1.1.1 Brennan and Schwartz (1980)

Possibly the most influential paper on the analysis of convertible bonds is due to Brennan and Schwartz (1980). In there, they continue the theme of valuing convertible securities by utilising contingent claims technique, as initiated by Ingersoll (1976) and, most importantly, they ameliorate several of the restrictive assumptions and conditions of previous research.

The distinguishing features of their model are:

a) Interest rates, which influence both the "straight bond" and the "converted stock" components of a convertible bond, are no longer assumed constant; they are allowed to change through time in a random manner.
b) The firms issuing convertible bonds are allowed to have both common equity and ordinary debt in their capital structure.

What follows is a detailed account of the Brennan and Schwartz (1980) analysis.

1.1.1.1 Notation and basic setting

The following notation is used:

V : Total firm market value

B : Market value of straight bonds (non-callable, non-puttable, non-convertible)

C : Market value of (callable) convertible bonds

N_B : Number of straight bonds outstanding

N_C: Number of (callable) convertible bonds outstanding

N_O: Number of ordinary shares outstanding (*before* any possible conversion)

S^{BC}: Ordinary share price *before* any possible conversion

S^{AC}: Ordinary share price *after* any possible conversion

F: Face value of each convertible bond

c: Coupon rate of convertible bond

$ConP$: Conversion price

CP: Price at which convertible bonds can be called by the firm

$q = \frac{F}{ConP}$: Conversion ratio

A firm with debt, equity and convertibles in its capital structure has a market value given by

$$V = N_B \cdot B + N_C \cdot C + N_0 \cdot S^{BC} \tag{1}$$

If the convertible bonds are converted to equity, then equation (1) becomes

$$V = N_B \cdot B + N_C \cdot C + (N_0 + \Delta N) \cdot S^{AC} \tag{2}$$

where ΔN is the number of new ordinary shares issued due to conversion of the bonds. With the established notation this will be $\Delta N = q \cdot N_C$. Once converted, the value of the convertible bonds will be

$$q \cdot S^{AC} = \frac{q}{N_0 + \Delta N}(V - N_B \cdot B) = Z(V - N_B \cdot B) \tag{3}$$

where the first equality comes from substituting S^{AC} from (2), and the second equality from defining $Z = \dfrac{q}{N_0 + \Delta N}$ as the fraction of total ordinary shares owned by convertible bond holders *after* conversion.

The value of a callable, convertible bond will depend on the optimal *conversion* (from bondholders) and *call* (from shareholders) strategies. The bonds will be called by management[1] so that equity value,

$$N_0 \cdot S = V - N_B \cdot B - N_C \cdot C$$

is maximised. As evident from the above equation, maximisation of $N_0 \cdot S$ is equivalent to minimisation of $N_C \cdot C$; and since N_C is constant, management will try to minimise the value of the convertibles. This will be accomplished if C is not allowed to rise to above CP, i.e. $C \le CP$. On the other hand, convertible bonds will not be called when below CP, since this will mean a direct transfer of value from shareholders. These conditions imply that the optimal call strategy for the firm is to call when

$$C = CP \tag{4}$$

Regarding conversion, bondholders will always find it optimal to convert to ordinary shares if C falls below its conversion value in (3). This implies that $C \geq q \cdot S^{AC}$. Moreover, if C exceeds the conversion value, bondholders would not convert since this would entail a loss of value to shareholders. Thus, the optimal conversion strategy is when

$$C = q \cdot S^{AC} = Z(V - N_B \cdot B) \tag{5}$$

The value of the (callable) convertible bond, C, needs to be determined simultaneously, and subject to the optimal strategies in (4) and (5).[2] This is done in the following section.

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[1] Conditional on absence of agency problems.

1.1.1.2 The model

Much like Black and Scholes (1973) and Ingersoll (1976), Brennan and Schwartz (1980) envisage corporate liabilities like (callable, convertible or not) bonds as (a portfolio of) options, written on the firm market value and the interest rate (r). Thus the value of straight and callable/convertible bonds will depend on V and r, i.e. $B(V,r)$ and $C(V,r)$.

