Induction Course

In

Quantitative Methods

M.Sc. in Accounting and Finance

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I. Linear Algebra

1. Introduction - Notation

A **matrix** is a rectangular array of numbers, such as:

$$\begin{pmatrix} 2 & -5 & 6 \\ 20 & 1 & 0 \end{pmatrix}$$

An item in a matrix is called **entry** or **element**. The example has entries: 2, -5, 6, 20, 1 and 0.
The horizontal and vertical lines in a matrix are called **rows** and **columns**, respectively. To specify a matrix size, a matrix with m rows and n columns is called a $m \times n$ matrix, while m and n are its **dimensions**. The example is 2×3 matrix.

A matrix with one row (a $1 \times n$ matrix) is called a **row vector** and a matrix with one column (a $m \times 1$ matrix) a **column vector**. For example the first row of the above example

$$\begin{pmatrix} 2 & -5 & 6 \end{pmatrix}$$

is a row vector, while its second column

$$\begin{pmatrix} -5 \\ 1 \end{pmatrix}$$

is a column vector.

In general a $m \times n$ matrix is represented as:

$$A_{mn} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

The elements of the matrix are represented as:

$$a_{ij}, i = 1,2,...,m \text{ and } j = 1,2,...,n$$

The subscripts of each element show its position in the matrix. Thus, the element a_{12} lies on the first row and the second column of the matrix.

The **transpose** of a matrix A is another matrix A' created by any one of the following equivalent actions:

- Write the rows of A as the columns of A'
- Write the columns of A as the rows of A'

Formally, the (i,j) element of A is the (j,i) element of A'. Thus if A is a $m \times n$ matrix then A' is a $n \times m$ matrix.

A **square matrix** has an equal number of columns and rows.
A **zero matrix** has all its elements equal to zero.
A **diagonal matrix** is a square matrix with all the elements equal to zero except those of the main diagonal.
An identity matrix is a diagonal matrix with all the elements of the main diagonal equal to 1, i.e,

$$I_{nn} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}$$

In a **symmetric matrix** the elements are symmetric with respect to the main diagonal. Thus a symmetric matrix is a square matrix with: $a_{ij} = a_{ji}, \forall\, i,j \text{ with } i \neq j$.

Matrices are useful for:

- Solving a linear system of equations, or finding whether or not this system has a (unique) solution

- An input-output analysis which describes a production process in macroeconomics
- A multiple regression analysis in econometrics
- A portfolio optimization problem in investments.

2. Matrix Operations

When two or more matrices are **added** or **subtracted**, these matrices should be **compatible**. This means they should have the same dimensions.

The rule of addition and subtraction of a matrix states that the respective elements of the matrices are added or subtracted.
The new matrix that will come up after the addition or subtraction of two or more matrices will have the same dimensions with them.

Properties of matrix addition:

- Commutative law: A + B = B + A
- Associative law: (A + B) + C = A + (B + C)

The multiplication of a matrix with a real number is called **scalar multiplication**.
The rule of the scalar multiplication states that the real number should be multiplied with each element of the matrix.

The **inner product** of a $1\times n$ row vector $a' = \begin{pmatrix} a_1 & a_2 & \dots & a_n \end{pmatrix}$ and a $n\times 1$ column vector $b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$ is equal to the number:

$$a'b = a_1b_1 + a_2b_2 + \dots + a_nb_n = \sum_{i=1}^{n} a_ib_i$$

In order to define the inner product of two vectors the number of the rows of the first vector should be equal to the number of the columns of the second.

Example 1: In a linear regression model the dependent variable is linearly related to several other variables, called the independent variables. Let y_i be the i-th observation of the dependent variable in question and let $x_i' = \begin{pmatrix} x_{i1} & x_{i2} & \dots & x_{iK} \end{pmatrix}$ be the i-th observation of the K independent variables. Then,

$$y_i = \beta_1 x_{i1} + \beta_2 x_{i2} + \dots + \beta_K x_{iK} + \varepsilon_i$$

where $\beta' = \begin{pmatrix} \beta_1 & \beta_2 & \dots & \beta_K \end{pmatrix}$ are unknown parameters to be estimated and ε_i is an unobserved error term. The above linear regression model can be written as an inner product as:

