

- Recall that the distribution of the sample proportion is approximately normal if the sample size is large, with standard deviation

$$\sigma_P = \sqrt{\frac{P(1-P)}{n}}$$

- We will estimate this with sample data:

$$\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

Confidence Interval Endpoints

- Upper and lower confidence limits for the population proportion are calculated with the formula

$$\hat{p} - z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} < P < \hat{p} + z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

- where
  - $z_{\alpha/2}$ is the standard normal value for the level of confidence desired
  - $\hat{p}$ is the sample proportion
  - n is the sample size

Example

- A random sample of 100 firms shows that 25 did not pay dividend
- Form a 95% confidence interval for the true proportion of non-paying firms

Example

(continued)

$$\hat{p} - z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} < P < \hat{p} + z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

$$\frac{25}{100} - 1.96\sqrt{\frac{.25(.75)}{100}} < P < \frac{25}{100} + 1.96\sqrt{\frac{.25(.75)}{100}}$$

$$0.1651 < P < 0.3349$$

Interpretation

- We are 95% confident that the true percentage of non-paying firms in the population is between

  16.51% and 33.49%.

- Although the interval from 0.1651 to 0.3349 may or may not contain the true proportion, 95% of intervals formed from samples of size 100 in this manner will contain the true proportion.

Dependent Samples

Dependent samples

Tests Means of 2 Related Populations

- Paired or matched samples
- Repeated measures (before/after)
- Use difference between paired values:

$$d_i = x_i - y_i$$

- Assumptions:
  - Both Populations Are Normally Distributed

Mean Difference

Dependent samples

The i^{th} paired difference is d_i , where

$$d_i = x_i - y_i$$

The point estimate for the population mean paired difference is $\bar{d}$:

$$\bar{d} = \frac{\sum_{i=1}^{n} d_i}{n}$$

The sample standard deviation is:

$$S_d = \sqrt{\frac{\sum_{i=1}^{n} (d_i - \bar{d})^2}{n-1}}$$

n is the number of matched pairs in the sample

Confidence Interval for Mean Difference

Dependent samples

The confidence interval for difference between population means, μ_d , is

$$\bar{d} - t_{n-1,\alpha/2} \frac{S_d}{\sqrt{n}} < \mu_d < \bar{d} + t_{n-1,\alpha/2} \frac{S_d}{\sqrt{n}}$$

Where
n = the sample size
(number of matched pairs in the paired sample)

Confidence Interval for Mean Difference

(continued)

Dependent samples

- The margin of error is

$$ME = t_{n-1,\alpha/2} \frac{s_d}{\sqrt{n}}$$

- $t_{n-1,\alpha/2}$ is the value from the Student's t distribution with (n – 1) degrees of freedom for which

$$P(t_{n-1} > t_{n-1,\alpha/2}) = \frac{\alpha}{2}$$

Paired Samples Example

- Six people sign up for a weight loss program. You collect the following data:

Person	Weight: Before (x)	After (y)	Difference, d_i
1	136	125	11
2	205	195	10
3	157	150	7
4	138	140	- 2
5	175	165	10
6	166	160	6
			42

$$\bar{d} = \frac{\sum d_i}{n}$$

$$= 7.0$$

$$S_d = \sqrt{\frac{\sum (d_i - \bar{d})^2}{n-1}}$$

$$= 4.82$$

Paired Samples Example

(continued)

- For a 95% confidence level, the appropriate t value is $t_{n-1,\alpha/2} = t_{5,.025} = 2.571$
- The 95% confidence interval for the difference between means, μ_d, is

$$\bar{d} - t_{n-1,\alpha/2}\frac{S_d}{\sqrt{n}} < \mu_d < \bar{d} + t_{n-1,\alpha/2}\frac{S_d}{\sqrt{n}}$$

$$7 - (2.571)\frac{4.82}{\sqrt{6}} < \mu_d < 7 + (2.571)\frac{4.82}{\sqrt{6}}$$

$$-1.94 < \mu_d < 12.06$$

Since this interval contains zero, we cannot be 95% confident, given this limited data, that the weight loss program helps people lose weight

Difference Between Two Means

Population means, independent samples

Goal: Form a confidence interval for the difference between two population means, $\mu_x - \mu_y$

- Different data sources
  - Unrelated
  - Independent
    - Sample selected from one population has no effect on the sample selected from the other population
- The point estimate is the difference between the two sample means:

$$\bar{x} - \bar{y}$$

Difference Between Two Means

(continued)

- Population means, independent samples
  - σ_x^2 and σ_y^2 known → Confidence interval uses $z_{\alpha/2}$
  - σ_x^2 and σ_y^2 unknown
    - σ_x^2 and σ_y^2 assumed equal
    - σ_x^2 and σ_y^2 assumed unequal

    → Confidence interval uses a value from the Student's **t** distribution

σ_x^2 and σ_y^2 Known

Population means, independent samples

- σ_x^2 and σ_y^2 known *
- σ_x^2 and σ_y^2 unknown

Assumptions:

- Samples are randomly and independently drawn
- both population distributions are normal
- Population variances are known

σ_x^2 and σ_y^2 Known

(continued)

Population means, independent samples

- σ_x^2 and σ_y^2 known *
- σ_x^2 and σ_y^2 unknown

When σ_x and σ_y are known and both populations are normal, the variance of $\bar{X} - \bar{Y}$ is

$$\sigma^2_{\bar{X}-\bar{Y}} = \frac{\sigma_x^2}{n_x} + \frac{\sigma_y^2}{n_y}$$

…and the random variable

$$Z = \frac{(\bar{x} - \bar{y}) - (\mu_X - \mu_Y)}{\sqrt{\frac{\sigma_x^2}{n_X} + \frac{\sigma_y^2}{n_Y}}}$$

has a standard normal distribution

Confidence Interval, σ_x^2 and σ_y^2 Known

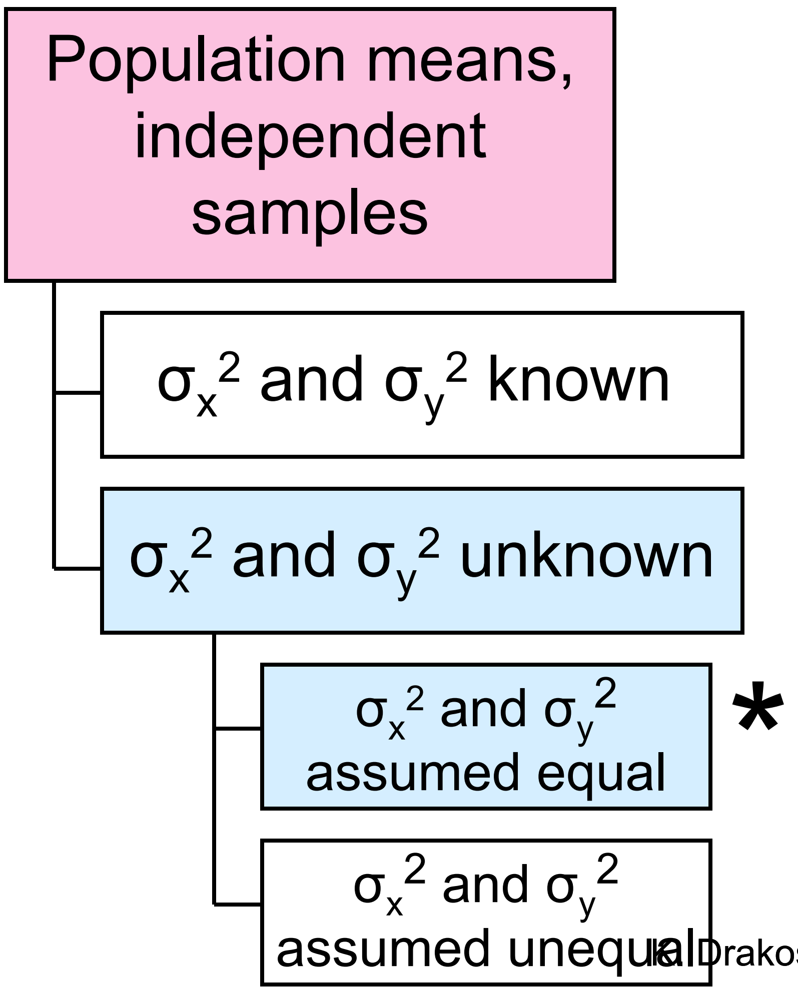
Population means, independent samples

- σ_x^2 and σ_y^2 known
- σ_x^2 and σ_y^2 unknown

* The confidence interval for $\mu_x - \mu_y$ is:

$$(\bar{x}-\bar{y}) - z_{\alpha/2}\sqrt{\frac{\sigma_X^2}{n_x}+\frac{\sigma_Y^2}{n_y}} < \mu_X - \mu_Y < (\bar{x}-\bar{y}) + z_{\alpha/2}\sqrt{\frac{\sigma_X^2}{n_x}+\frac{\sigma_Y^2}{n_y}}$$

σ_x^2 and σ_y^2 Unknown, Assumed Equal



Assumptions:

- Samples are randomly and independently drawn
- Populations are normally distributed
- Population variances are unknown but assumed equal