They assume that both the firm market value and the interest rate are random; specifically,

$$dV = [\mu_V V - Q(V,t)]dt + \sigma_V V dZ_V \tag{6}$$

$$dr = a(\mu_r - r)dt + \sigma_r r dZ_r \tag{7}$$

where

μ_V:	expected total rate of return of firm value
$Q(V,t) = I_B + I_C + D(V,t)$:	is the rate of cash payments made by the firm to all stakeholders; I_B, I_C are the coupons paid to straight and convertible bondholders respectively, and $D(V,t)$ is the dividends distributed to shareholders
σ_r:	volatility of firm market value changes
dZ_V, dZ_r:	two correlated (correlation ρ) random components (Wiener processes)
μ_r:	the long run interest rate level

[2] Such conversion and call strategies are also discussed in Ingersoll (1976).

a:　　　　　　　　　　speed of mean-reversion

σ_r:　　　　　　　　　　volatility of interest rate changes

Equations (6) and (7) are continuous-time equivalents of difference equations. Basically they imply that changes in firm market value are random and lognormally distributed while changes in the interest rate are pulled towards a level μ_r in the long run with speed a.

Applying Itô's lemma (1951) on $C(V,r)$, Brennan and Schwartz (1980) show that the value of a convertible bond satisfies the following partial differential equation

$$\frac{1}{2}\sigma_V^2 V^2 C_{VV} + \frac{1}{2}\sigma_r^2 r^2 C_{rr} + \sigma_V \sigma_r \rho V r C_{Vr} + \left[a(\mu_r - r) - \lambda r \sigma_r\right] C_r$$

$$+ \left[rV - Q(V,t)\right] C_V - rC + cF + C_t = 0 \tag{8}$$

In this equation, λ is the market price of interest rate risk and subscripts on C denote partial derivatives with respect to the respective arguments. This partial differential equation, when solved subject to specific boundary conditions, will determine the value of the convertible bond.

These boundary conditions are:

Conversion condition　　　$C(V,r,t) \geq Z(V - N_B B(V,r,t))$　　　(9)

Call condition　　　$C(V,r,t) \leq CP(t)$　　　(10)

Maturity condition

$$C(V,r,T)=\begin{cases} Z(V-N_B B(V,r,T)) & \text{if } Z(V-N_B B(V,r,T))\geq F \\ F & \text{if } Z(V-N_B B(V,r,T))\leq F\leq \frac{1}{N_C}\left(V-N_B B^0\right) \\ \frac{1}{N_C}\left(V-N_B B^0\right) & \text{if } F\geq \frac{1}{N_C}\left(V-N_B B^0\right)\geq 0 \\ 0 & \text{if } V<N_B B^0 \end{cases}$$

(11)

Bankruptcy condition $C(V,r,t)=kF \quad \text{if } V=N_B B^0+kN_C F$

(12)

Boundary conditions (9) and (10) state what happens due to conversion of the bonds to equity and due to the bonds been called by management respectively. If the convertible bonds are assumed to mature at time T and prior to senior debt maturity, then boundary condition (11) states that "convertible holders receive the conversion value if it exceeds the par value of the bond; then they receive the par value provided that this does not exceed the value of the firm less the par value of the senior debt (B^0). If this condition is not satisfied, the firm goes bankrupt and the convertible bondholders are paid after the senior bondholders" (Brennan and Schwartz (1980), p. 913). Finally, boundary condition (12) assumes that convertible bondholders will receive a fraction k of the par value in case of bankruptcy.

Solving equation (8) subject to (9)-(12) uniquely determines the value of the convertible bond. Unfortunately, this system can not be solved in closed-form, thus some numerical method must be employed. Brennan and Schwartz (1980) used the finite difference method[3] to solve this system for a number of different parameters for convertible, callable bonds.

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[3] See Hull (2003), pp. 418-427 for an introduction to the finite difference method. I describe my application of the method in subsequent sections.

Apart from providing the most sound valuation framework for convertible bonds, Brennan and Schwartz (1980) in their numerical example make a very powerful point: for a reasonable range of interest rates (0-20%), convertible bond values under constant interest rates are very close to those implied by their more complex model and thus, for practical reasons, it is preferable to use a constant interest rate assumption for valuing convertible bonds. This finding was subsequently exploited by Tsiveriotis and Fernandes (1998), a paper which is reviewed in the following section.

1.1.2 Tsiveriotis and Fernandes (1998)

In a paper targeted to practitioners, Tsiveriotis and Fernandes suggest ways to improve the convertible bonds valuation framework of Brennan and Schwartz (1980). Much like the latter authors, Tsiveriotis and Fernandes agree that convertible bonds can be accurately valued only by simultaneous pricing of the equity and fixed-income parts. However, they highlight a few aspects of previous models that seem unsatisfactory for practical applications.