$$y_i = x_i'\beta_i + \varepsilon_i$$

Example 2: Assume an investor that invested in a portfolio of n stocks. The weights of the stocks in the portfolio are represented by the row vector $w' = \begin{pmatrix} w_1 & w_2 & \dots & w_n \end{pmatrix}$ with $\sum_{i=1}^{n} w_i = 1$. The expected annual returns of the n stocks are represented by the column vector $r=\begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{pmatrix}$. Then, the expected annual return of the portfolio is given by the inner product of vectors w' and r, i.e.,

$$w'r=\sum_{i=1}^{n} w_i r_i$$

If for example the portfolio contains 4 stocks with weights: $w'=\begin{pmatrix} 0.2 & 0.3 & 0.1 & 0.4 \end{pmatrix}$ and expected annual returns: $r=\begin{pmatrix} 0.1 \\ 0.2 \\ 0.15 \\ 0.05 \end{pmatrix}$, then the expected annual return of the portfolio is equal to:

$$w'r=0.2\times 0.1+0.3\times 0.2+0.1\times 0.15+0.4\times 0.05=0.115$$

Multiplication of two matrices is defined only if the number of columns of the left matrix is the same as the number of rows of the right matrix. If A is a $m\times n$ matrix and B is a $n\times p$ matrix, then their **matrix product** C = AB is the $m\times p$ matrix whose entries are given by the inner product of the corresponding row of A and the corresponding column of B:

$$c_{ij}=\sum_{k=1}^{n} a_{ik}b_{kj}=a_{i1}b_{1j}+a_{i2}b_{2j}+...+a_{in}b_{nj}$$

For example the multiplication of two 2×2 matrices A and B yields:

$$AB = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} =$$

$$= \begin{pmatrix} \begin{pmatrix} a_{11} & a_{12} \end{pmatrix} \begin{pmatrix} b_{11} \\ b_{21} \end{pmatrix} & \begin{pmatrix} a_{11} & a_{12} \end{pmatrix} \begin{pmatrix} b_{12} \\ b_{22} \end{pmatrix} \\ \begin{pmatrix} a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11} \\ b_{21} \end{pmatrix} & \begin{pmatrix} a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{12} \\ b_{22} \end{pmatrix} \end{pmatrix} =$$

$$= \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{pmatrix}$$

Example 3: Assume an investor that invested in a portfolio with n stocks. The weights of the stocks in the portfolio are represented by the row vector $w' = \begin{pmatrix} w_1 & w_2 & \dots & w_n \end{pmatrix}$ with $\sum_{i=1}^{n} w_i = 1$. The variance-covariance matrix V of the n stocks is a $n \times n$ matrix with the (i,i) element being the annual return variance of the i-th stock and the (i,j) element being the covariance between the i-th and the j-th stock. Then, the variance of the portfolio annual return is given as:

$$w'Vw$$

If $w' = \begin{pmatrix} 0.2 & 0.1 & 0.1 & 0.35 & 0.15 & 0.05 & 0.05 \end{pmatrix}$ and the variance-covariance matrix is equal to:

	ΕΜΠ	**ΑΕΓΕΚ**	**ΑΛΦΑ**	**ΒΙΟΧΚ**	**ΕΛΑΙΣ**	**ΕΤΕ**	**ΟΤΕ**
ΕΜΠ	0.00473	0.00199	0.00262	0.00132	0.00196	0.00362	0.00213
ΑΕΓΕΚ	0.00199	0.00498	0.00142	0.00173	0.00201	0.00144	0.00143
ΑΛΦΑ	0.00262	0.00142	0.00239	0.00152	0.0015	0.00239	0.00159
ΒΙΟΧΚ	0.00132	0.00173	0.00152	0.0036	0.00176	0.00174	0.00141
ΕΛΑΙΣ	0.00196	0.00201	0.0015	0.00176	0.00438	0.00161	0.00117
ΕΤΕ	0.00362	0.00144	0.00239	0.00174	0.00161	0.00351	0.00198
ΟΤΕ	0.00213	0.00143	0.00159	0.00141	0.00117	0.00198	0.00316

Then, the variance of the portfolio return is equal to 0.0022.