$\sigma_x 2$ and σ_y^2 Unknown, Assumed Equal

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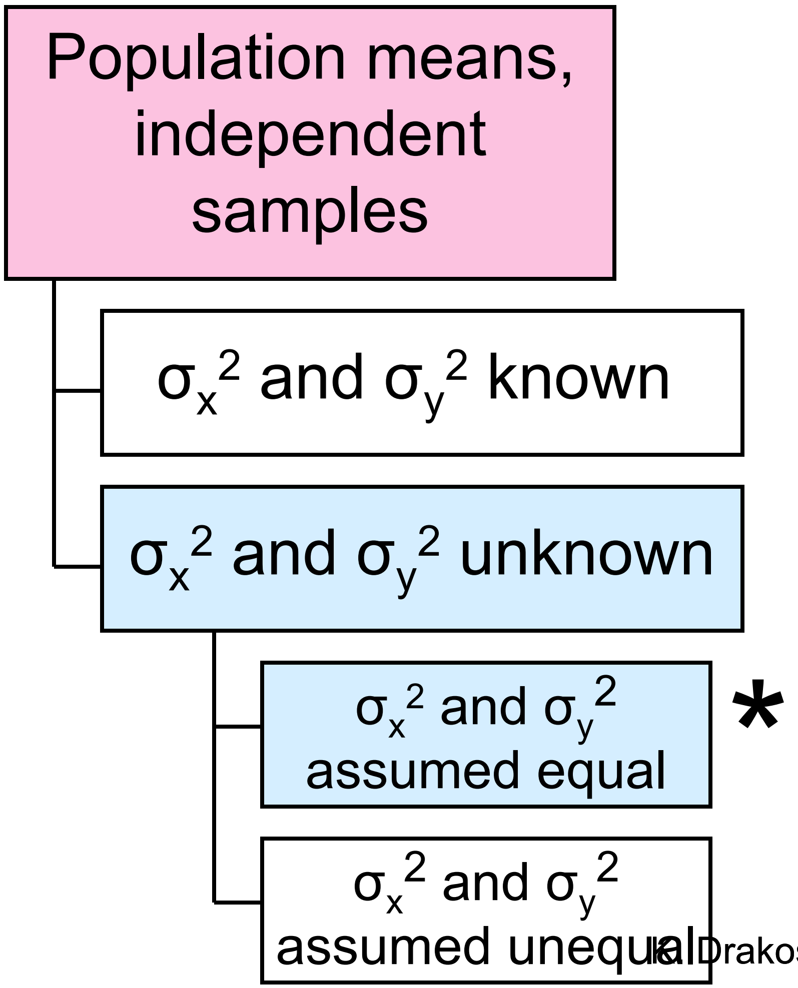


Forming interval estimates:

- The population variances are assumed equal, so use the two sample standard deviations and pool them to estimate σ
- use a t value with $(n_x + n_y - 2)$ degrees of freedom

σ_x^2 and σ_y^2 Unknown, Assumed Equal

(continued)

Population means, independent samples

- σ_x^2 and σ_y^2 known
- σ_x^2 and σ_y^2 unknown
  - σ_x^2 and σ_y^2 assumed equal *
  - σ_x^2 and σ_y^2 assumed unequal

The pooled variance is

$$s_p^2 = \frac{(n_x - 1)s_x^2 + (n_y - 1)s_y^2}{n_x + n_y - 2}$$

Confidence Interval, σ_x^2 and σ_y^2 Unknown, Equal

σ_x^2 and σ_y^2 unknown

- σ_x^2 and σ_y^2 assumed equal *
- σ_x^2 and σ_y^2 assumed unequal

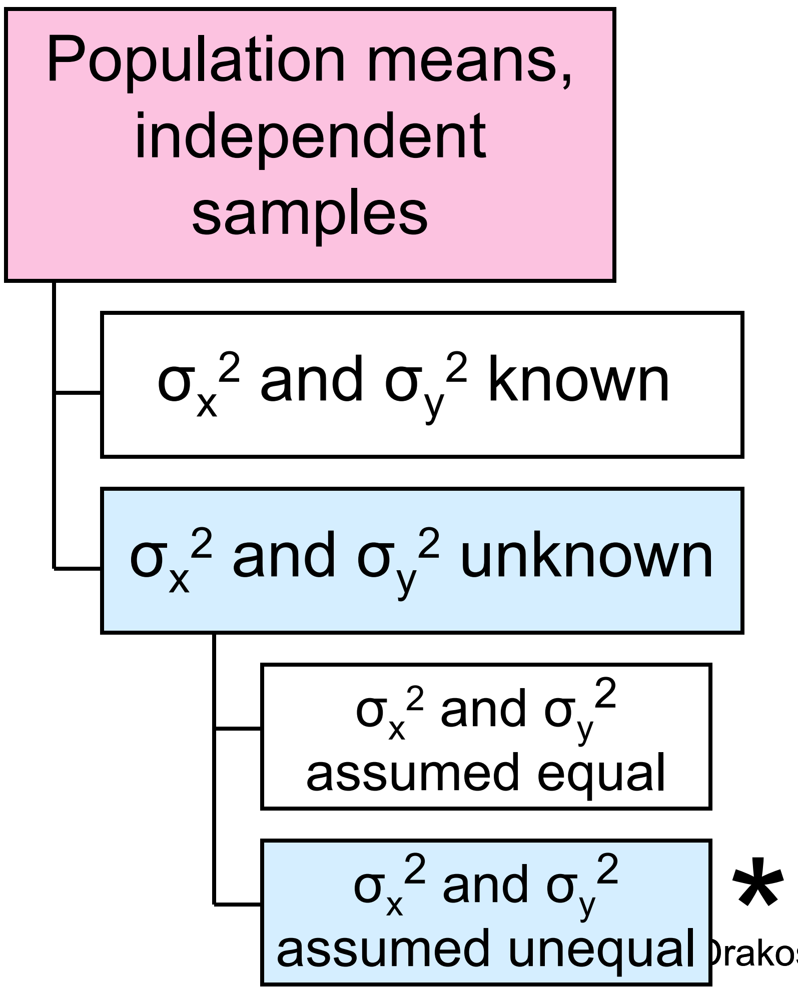
The confidence interval for $\mu_1 - \mu_2$ is:

$$(\bar{x}-\bar{y}) - t_{n_x+n_y-2,\alpha/2}\sqrt{\frac{s_p^2}{n_x}+\frac{s_p^2}{n_y}} < \mu_X - \mu_Y < (\bar{x}-\bar{y}) + t_{n_x+n_y-2,\alpha/2}\sqrt{\frac{s_p^2}{n_x}+\frac{s_p^2}{n_y}}$$

Where

$$s_p^2 = \frac{(n_x-1)s_x^2 + (n_y-1)s_y^2}{n_x+n_y-2}$$

σ_x^2 and σ_y^2 Unknown, Assumed Unequal

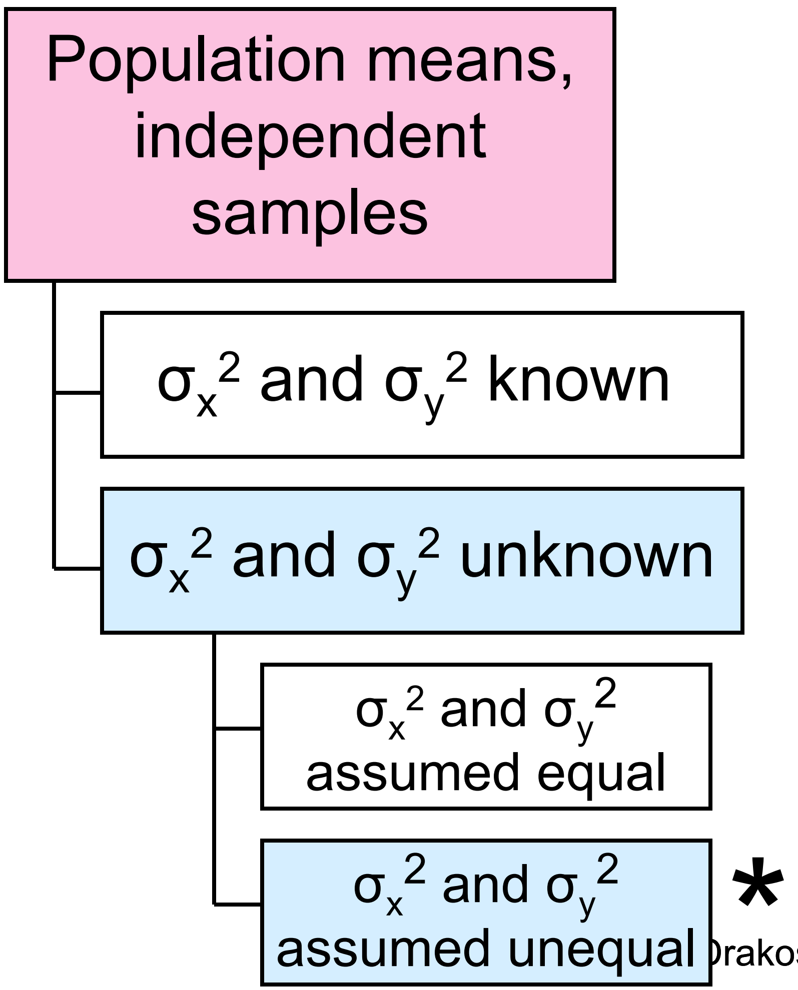


Assumptions:

- Samples are randomly and independently drawn
- Populations are normally distributed
- Population variances are unknown and assumed unequal