One such aspect is credit risk. In practice, corporate securities are priced in such a manner so as to implicitly incorporate the possibility that the issuing corporation may not be able to cover future cash obligations. The Brennan and Schwartz (1980) framework does not account for that, thus an appropriate adjustment to this direction is needed if our valuation framework is to be of any value to practitioners. However, convertible bonds in particular present the following difficulty with respect to credit risk adjustments: their equity and bond components have different default risk exposures. For example, the equity into which a bond can be converted-since issued by the same corporation-will have zero credit risk (the issuer can always deliver its own stock), while coupon and principal payments and any put provisions will depend on the issuer's access to the required cash in time and thus introduce credit risk. Therefore, any differential equation like (8) that describes the value of a convertible bond can not be adjusted for a credit risk spread in a theoretical sound manner, since only part of the security is influenced by the possibility of default.

Another unsatisfactory aspect of previous treatments is that they involve variables like the total firm market value (V in the previous section) and its volatility (c_V) that are unobservable, difficult to estimate or proxy and impractical for front desk traders.

Tsiveriotis and Fernandes (1998) propose a model that can directly incorporate market-observed credit spreads of straight bonds in the valuation of convertibles. The approach is based on the fact that the value of the future cash payments a rational convertible bondholder will choose to receive is itself a derivative of the underlying equity and interest rates, and therefore amenable to the same valuation tools as those employed by Brennan and Schwartz (1980). This however allows the adjustment of the different cash payment components (from converted equity and from the bond) for their different credit risk exposure, making the valuation more realistic.

1.1.2.1 The model

A convertible bond u is a contingent claim on time, the underlying equity of the issuing firm, S, and on the interest rate, r, i.e. $u(S,t,r)$. As Brennan and Schwartz (1980) suggested, the optionality due to random interest rates is only a small part of the value of a convertible bond, thus I follow Tsiveriotis and Fernandes (1998) and assume that $u(.)$ only depends on the underlying stock price S and time, t, i.e. $u(S,t)$.

Using similar arguments with those in Brennan and Schwartz (1980), they show that $u(S,t)$ will satisfy the partial differential equation

$$\frac{1}{2}\sigma^2 S^2 u_{SS} + r_g S u_S + u_t - (r + r_c)u + f(u,S,t) = 0 \qquad (13)$$

where c is the volatility of equity price changes, r_g is the growth rate of the stock price, r_c is a credit spread reflecting payoff default risk and $f(u,S,t)$ describes various predetermined external flows to the derivative (e.g. coupons c). Once again, subscripts of u denote partial derivatives with respect to the respective argument.

As we have seen, equation (13) can be used to solve any convertible, puttable, callable bond once necessary boundary conditions are imposed; however one cannot assign a predetermined value for the credit spread r_c since (13) applies to both components of a convertible bond, and those have different credit risks as we have discussed.

To overcome this, Tsiveriotis and Fernandes (1998) define a security v, related to u, which is referred to as the value of the "cash-only part of the convertible bond" (COCB). The holder of a COCB is entitled to all cash flows, and no equity flows, that an optimally behaving holder of the corresponding convertible bond would receive.

By definition, the value of v will be determined by the behaviour of the stock price S and time t, $v(S,t)$, and will satisfy a similar partial differential equation

$$\frac{1}{2}\sigma^2 S^2 v_{SS} + r_g S v_S + v_t - (r + r_c)v + f(t) = 0 \tag{14}$$

and equation (13) for the value of the convertible bond $u(S,t)$ can now be written as

$$\frac{1}{2}\sigma^2 S^2 u_{SS} + r_g S u_S + u_t - r(u - v) - (r + r_c)v + f(t) = 0 \tag{15}$$

What this accomplishes is that the component of the convertible bond related to equity, $(u - v)$, is discounted with the risk-free rate, while the component related to the bond payments, v, will be discounted with a rate augmented by the necessary credit spread, $(r + r_c)$.

Equations (14) and (15) are two partial differential equations that need to be solved simultaneously for $u(S,t)$ and $v(S,t)$ subject to the appropriate boundary conditions. The boundary conditions that conclude the problem are:

Maturity conditions

$$u(S,T) = \begin{cases} aS & \text{for } S \geq B/a \\ B & \text{otherwise} \end{cases} \tag{16}$$

$$v(S,T) = \begin{cases} 0 & \text{for } S \geq B/a \\ B & \text{otherwise} \end{cases} \tag{17}$$

Conversion conditions

$$u(S,t) \geq aS \qquad \text{for } t \in [0,T] \tag{18}$$

$$v(S,t) = 0 \quad \text{if } u(S,t) : \ aS \quad \text{for } t \in [0,T] \tag{19}$$

Call conditions

$$u(S,t) : \ \max(B_c, aS) \qquad \text{for } t \in [T_c, T] \tag{20}$$

$$v(S,t) = 0 \quad \text{if } u(S,t) \geq B_c \quad \text{for } t \in [T_c, T] \tag{21}$$