Properties of matrix multiplication:

- The commutative law does not hold: $AB \neq BA$
- Associative law: (AB)C = A(BC)
- Distributive law: (A + B)C = AC + BC
- Multiply with the identity matrix: AI = IA = A

Properties of transpose matrices:

- $(A')' = A$
- $(A+B)' = A' + B'$
- $(AB)' = B'A'$
- $(cA)' = cA'$

3. The determinant of a matrix

The determinant is a special number associated with any square matrix. The fundamental geometric meaning of a determinant is a scale factor for measure when the matrix is regarded as a linear transformation.

The **Laplace expansion** of a $n \times n$ square matrix A gives the determinant, denoted as $|A|$, of the matrix. This is the sum of n determinants of $(n-1)\times(n-1)$ sub-matrices of A. There are n^2 such expressions, one for each row and column of A.
Define the (i,j) **minor matrix** M_{ij} of A as the $(n-1)\times(n-1)$ matrix that results from deleting the i-th row and the j-th column of A, and C_{ij} the **cofactor** of A as $C_{ij} = (-1)^{i+j}|M_{ij}|$.

Then the Laplace expansion gives the determinant of A as:

$$|A| = a_{i1}C_{i1} + a_{i2}C_{i2} + ... + a_{in}C_{in} =$$
$$= a_{1j}C_{1j} + a_{2j}C_{2j} + ... + a_{nj}C_{nj}$$

Example 4: The determinant of a 2×2 matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ is equal to: $|A| = a_{11}a_{22} - a_{12}a_{21}$.

Example 5: Consider the matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$. The determinant of this matrix can be computed using the Laplace expansion along the first row:

$$|A| = 1\times\begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2\times\begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3\times\begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} =$$
$$= 1\times(-3) - 2\times(-6) + 3\times(-3) = 0$$

Properties of determinants:

- If all the elements of a row or column of a matrix are zero then the determinant of the matrix is equal to zero
- If B results from A by interchanging two rows or two columns then $|B| = -|A|$
- If B results from A by multiplying one row or one column with a number c, then $|B| = c|A|$
- If B results from A by adding a multiple of one row to another row, or a multiple of one column to another column, then $|B| = |A|$

- $|cA_{nn}| = c^n |A_{nn}|$
- $|AB| = |A||B|$
- $|A'| = |A|$

Consider a $n \times n$ matrix A and consider that there exists a vector $\lambda' = (\lambda_1 \quad \lambda_2 \quad \dots \quad \lambda_n) \neq 0$ for which:

$$\lambda_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{pmatrix} + \lambda_2 \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{n2} \end{pmatrix} + \ldots + \lambda_n \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{nn} \end{pmatrix} = 0$$

Then the vectors $(a_1 \quad a_2 \quad \dots \quad a_n)$ are **linearly dependent.** In this case the determinant of A is equal to zero.[1] This matrix is called **singular**.

This is exactly what happens in Example 5 where the sum of the first and third column is twice the second column.

4. The inverse of a matrix

The **inverse** of a square matrix is a new matrix A^{-1} (with the same dimensions with A) such that:

$$AA^{-1} = A^{-1}A = I$$

The calculation of the inverse of a matrix is required when we need to "move" the matrix from one side of the equation to the other in order to solve a linear system of equations. For example, if $AX = B$, then by multiplying in the left both sides of the equation with A^{-1} we obtain:

$$A^{-1}AX = A^{-1}B \Rightarrow IX = A^{-1}B \Rightarrow X = A^{-1}B$$

The inverse of a matrix can be calculated as:

[1] The same holds when the columns of the matrix are linearly dependent.

$$A^{-1} = \frac{1}{|A|} adj(A)$$

where adj(A) is the **adjoint** matrix of A.
If $C_{ij} = (-1)^{i+j} |M_{ij}|$ defines the cofactor matrix of A, then

$$adj(A) = C'$$

From the above equation it becomes apparent that the inverse of the matrix is defined only when the determinant $|A| \neq 0$. If this property holds then the matrix is called **invertible** or **non-singular**.

Example 6: The inverse of a 2×2 matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ is:

$$A^{-1} = \frac{1}{(a_{22}a_{11} - a_{12}a_{21})} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

Properties of the inverse of a matrix:

- $(A^{-1})^{-1} = A$
- $(cA)^{-1} = c^{-1}A^{-1}$
- $(A')^{-1} = (A^{-1})'$
- $(AB)^{-1} = B^{-1}A^{-1}$
- $|A^{-1}| = 1/|A|$

II. System of Linear Equations

1. Introduction – Notation

A **system of linear equations** is a collection of linear equations involving the same set of variables.