σ_x^2 and σ_y^2 Unknown, Assumed Unequal

(continued)



Forming interval estimates:

- The population variances are assumed unequal, so a pooled variance is not appropriate
- use a t value with ν degrees of freedom, where

$$\nu = \frac{\left[\left(\frac{s_x^2}{n_x}\right)+\left(\frac{s_y^2}{n_y}\right)\right]^2}{\left(\frac{s_x^2}{n_x}\right)^2/(n_x-1)+\left(\frac{s_y^2}{n_y}\right)^2/(n_y-1)}$$

Confidence Interval, σ_x^2 and σ_y^2 Unknown, Unequal

σ_x^2 and σ_y^2 unknown

- σ_x^2 and σ_y^2 assumed equal
- σ_x^2 and σ_y^2 assumed unequal *

The confidence interval for $\mu_1 - \mu_2$ is:

$$(\bar{x}-\bar{y}) - t_{\nu,\alpha/2}\sqrt{\frac{s_x^2}{n_x}+\frac{s_y^2}{n_y}} < \mu_X - \mu_Y < (\bar{x}-\bar{y}) + t_{\nu,\alpha/2}\sqrt{\frac{s_x^2}{n_x}+\frac{s_y^2}{n_y}}$$

Where

$$\nu = \frac{\left[\left(\frac{s_x^2}{n_x}\right)+\left(\frac{s_y^2}{n_y}\right)\right]^2}{\left(\frac{s_x^2}{n_x}\right)^2/(n_x-1)+\left(\frac{s_y^2}{n_y}\right)^2/(n_y-1)}$$

Two Population Proportions

Population proportions

Goal: Form a confidence interval for the difference between two population proportions, $P_x - P_y$

Assumptions:

Both sample sizes are large (generally at least 40 observations in each sample)

The point estimate for the difference is $\hat{p}_x - \hat{p}_y$

Two Population Proportions

(continued)

Population proportions

- The random variable

$$Z = \frac{(\hat{p}_x - \hat{p}_y) - (p_x - p_y)}{\sqrt{\frac{\hat{p}_x(1-\hat{p}_x)}{n_x} + \frac{\hat{p}_y(1-\hat{p}_y)}{n_y}}}$$

is approximately normally distributed

Confidence Interval for Two Population Proportions

Population proportions

The confidence limits for $P_x - P_y$ are:

$$(\hat{p}_x - \hat{p}_y) \pm Z_{\alpha/2}\sqrt{\frac{\hat{p}_x(1-\hat{p}_x)}{n_x} + \frac{\hat{p}_y(1-\hat{p}_y)}{n_y}}$$

Example: Two Population Proportions

Form a 90% confidence interval for the difference between the proportion of retail firms and the proportion of industrial firms who went bankrupt last year.

- In a random sample, 26 of 50 retail and 28 of 40 industrial firms had gone bankrupt

Example: Two Population Proportions

(continued)

Retail:

$$\hat{p}_x = \frac{26}{50} = 0.52$$

Industrial:

$$\hat{p}_y = \frac{28}{40} = 0.70$$

$$\sqrt{\frac{\hat{p}_x(1-\hat{p}_x)}{n_x} + \frac{\hat{p}_y(1-\hat{p}_y)}{n_y}} = \sqrt{\frac{0.52(0.48)}{50} + \frac{0.70(0.30)}{40}} = 0.1012$$

For 90% confidence, $Z_{\alpha/2} = 1.645$

Example: Two Population Proportions

(continued)

The confidence limits are:

$$(\hat{p}_x - \hat{p}_y) \pm Z_{\alpha/2} \sqrt{\frac{\hat{p}_x(1-\hat{p}_x)}{n_x} + \frac{\hat{p}_y(1-\hat{p}_y)}{n_y}}$$

$$= (.52 - .70) \pm 1.645(0.1012)$$

so the confidence interval is

$$-0.3465 < P_x - P_y < -0.0135$$

Since this interval does not contain zero we are 90% confident that the two proportions are not equal

Confidence Intervals for the Population Variance

Population Variance

- Goal: Form a confidence interval for the population variance, σ^2

- The confidence interval is based on the sample variance, s^2

- Assumed: the population is normally distributed

Confidence Intervals for the Population Variance

(continued)

Population Variance

The random variable

$$\chi^2_{n-1} = \frac{(n-1)s^2}{\sigma^2}$$

follows a chi-square distribution with (n – 1) degrees of freedom

The chi-square value $\chi^2_{n-1,\alpha}$ denotes the number for which

$$P(\chi^2_{n-1} > \chi^2_{n-1,\alpha}) = \alpha$$

Confidence Intervals for the Population Variance

(continued)

Population Variance

The (1 - α)% confidence interval for the population variance is

$$\frac{(n-1)s^2}{\chi^2_{n-1,\,\alpha/2}} < \sigma^2 < \frac{(n-1)s^2}{\chi^2_{n-1,\,1-\alpha/2}}$$

Hypothesis Testing

What is a Hypothesis?

- A hypothesis is a claim (assumption) about a population parameter:
- population mean / population proportion

Example: The mean monthly cell phone bill of this city is $\mu = \$42$

Example: The proportion of adults in this city with cell phones is $p = .68$

The Null Hypothesis, H_0

- States the assumption (numerical) to be tested

Example: The average number of TV sets in U.S. Homes is equal to three ($H_0: \mu = 3$)

- Is always about a population parameter, not about a sample statistic

$$H_0: \mu = 3$$

$$H_0: \bar{X} = 3$$

The Null Hypothesis, H_0

(continued)

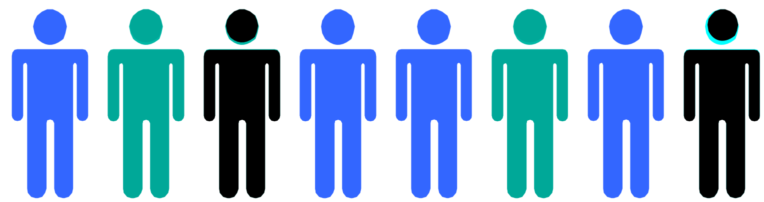
- Begin with the assumption that the null hypothesis is true
  - Similar to the notion of innocent until proven guilty
- Refers to the status quo
- Always contains "=" , "≤" or "≥" sign
- May or may not be rejected

The Alternative Hypothesis, H_1

- Is the opposite of the null hypothesis
  - e.g., The average number of TV sets in U.S. homes is not equal to 3 ($H_1: \mu \neq 3$)
- Challenges the status quo
- May or may not be supported
- Is generally the hypothesis that the researcher is trying to support

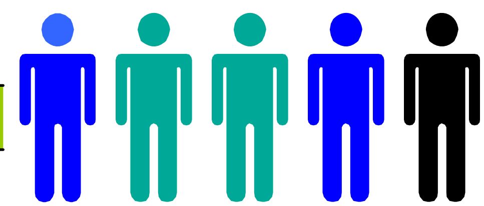
Hypothesis Testing Process

Claim: the population mean age is 50. (Null Hypothesis: $H_0: \mu = 50$)



Population

Now select a random sample



Sample

Suppose the sample mean age is 20: $\bar{X} = 20$

Is $\bar{X} = 20$ likely if $\mu = 50$?

If not likely,

REJECT Null Hypothesis

Hypothesis Tests Design

- Is $\overline{X}$ likely given that μ = 50? If we believe that this not likely we will reject H_0.
- How can we determine if the event is likely to occur given that H_0 is true?
- We define a *rejection region* of the sampling distribution, $\overline{X} < c$.

$$P\left(\overline{X} < c \,\middle|\, \mu = 50\right) = P\left(\text{Re ject } H_0 \,\middle|\, H_0 \text{ true}\right) = \alpha$$

- So we want small values of α (*significance level*). According to α if we know the distribution of X we can determine c (*critical value*)