Put conditions

$$u(S,t) \geq B_p \qquad \text{for } t \in [T_p, T] \tag{22}$$

$$v(S,t) = B_p \quad \text{if } u(S,t) : \ B_p \quad \text{for } t \in [T_p, T] \tag{23}$$

where

T: maturity of convertible bond

a: Conversion ratio, number of ordinary shares a convertible bond can be converted to

B: Face value of each convertible bond

c: coupon paid by convertible bond

B_c: Price at which the convertible bond can be called by firm management

T_c: Time *after* which firm management can call the convertible bond

B_p: Cash amount received by convertible bondholder when deciding to put the bond back to the issuing firm

T_c: Time *after* which convertible bondholders can put the convertible bond back to the issuing firm

As evident from the boundary conditions, the COCB value $v(S,t)$ assumes non-zero values only when a cash payment takes place, either at maturity (equation (17)) or when put (equation (23)).

The system of equations (14)-(15) subject to (16)-(23) completely describes the valuation problem and needs to be solved for the unknown functions $u(S,t)$, $v(S,t)$ and the levels of the stock price S at which conversion, call or put exercises take place. Unfortunately, the system is highly non-linear and can not be solved in closed-form as the Black and Scholes (1973) European option problem. Thus, some numerical scheme must be employed in order to determine the convertible bond price for a set of parameter values. In the next section we discuss how to employ the finite difference method to numerically evaluate the model proposed by Tsiveriotis and Fernandes (1998).

1.3 Finite Difference method

The finite difference method was brought to finance from engineering by Brennan and Schwartz (1977) who first used it to price American style options. Since then, this method and the binomial asset pricing model of Cox, Ross and Rubinstein (1979) are considered the most widely applied numerical pricing methods.

The finite difference method involves approximating the differential equation that a contingent claim satisfies by discrete-time difference equations which are solved iteratively, from known boundary conditions and backwards in time, until the price of the contingent claim is determined. Two finite difference schemes have been proposed in the literature: the *implicit* method, which while very robust, is very computationally demanding and the *explicit* method which is easier to compute but less stable.

Following Tsiveriotis and Fernandes (1998) I will employ the explicit finite difference method to tackle the problem at hand. First, I apply the transformations $x = \ln(S)$ and $\tau = T - t$ to differential equations (14) and (15), which now become

$$\frac{1}{2}\sigma^2 u_{xx} + \left(r_g - \frac{\sigma^2}{2} \right) u_S - r(u - v) - (r + r_c)v + f(t) - u_\tau = 0 \tag{24}$$

$$\frac{1}{2}\sigma^2 v_{xx} + \left(r_g - \frac{\sigma^2}{2} \right) v_S - (r + r_c)v + f(t) - v_\tau = 0 \tag{25}$$

The above differential equations are discretised in the following fashion: Define N equally spaced prices x_i, $i = 1, \ldots, N$, each with a distance h. Divide the time to maturity of the convertible bond T into steps of length $\Delta\tau$. Under this discretisation scheme, equations (24) and (25) become

$$\frac{u_i^{k+1} - u_i^k}{\Delta\tau} = \frac{\sigma^2}{2} \frac{u_{i+1}^k - 2u_i^k + u_{i-1}^k}{h^2} + \left(r_g - \frac{\sigma^2}{2} \right) \frac{u_{i+1}^k - u_{i-1}^k}{2h} - r\left(u_i^k - v_i^k\right) - (r + r_c)v_i^k + f(\Delta\tau) \tag{26}$$

$$\frac{v_i^{k+1} - v_i^k}{\Delta\tau} = \frac{\sigma^2}{2} \frac{v_{i+1}^k - 2v_i^k + v_{i-1}^k}{h^2} + \left(r_g - \frac{\sigma^2}{2} \right) \frac{u_{i+1}^k - u_{i-1}^k}{2h} - (r + r_c)v_i^k + f(\Delta\tau) \tag{27}$$

The solution proceeds as follows: At the maturity of the convertible bond T, conditions (16) and (17) are applied and $\left(u^0, v^0\right)$ are determined. After that, we work backwards in time: At time step $k + 1$ (time $\tau = (k + 1)\Delta\tau$) we start with $\left(u^k, v^k\right)$, and using (26) we calculate u^{k+1} and check for early conversion, call and/or put exercises using (18), (20) and (22). Then equation (27) is used to estimate v^{k+1} and early conversion, call and/or put exercises are checked via (19), (21) and (23). Iteratively, the steps are repeated until the initial prices are determined.

In the next chapter we apply the above-described finite difference algorithm is applied to the valuation of the convertible bonds issued by XXX Ltd.

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