Example 7: Consider an economic model that describes demand and supply of an asset. We can write that the quantity of the asset that the market demand, denoted as Q_d is:

$$Q_d = a - bP, a, b > 0$$

where P is the price of the asset. The above equation implies that as the price decreases the demand increases. The quantity supplied by the market, denoted as Q_s is:

$$Q_s = -c + dP, c, d > 0$$

Now, price and supply are positively related.
For the market to be in equilibrium we must also impose:

$$Q_s = Q_d \equiv Q$$

These three equations constitute a system of two linear equations with two unknowns that describe demand, supply and the asset price in equilibrium. We can therefore write:

$$Q = a - bP$$

$$Q = -c + dP$$

Solving this system with respect to P and Q we determine the market equilibrium.

In general, a system of linear equations is written as:

$$
\begin{aligned}
&a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\
&a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\
&\vdots \\
&a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m
\end{aligned}
$$

Here $x_1, x_2, \ldots, x_n$ are the unknowns (endogenous variables), $a_{11}, a_{12}, \ldots, a_{mn}$ are the coefficients of the system and $b_1, b_2, \ldots, b_m$ are the constant terms (exogenous variables). This is system with m linear equations and n unknowns.

Every system of linear equations can be written in matrix form as follows:

$$AX = B \tag{1}$$

with

$$A_{(m\times n)} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}, X_{(n\times 1)} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, B_{(m\times 1)} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

A is the **coefficient matrix**, X is the **solution vector** and B is the **vector of constant terms**.

For example the system of equations that determines the market equilibrium can be written as:

$$\underbrace{\begin{pmatrix} 1 & b \\ 1 & -d \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} Q \\ P \end{pmatrix}}_{X} = \underbrace{\begin{pmatrix} a \\ -c \end{pmatrix}}_{B}$$

2. The solution of linear system of equations

A linear system may behave in any one of three possible ways:

- The system has an infinite number of solutions
- The system has a unique solution
- The system has no solution (**inconsistency**)

The **column rank** of matrix A is the maximal number of linearly independent columns of A. Likewise, the **row rank** of A is the maximal number of linearly independent rows of A. Since the column rank and the row rank are always equal, they are simply called the **rank** of A.
The rank of a $m \times n$ matrix is at most min(m,n), i.e., $r(A_{mn}) \leq \min(m,n)$. When $r(A_{mn}) = \min(m,n)$ the matrix is said to have **full rank**.
Also define the **augmented matrix** as $C_{m(n+1)} = \begin{pmatrix} A & B \end{pmatrix}$.

How can we calculate the rank of a matrix? To do so, we need to transform it to its **row echelon form**. This form should satisfy the following conditions:

- The first non-zero element in each row, called the leading entry, is 1.
- Each leading entry is in a column to the right of the leading entry in the previous row.
- Rows with all zero elements, if any, are below rows having non-zero elements.

To transform a matrix to its echelon form we need to do a series of elementary row operations as follows:

1. Pivot the matrix.
    - Start with the first row of the matrix if the entry of the first column is different than zero. This entry is known as the pivot. Otherwise, interchange it with another row.

- Multiply each element in the pivot row by the inverse of the pivot, so the pivot equals 1.
- Add multiples of the pivot row to each of the lower rows, so every element in the pivot column is equal to 0.

2. Repeat step 1.
   - Repeat the procedure from step 1 above, ignoring previous pivot rows.
   - Continued until there are no more pivots to be processed.

Once the row echelon form is found, the rank of the matrix is equal to the number of non-zero rows in its echelon form.

Example 8: Consider the following matrix $A = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 7 & 8 \end{pmatrix}$. To transform this matrix to its echelon form we do the following series of elementary row operations:

$$\begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 7 & 8 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 2 & 7 & 8 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 3 & 6 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

The last matrix above is the echelon form. It has two non-zero rows so the rank of matrix A is equal to 2, r(A) = 2.

Proposition 1: The linear system of equations (1) has a solution if and only if r(A) = r(C).

Example 9: Assume that $A = \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 8 \\ 9 \end{pmatrix}$. Then we can show that r(A) = 1 and r(C) = 2, thus the system does not have a solution.