Hypothesis Tests Design

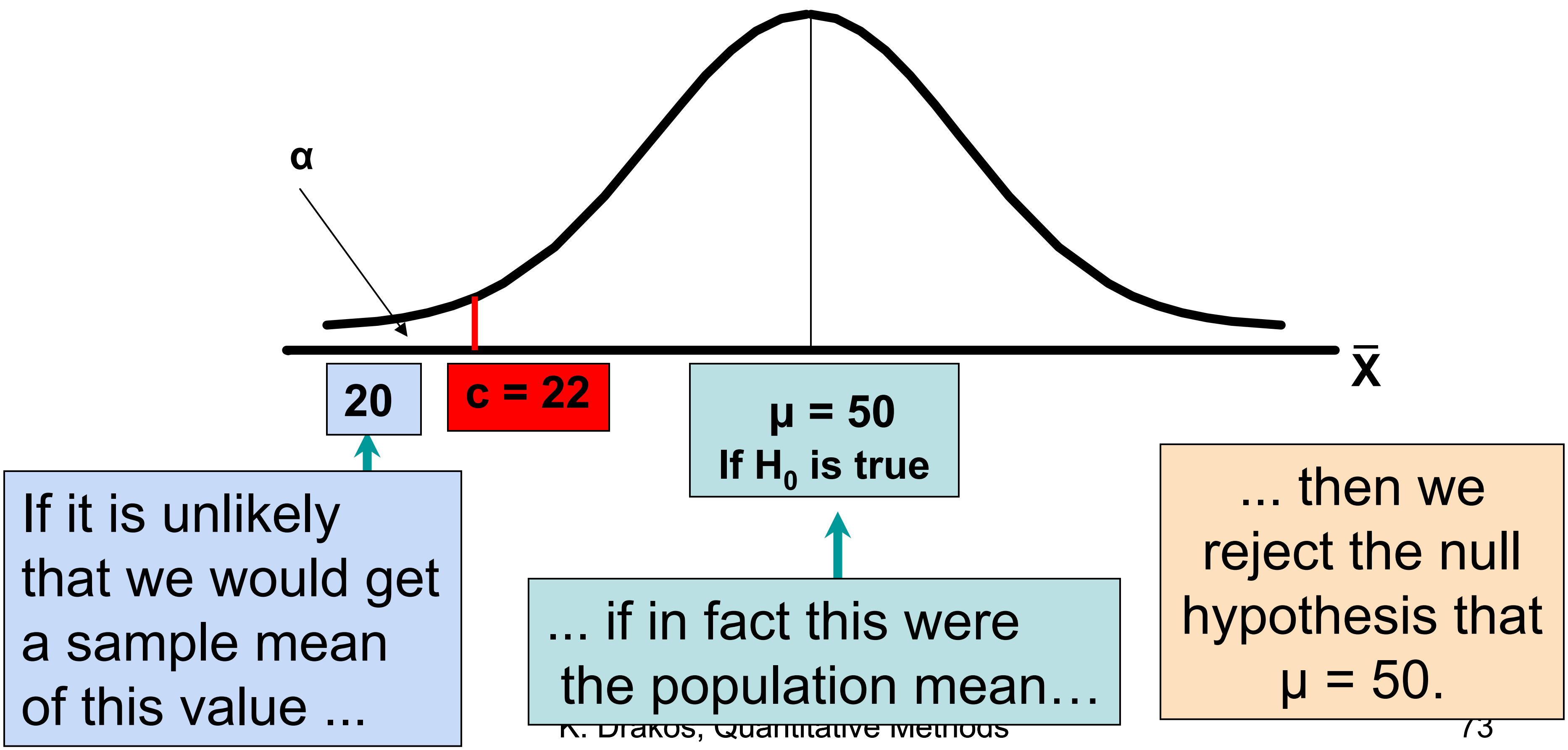
(continued)

- If we find that for α = 1% the c = 22 then since 20 < 22 *we will reject H_0 at the 1% significance level*.
- So for H_0 being true the 1% of the samples would have $\overline{X}$ < c. The rest 99% would have a sample mean $\overline{X}$ > c. So we are 99% confident that H_0 should be rejected.

Hypothesis Tests Design

(continued)

Sampling Distribution of $\bar{X}$



Level of Significance, α

- Defines the unlikely values of the sample statistic if the null hypothesis is true
  - Defines rejection region of the sampling distribution
- Is designated by α , (level of significance)
  - Typical values are .01, .05, or .10
- Is selected by the researcher at the beginning
- Provides the critical value(s) of the test

Level of Significance and the Rejection Region

Level of significance = α

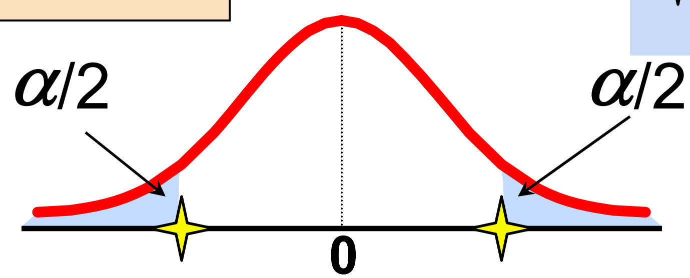
Represents critical value

Rejection region is shaded

H_0: $\mu = 3$

H_1: $\mu \neq 3$

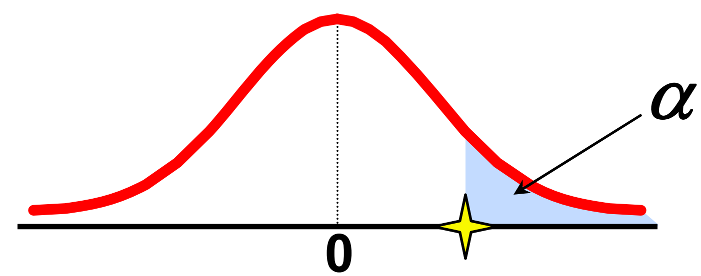
Two-tail test



H_0: $\mu \leq 3$

H_1: $\mu > 3$

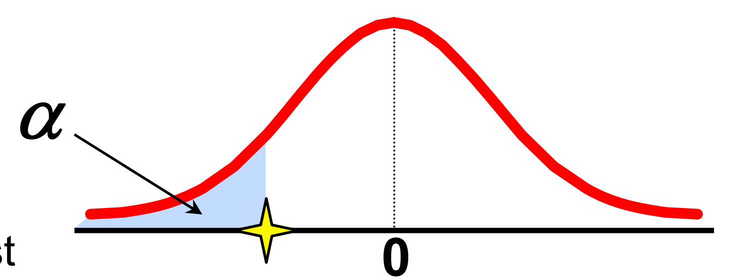
Upper-tail test



H_0: $\mu \geq 3$

H_1: $\mu < 3$

Lower-tail test



Errors in Making Decisions

- **Type I Error**
  - Reject a true null hypothesis
  - Considered a serious type of error

The probability of Type I Error is α

- Called level of significance of the test
- Set by researcher in advance

Errors in Making Decisions

(continued)

- **Type II Error**
  - Fail to reject a false null hypothesis

The probability of Type II Error is β

Outcomes and Probabilities

Possible Hypothesis Test Outcomes		
	Actual Situation	
Decision	H_0 True	H_0 False
Do Not Reject H_0	**No error** **(1 - α)**	**Type II Error** **(β)**
Reject H_0	**Type I Error** **(α)**	**No Error** **(1 - β)**

Key:
Outcome
(Probability)

Type I & II Error Relationship

- Type I and Type II errors can not happen at the same time
  - Type I error can only occur if H_0 is true
  - Type II error can only occur if H_0 is false

If Type I error probability (α) ⇧, then

Type II error probability (β) ⇩

Factors Affecting Type II Error

- All else equal,
  - β ↑ when the difference between hypothesized parameter and its true value ↓
  - β ↑ when α ↓
  - β ↑ when σ ↑
  - β ↑ when *n* ↓

Power of the Test

- The power of a test is the probability of rejecting a null hypothesis that is false

- i.e., Power = P(Reject H_0 | H_1 is true)

  – Power of the test increases as the sample size increases

Test of Hypothesis for the Mean (σ Known)

- Convert sample result ($\bar{x}$) to a z value

Hypothesis Tests for μ

σ Known | **σ Unknown**

Consider the test

$$H_0 : \mu = \mu_0$$
$$H_1 : \mu > \mu_0$$

(Assume the population is normal)

The decision rule is:

$$\text{Reject } H_0 \text{ if } \quad z = \frac{\bar{x} - \mu_0}{\frac{\sigma}{\sqrt{n}}} > z_\alpha$$

Decision Rule

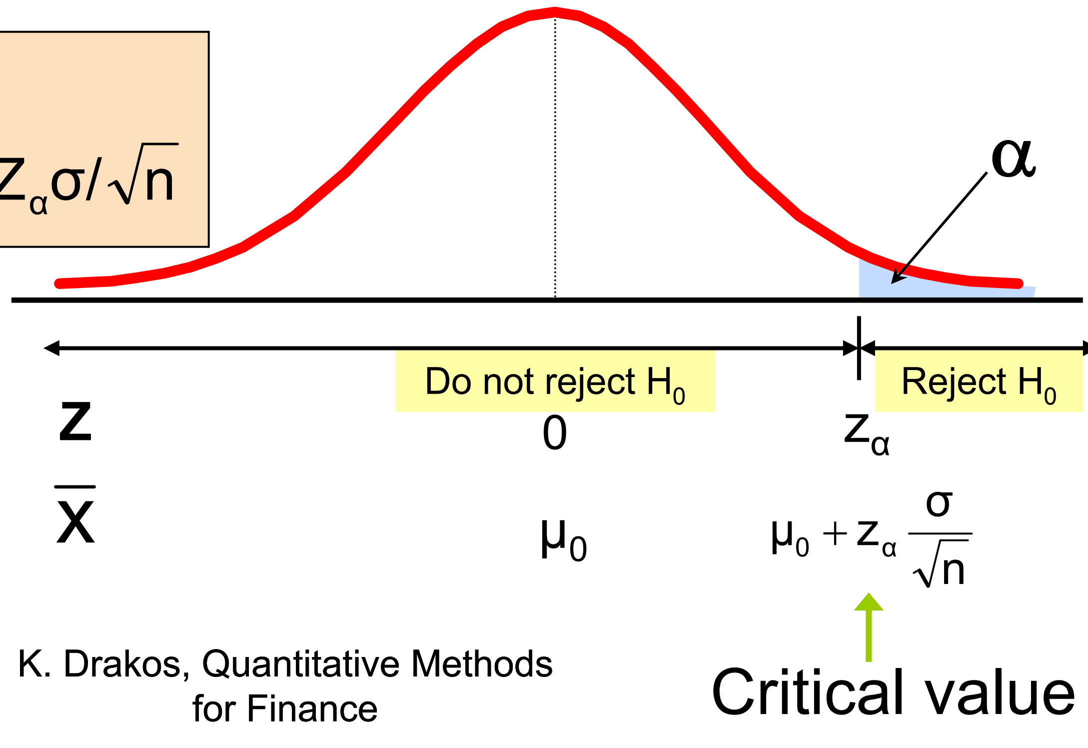
Reject H_0 if $z = \dfrac{\bar{x} - \mu_0}{\dfrac{\sigma}{\sqrt{n}}} > z_\alpha$