If $B = \begin{pmatrix} 8 \\ 4 \end{pmatrix}$, then r(C) = 1 and the system has a solution (though not unique).

Proposition 2: The linear system of equations (1) has a unique solution if r(A) = r(C) = n (where n is the number of unknowns).

Consider that m < n (more unknowns than equations). In this case $r(A) \leq \min(m, n) = m < n$ and the system cannot have a unique solution.

Proposition 3: If n = m and r(A) = n, then the system (1) has a unique solution.[2]

In this case the matrix A is invertible. This comes from the fact that A is a square matrix and $|A| \neq 0$ (since r(A) = n). Thus the solution of the system is:

$$X = A^{-1}B$$

Example 7 (continued): In this case we have that n = m and r(A) = 2, thus the system has a unique solution. The inverse of A is:

$$A^{-1} = -\frac{1}{d+b}\begin{pmatrix} -d & -b \\ -1 & 1 \end{pmatrix} = \frac{1}{d+b}\begin{pmatrix} d & b \\ 1 & -1 \end{pmatrix}$$

and

[2] This comes from Proposition 2. When n = m we can easily prove that r(C) = n.

$$X = \frac{1}{d+b}\begin{pmatrix} d & b \\ 1 & -1 \end{pmatrix}\begin{pmatrix} a \\ -c \end{pmatrix} = \frac{1}{d+b}\begin{pmatrix} ad-bc \\ a+c \end{pmatrix}.$$

Thus the equilibrium price and quantity are: $P = \frac{a+c}{b+d}$ and $Q = \frac{ad-bc}{b+d}$, respectively.

Example 10: Consider an economic model with two assets 1 and 2. The demand for the two assets is:

$$Q_{d1} = a_0 + a_1 P_1 + a_2 P_2$$

$$Q_{d2} = b_0 + b_1 P_1 + b_2 P_2$$

where P_1 and P_2 are the prices of asset 1 and 2, respectively. Similarly, the supply for the two assets is:

$$Q_{s1} = c_0 + c_1 P_1 + c_2 P_2$$

$$Q_{s2} = d_0 + d_1 P_1 + d_2 P_2$$

In equilibrium,

$$Q_{s1} = Q_{d1} \equiv Q_1$$

$$Q_{s2} = Q_{d2} \equiv Q_2$$

We can write the above problem as a system of linear equations as follows:

$$\underbrace{\begin{pmatrix} 1 & 0 & -a_1 & -a_2 \\ 0 & 1 & -b_1 & -b_2 \\ 1 & 0 & -c_1 & -c_2 \\ 0 & 1 & -d_1 & -d_2 \end{pmatrix}}_{A}\underbrace{\begin{pmatrix} Q_1 \\ Q_2 \\ P_1 \\ P_2 \end{pmatrix}}_{X} = \underbrace{\begin{pmatrix} a_0 \\ b_0 \\ c_0 \\ d_0 \end{pmatrix}}_{B}$$

Thus we have a 4×4 system of linear equations. Assume that r(A) = 4, then this system has a unique solution which determines the equilibrium in this two assets market. To solve this system one should calculate the inverse of a 4×4 matrix which is not an easy task to do without a computer program. However, we can simplify the above system of equations in order to give an explicit solution without using a computer. To do so, we will use results of matrix algebra presented in the previous section. First, write the above system as:

$$\begin{pmatrix} I & A_1 \\ I & A_2 \end{pmatrix}\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}$$

which implies:

$$X_1 + A_1 X_2 = B_1$$

$$X_1 + A_2 X_2 = B_2$$

Substituting X_1 from the first equation to the second we obtain:

$$(B_1 - A_1 X_2) + A_2 X_2 = B_2 \Rightarrow (A_2 - A_1)X_2 = B_2 - B_1$$

Thus we have a 2×2 system of equations which can be easily solved, with respect to X_2 as in the previous example. Finally, using one of the above equations we can solve for X_1.