H_0: $\mu = \mu_0$

H_1: $\mu > \mu_0$

Alternate rule:

Reject H_0 if $\bar{X} > \mu_0 + Z_\alpha \sigma / \sqrt{n}$



p-Value Approach to Testing

- p-value: Probability of obtaining a test statistic more extreme ($\leq$ or $\geq$) than the observed sample value given H_0 is true
  - Also called observed level of significance
  - Smallest value of α for which H_0 can be rejected

p-Value Approach to Testing

(continued)

- Convert sample result (e.g., $\bar{x}$) to test statistic (e.g., z statistic)
- Obtain the p-value
  - For an upper tail test:

$$\text{p-value} = P\left(Z > \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}, \text{ given that } H_0 \text{ is true}\right)$$

$$= P\left(Z > \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} \mid \mu = \mu_0\right)$$

- Decision rule: compare the p-value to α

– If p-value $< \alpha$, reject H_0

– If p-value $\geq \alpha$, do not reject H_0

Example: Upper-Tail Z Test for Mean (σ Known)

A phone industry manager thinks that customer monthly cell phone bill have increased, and now average over $52 per month. The company wishes to test this claim. (Assume $\sigma = 10$ is known)

Form hypothesis test:

H_0: $\mu \leq 52$ the average is not over $52 per month

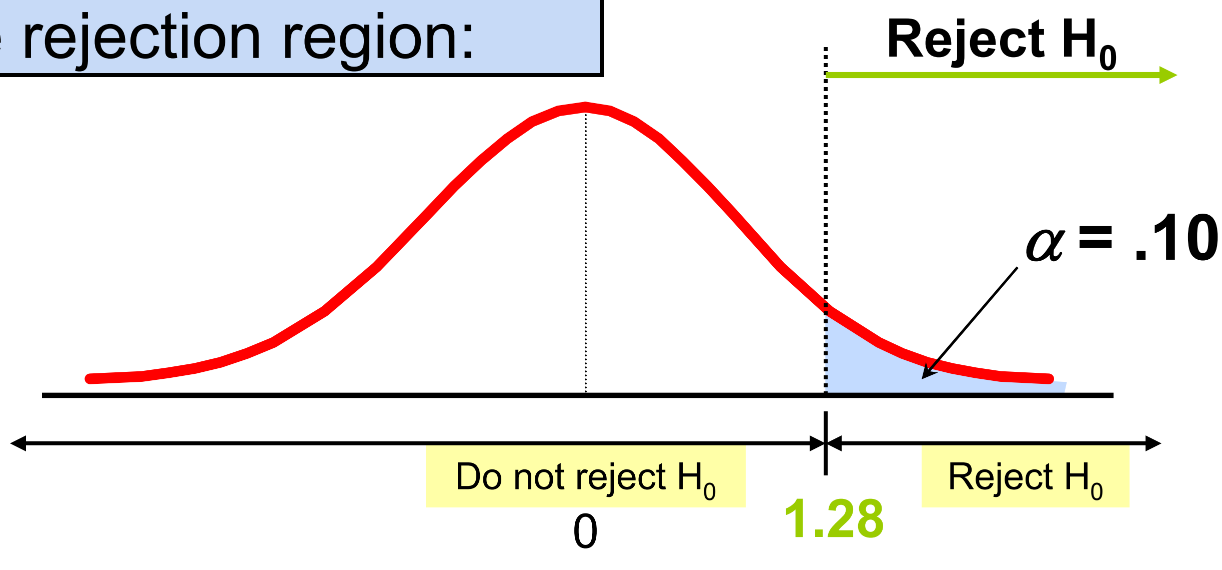
H_1: $\mu > 52$ the average **is** greater than $52 per month (i.e., sufficient evidence exists to support the manager's claim)

Example: Find Rejection Region

(continued)

- Suppose that $\alpha = .10$ is chosen for this test

Find the rejection region:



Reject H_0 if $z = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} > 1.28$

Example: Sample Results

(continued)

Obtain sample and compute the test statistic

Suppose a sample is taken with the following results: n = 64, $\bar{x}$ = 53.1 (σ=10 was assumed known)

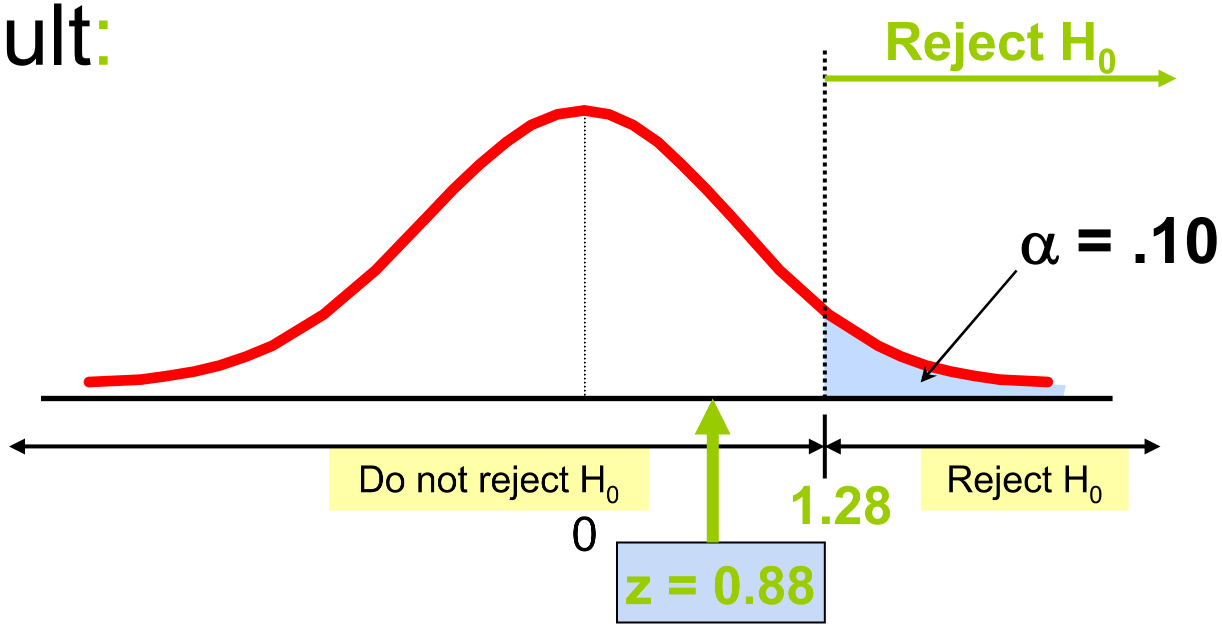
Using the sample results,

$$z = \frac{\bar{x} - \mu_0}{\frac{\sigma}{\sqrt{n}}} = \frac{53.1 - 52}{\frac{10}{\sqrt{64}}} = 0.88$$

Example: Decision

(continued)

Reach a decision and interpret the result:



Do not reject H_0 since z = 0.88 < 1.28

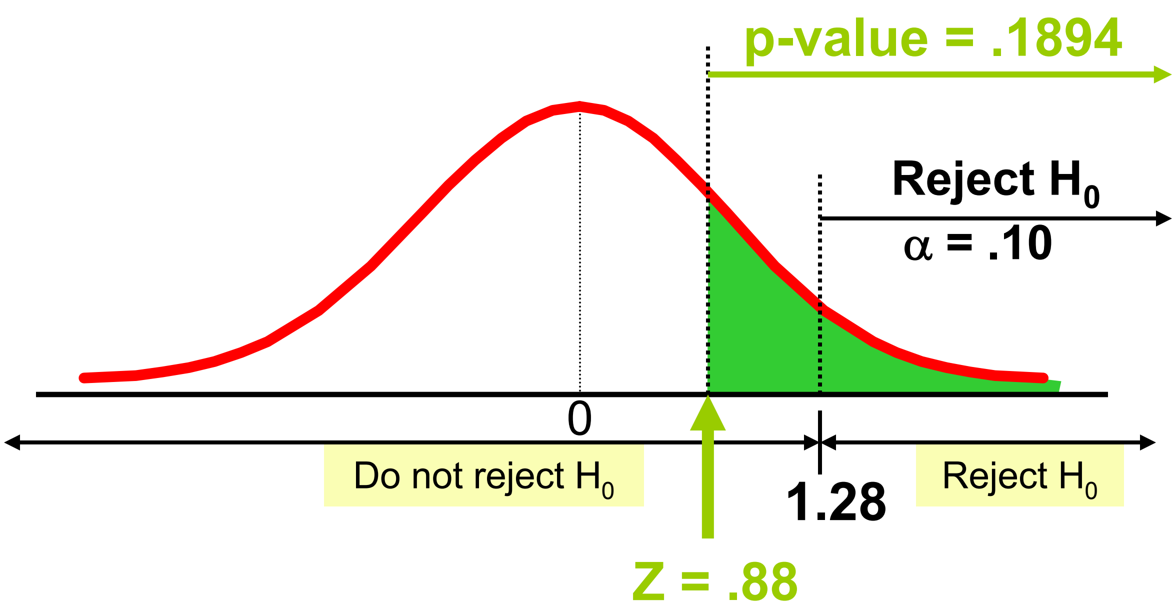
i.e.: there is not sufficient evidence that the mean bill is over $52

Example: p-Value Solution

(continued)

Calculate the p-value and compare to α

(assuming that $\mu = 52.0$)



$$P(\bar{x} \ge 53.1 \mid \mu = 52.0)$$

$$= P\left(z \ge \frac{53.1 - 52.0}{10/\sqrt{64}}\right)$$

$$= P(z \ge 0.88) = 1 - .8106$$

$$= .1894$$

Do not reject H_0 since p-value = .1894 > α = .10

One-Tail Tests

- In many cases, the alternative hypothesis focuses on one particular direction

H_0: $\mu \leq 3$
H_1: $\mu > 3$

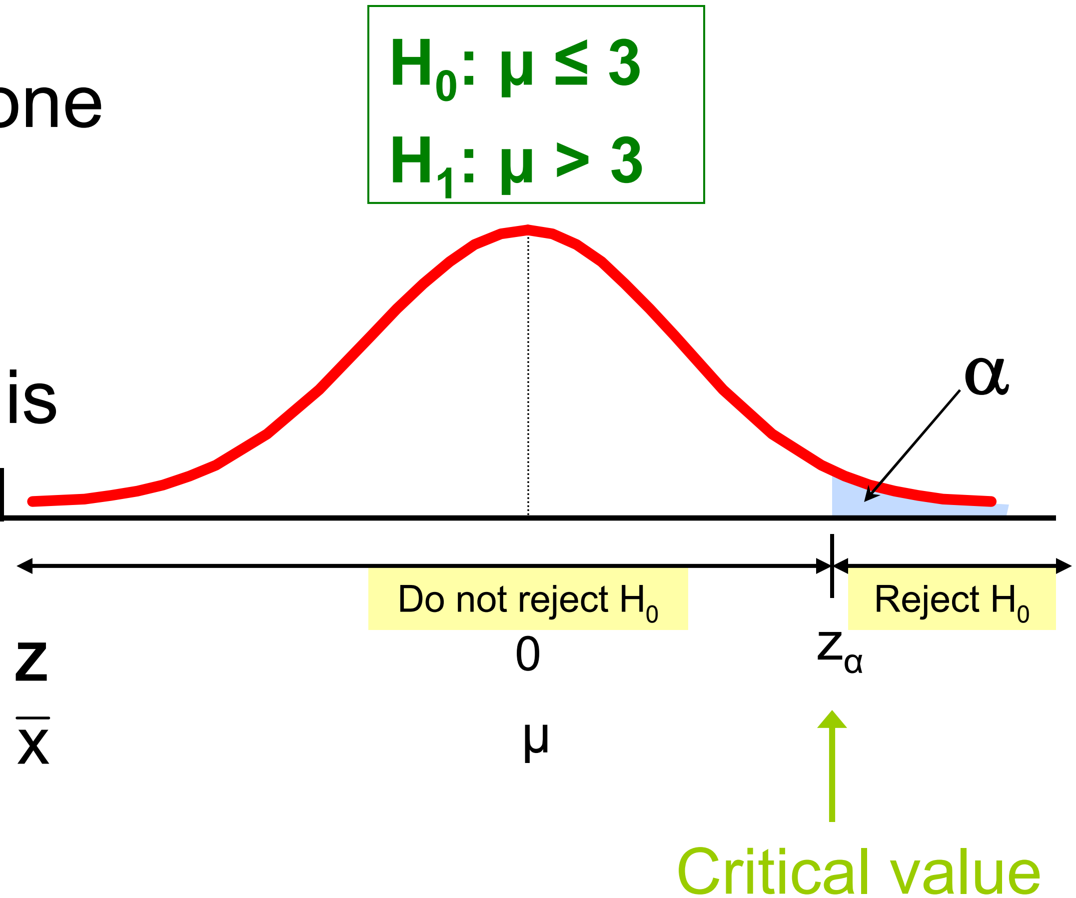
→ This is an upper-tail test since the alternative hypothesis is focused on the upper tail above the mean of 3

H_0: $\mu \geq 3$
H_1: $\mu < 3$

→ This is a lower-tail test since the alternative hypothesis is focused on the lower tail below the mean of 3