Example 11: **A pure, primitive or Arrow-Debreu security** is defined as a security that pays $1 at the end of a period if a given state occurs and nothing if any other state occurs. The concept of pure securities allows the decomposition of market securities into portfolios of pure securities. Thus every market security may be considered a combination of various pure securities. Of course these securities are not traded so they should be inferred from market traded securities.
Consider two securities j and k, whose payoff table is given below:

Payoff table for securities j and k

Security	State 1	State 2	
j	\$10	\$20	p_j = \$8
k	\$30	\$10	p_k = \$9

We want to determine the prices of the pure securities (1, 0) and (0 ,1). Note that we have two states of nature and two linearly independent market securities, thus the market is **complete**. This means that the pure securities and uniquely determined.

By buying 10 units of (1, 0) and 20 units of (0, 1) we can construct the end-of-period payoff of security j. Similarly, by buying 30 units of (1, 0) and 10 units of (0, 1) we can construct the end-of-period payoff of security k. Since the payoffs of these portfolios equate the payoffs of the two market securities, we can assume that the prices should do so. Thus we can write:

$$p_1 Q_{j1} + p_2 Q_{j2} = p_j$$

$$p_1 Q_{k1} + p_2 Q_{k2} = p_k$$

Substituting the values given in the table we obtain:

$$p_1 10 + p_2 20 = 8$$

$$p_1 30 + p_2 10 = 9$$

This is a 2×2 system of equations. Solving for p_1 and p_2 we obtain: p_1 = \$0.2 and p_2 = \$0.3.

III. Eigenvalues and eigenvectors

1. Positive definite matrices

A symmetric $n \times n$ matrix A is said to be **positive definite** if $x'Ax > 0$ for all non zero $n \times 1$ vectors x. Similarly, we can define **positive semi-definite** $(x'Ax \geq 0)$, **negative definite** $(x'Ax < 0)$ and **negative semi-definite** $(x'Ax \leq 0)$.

If the matrix A is positive definite then it is invertible.

Positive definite matrices are important for linear regressions analysis and the maximization or minimization of a multivariate function.

2. Eigenvalues and eigenvectors

Let know write the system of equations:

$$Ax = \lambda x$$

where A is a $n \times n$ square matrix, x is $n \times 1$ vector and λ is real number. This system can be also written as:

$$(A - \lambda I)x = 0$$

This is a **homogenous** system of equations (since the vector of constant terms is zero). A trivial solution to the above system (when $A - \lambda I$ is invertible) is X = 0. When, however,

$$\left| A - \lambda I \right| = 0 \qquad (2)$$

the system has also **non-trivial** solutions. The determination of these solutions is known as the

eigenvalue problem. The values of λ that satisfy equation (2) are known as the **eigenvalues** of matrix A. The non-trivial solutions of x are known as the **eigenvectors** of matrix A.

Example 12: Consider the matrix $A = \begin{pmatrix} -5 & 2 \\ 2 & -2 \end{pmatrix}$. Then, to find the eigenvalues we solve:

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} -5-\lambda & 2 \\ 2 & -2-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 + 7\lambda + 6 = 0$$

The solutions are λ = -6 and λ = -1. These are the eigenvalues of matrix A.
For each eigenvalue there is a corresponding eigenvector. This vector can be found by substituting one of the eigenvalues back into the original equation.
For λ = -6 we have that:

$$\begin{pmatrix} -5+6 & 2 \\ 2 & -2+6 \end{pmatrix} x = 0 \Rightarrow \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} x = 0 \Rightarrow x_1 = -2x_2$$

So the eigenvector corresponding to λ = -6 is any vector of the form $\begin{pmatrix} -2c \\ c \end{pmatrix}, c \neq 0$.
For λ = -1 we have that:

$$\begin{pmatrix} -5+1 & 2 \\ 2 & -2+1 \end{pmatrix} x = 0 \Rightarrow \begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix} x = 0 \Rightarrow x_2 = 2x_1$$

So the eigenvector corresponding to λ = -1 is any vector of the form $\begin{pmatrix} c \\ 2c \end{pmatrix}, c \neq 0$.
Eignevalues are directly related to whether or not a matrix is positive definite. More specifically,

- When all eigenvalues are positive the matrix is positive definite
- When all eigenvalues are negative the matrix is negative definite (as in the Example 11)
- When all eigenvalues are non-negative (for example all are positive and one is equal to zero) the matrix is positive semi-definite
- When all eigenvalues are non-positive (for example all are negative and one is equal to zero) the matrix is negative semi-definite
- When eignevalues are both positive and negative the matrix is **indefinite**.