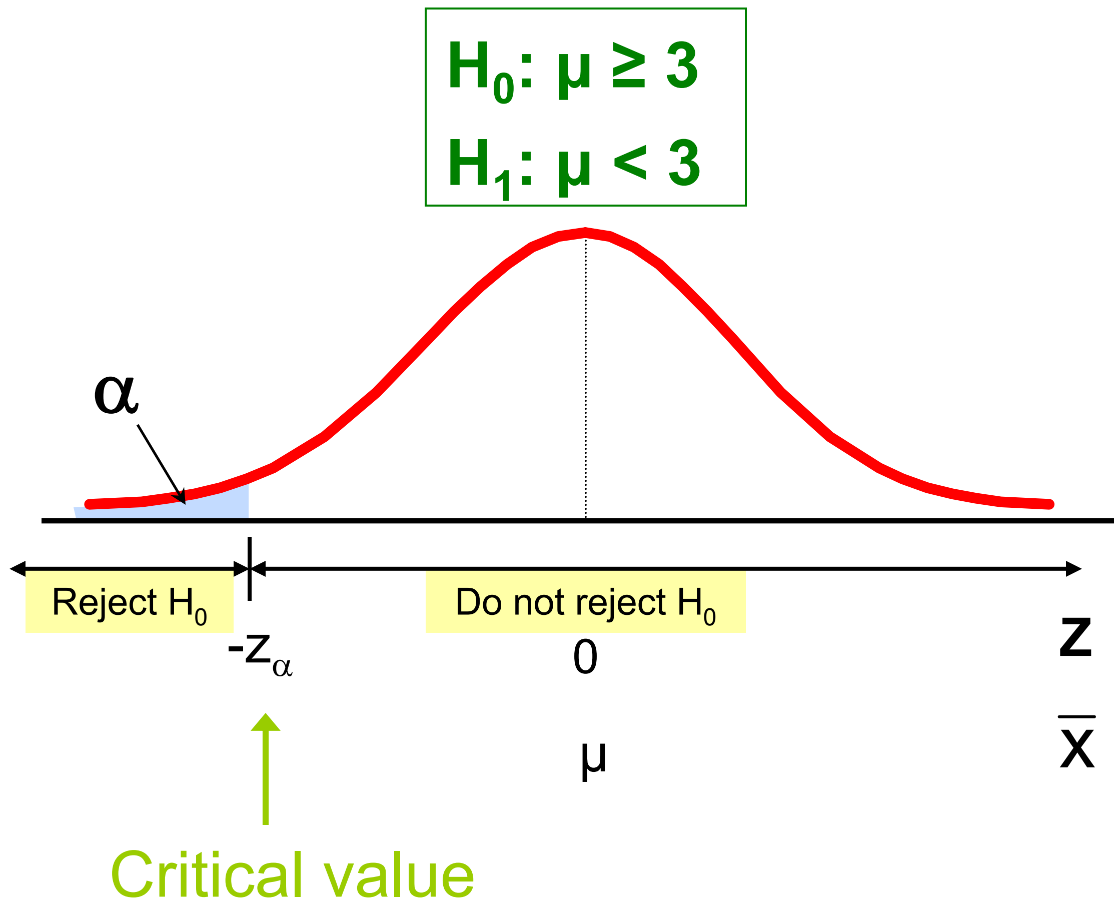
Upper-Tail Tests

- There is only one critical value, since the rejection area is in only one tail



Lower-Tail Tests

- There is only one critical value, since the rejection area is in only one tail

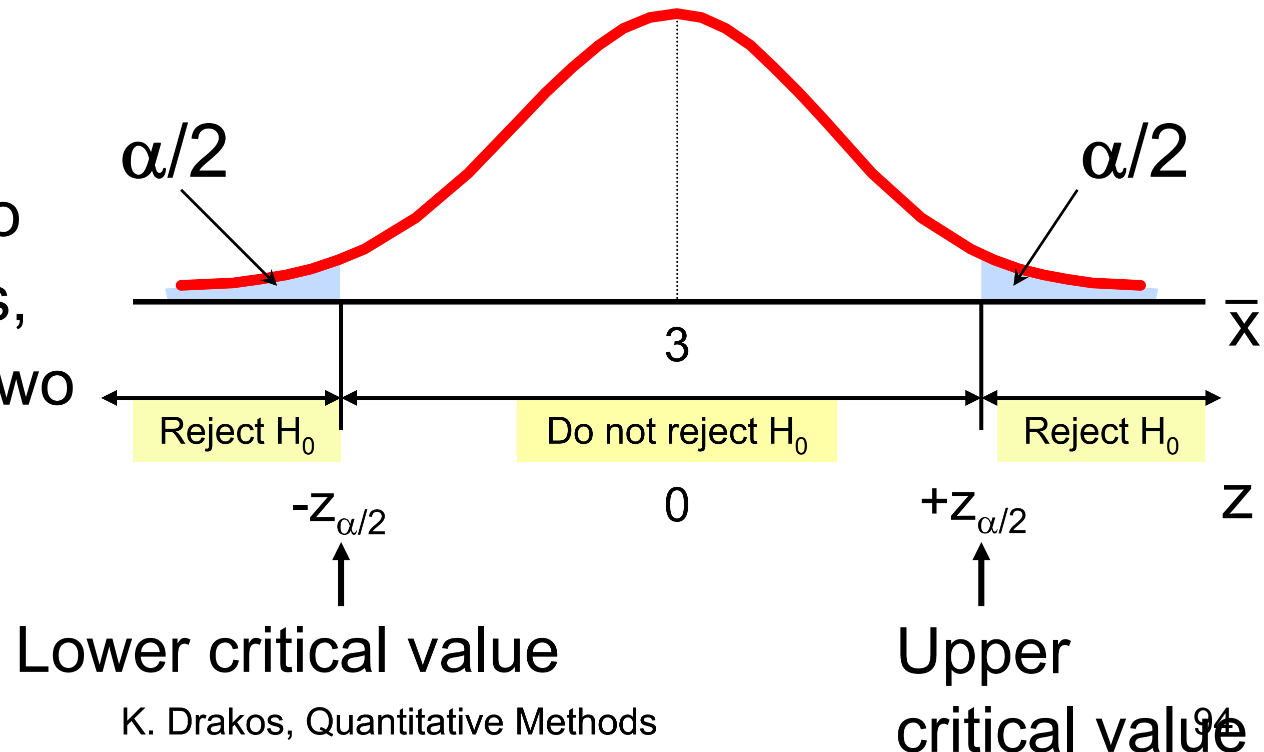


Two-Tail Tests

- In some settings, the alternative hypothesis does not specify a unique direction

H_0: $\mu = 3$
H_1: $\mu \neq 3$

- There are two critical values, defining the two regions of rejection



Hypothesis Testing Example

Test the claim that the true mean # of TV sets in US homes is equal to 3. (Assume σ = 0.8)

- State the appropriate null and alternative hypotheses
  - H_0: μ = 3 , H_1: μ ≠ 3 (This is a two tailed test)
- Specify the desired level of significance
  - Suppose that α = .05 is chosen for this test
- Choose a sample size
  - Suppose a sample of size n = 100 is selected

Hypothesis Testing Example

(continued)

- Determine the appropriate technique
  - σ is known so this is a z test
- Set up the critical values
  - For $\alpha = .05$ the critical z values are ±1.96
- Collect the data and compute the test statistic
  - Suppose the sample results are

    $n = 100$, $\bar{x} = 2.84$ ($\sigma = 0.8$ is assumed known)

So the test statistic is:

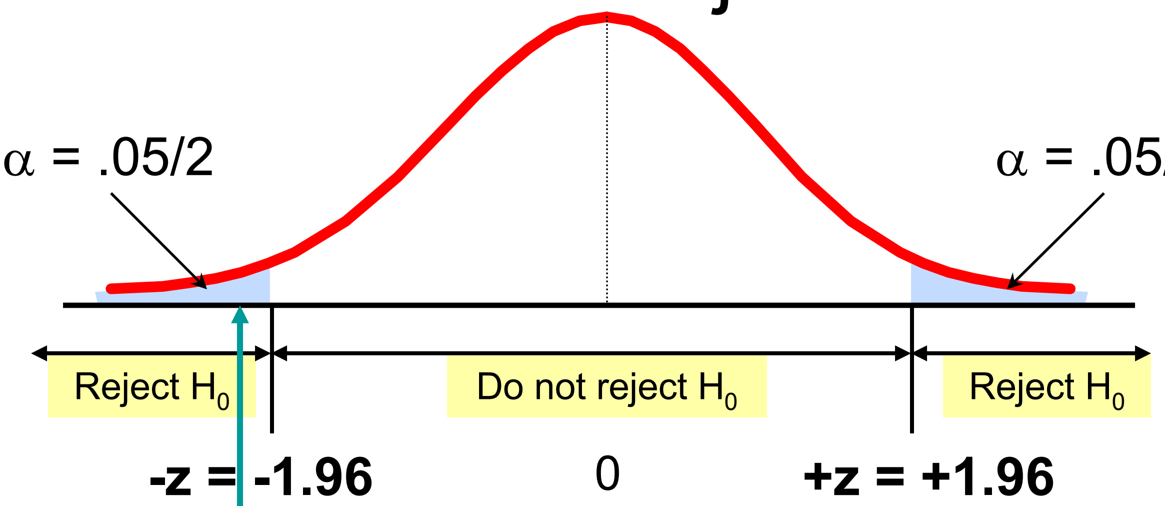
$$z = \frac{\bar{X} - \mu_0}{\frac{\sigma}{\sqrt{n}}} = \frac{2.84 - 3}{\frac{0.8}{\sqrt{100}}} = \frac{-.16}{.08} = -2.0$$

Hypothesis Testing Example

(continued)

- Is the test statistic in the rejection region?

Reject H_0 if $z < -1.96$ or $z > 1.96$; otherwise do not reject H_0

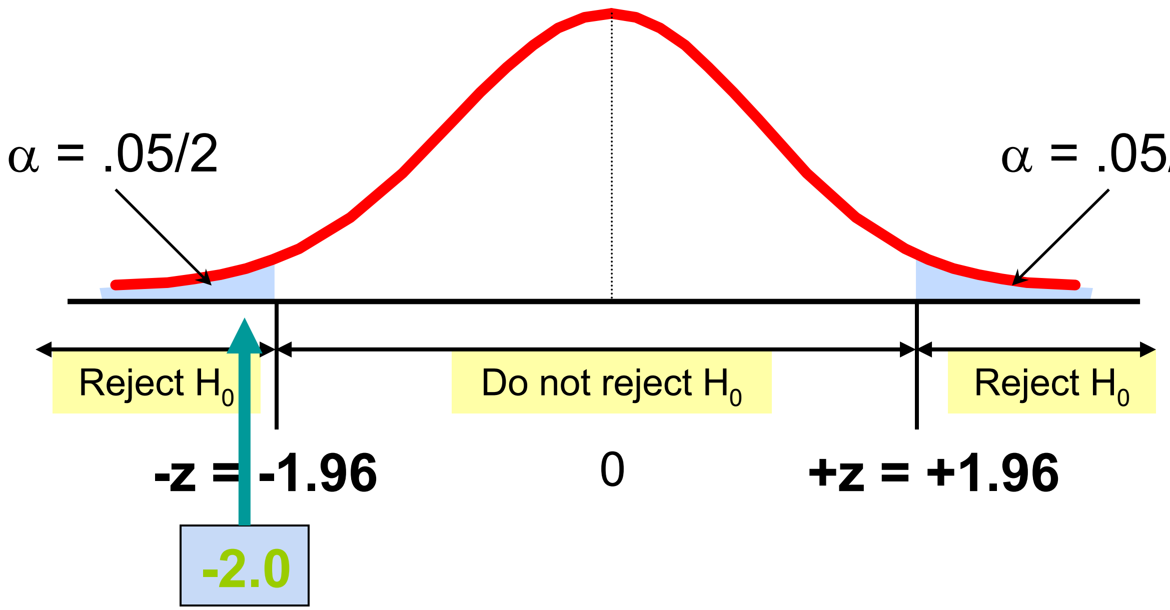


Here, $z = -2.0 < -1.96$, so the test statistic is in the rejection region

Hypothesis Testing Example

(continued)

- Reach a decision and interpret the result



Since $z = -2.0 < -1.96$, we reject the null hypothesis and conclude that there is sufficient evidence that the mean number of TVs in US homes is not equal to 3

Example: p-Value

- Example: How likely is it to see a sample mean of 2.84 (or something further from the mean, in either direction) if the true mean is $\mu = 3.0$?

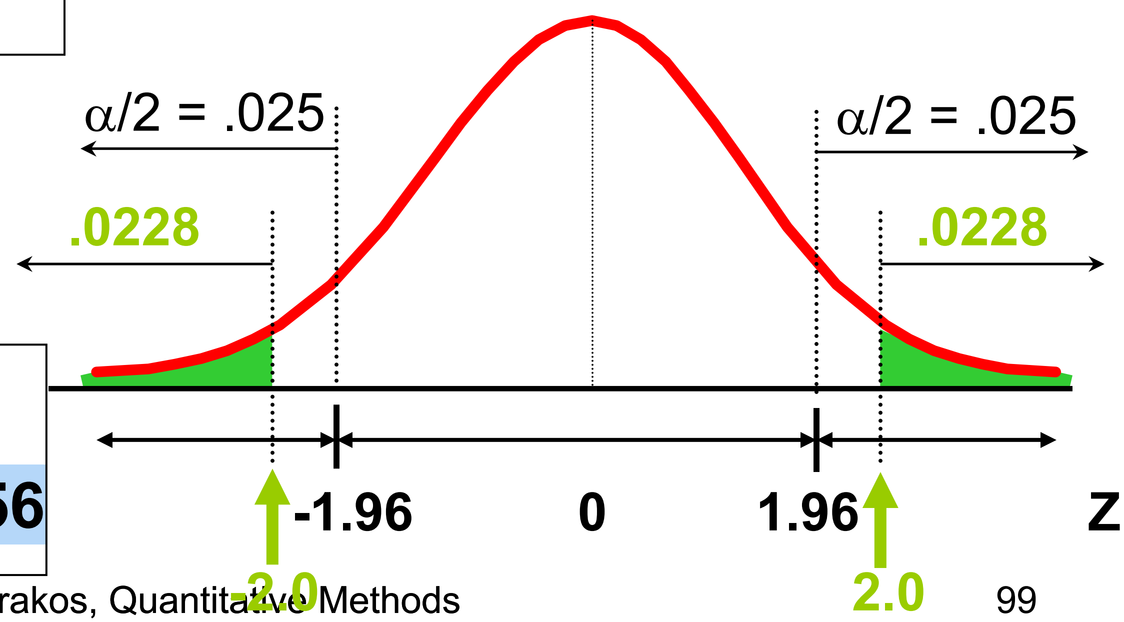
$\bar{x} = 2.84$ is translated to a z score of z = -2.0

$P(z < -2.0) = .0228$

$P(z > 2.0) = .0228$

p-value

= .0228 + .0228 = .0456



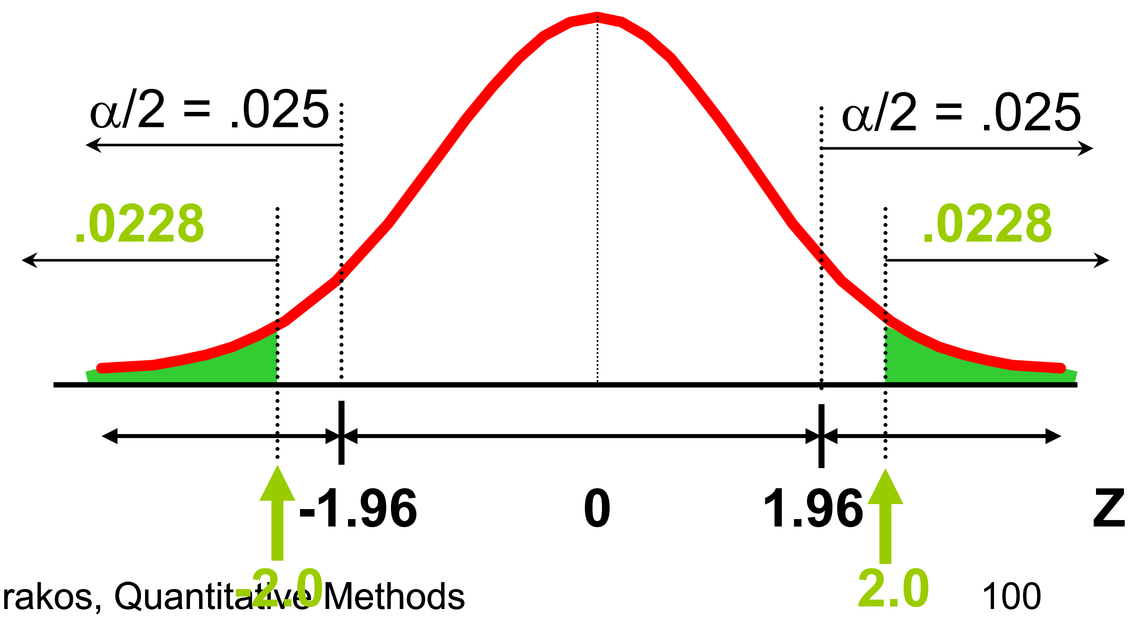
Example: p-Value

(continued)

- Compare the p-value with α
  - If p-value $<$ α , reject H_0
  - If p-value $\geq$ α , do not reject H_0

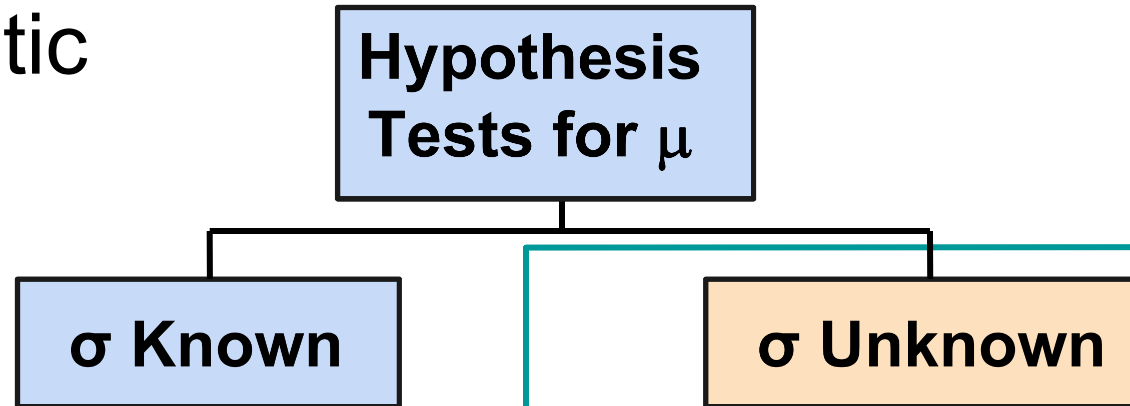
Here: p-value = .0456
α = .05

Since .0456 < .05, we reject the null hypothesis



t Test of Hypothesis for the Mean (σ Unknown)

- Convert sample result ($\bar{x}$) to a t test statistic



Consider the test

$$H_0 : \mu = \mu_0$$
$$H_1 : \mu > \mu_0$$

(Assume the population is normal)

The decision rule is:

$$\text{Reject } H_0 \text{ if } t = \frac{\bar{x} - \mu_0}{\frac{s}{\sqrt{n}}} > t_{n-1,\,\alpha}$$

t Test of Hypothesis for the Mean (σ Unknown)

(continued)

- For a two-tailed test:

Consider the test

$$H_0 : \mu = \mu_0$$
$$H_1 : \mu \neq \mu_0$$

(Assume the population is normal, and the population variance is unknown)

The decision rule is:

Reject H_0 if $t = \frac{\bar{x} - \mu_0}{\frac{s}{\sqrt{n}}} < -t_{n-1,\,\alpha/2}$ or if $t = \frac{\bar{x} - \mu_0}{\frac{s}{\sqrt{n}}} > t_{n-1,\,\alpha/2}$

Example: Two-Tail Test (σ Unknown)

The average cost of a 5-star hotel room in Athens is said to be 168 euros per night. A random sample of 25 hotels resulted in $\bar{x}$ = 172.50 euros and s = 15.40 euros. Test at the

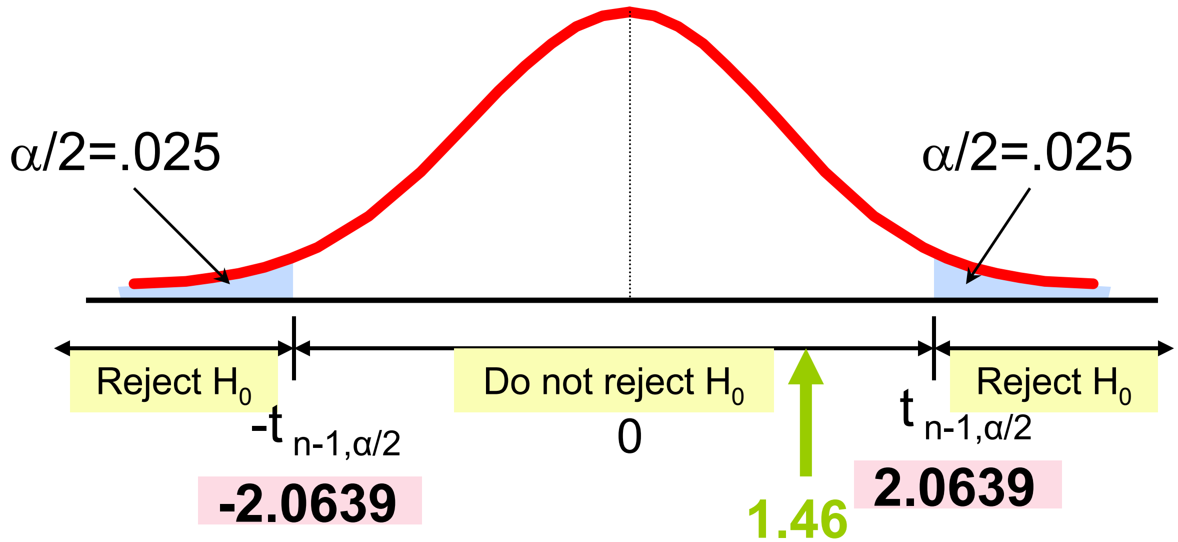
α = 0.05 level.

(Assume the population distribution is normal)

H_0: μ = 168
H_1: μ ≠ 168

Example Solution: Two-Tail Test

H_0: μ = 168
H_1: μ ≠ 168



- **n = 25**
- **a = 0.05**
- **σ is unknown, so use a t statistic**
- **Critical Value:**

$$t_{24\,,\,0.025} = \pm\, 2.0639$$

$$t_{n-1} = \frac{\bar{x} - \mu}{\frac{s}{\sqrt{n}}} = \frac{172.50 - 168}{\frac{15.40}{\sqrt{25}}} = 1.46$$

Do not reject H_0: not sufficient evidence that true mean cost is different than $168