6.5

Friday, May 10, 2024 12:44 PM

ΠΑΡΑΔΕΙΓΜΑ 6.25

$X =$ ΖΑΡΙ 1

$Y =$ ΖΑΡΙ 2

$Z =$ ΖΑΡΙ 3

$6 \cdot 6 \cdot 6 = 6^3 = 216$

$$P(P_1 = k_1, P_2 = k_2, P_3 = k_3) = P(P_1 = k_1)P(P_2 = k_2)P(P_3 = k_3)$$
$$= \frac{1}{6} \cdot \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{216}$$

$$P(X+Y+Z = 18) = \frac{1}{216}$$

$$P(X+Y+Z = 17) = P((6,6,5), (6,5,6), (5,6,6)) = \frac{3}{216}$$

$$P(X+Y+Z = 16) = P((6,6,4)\ (6,4,6), (4,6,6)$$
$$(6,5,5), (5,6,5), (5,5,6)) = \frac{6}{216}$$

$$P_{X+Y}(2) = P_X(1)P_Y(1)$$

$$P_{X+Y}(3) = P_X(2)P_Y(1) + P_X(1)P_Y(2)$$

$$P_{X+Y}(4) = P_X(3)P_Y(1) + P_X(2)P_Y(2) + P_X(1)P_Y(3)$$

$$P_{X+Y}(5) = P_X(4)P_Y(1) + P_X(3)P_Y(2) + P_X(2)P_Y(3) + P_X(1)P_Y(4)$$

$$P_{X+Y}(7) = P_X(6)P_Y(1) + \dots + P_X(1)P_Y(6)$$

$$P_{X+Y}(12) = P_X(6)P_Y(6)$$

$W = X+Y$ $\qquad P_W(w) \qquad w = 2, 12$

$$P_{X+Y+Z}(11) = P_{W+Z}(11)$$

$$P(W=10)P(Z=1) + P(W=9)P(Z=2) + P(W=8, Z=3)$$
$$+ \dots\ P(W=5)P(Z=6) + P(W=4)\cancel{P(Z=7)} \to 0$$

$P(W=11)P(Z=0)$
$\quad\quad\quad\quad\quad\quad\ \ \shortparallel$
$\quad\quad\quad\quad\quad\quad\ \ 0$

ΠΑΡΑΔΕΙΓΜΑ 6.22

X, Y ΣΥΝΕΛΙΞΗ

CONVOLUTION

$S_X = \mathbb{Z} = S_Y = \mathbb{Z}$

$$P_{X+Y}(m) = \sum_{k=-\infty}^{\infty} P_X(k) P_Y(m-k), \quad m \in \mathbb{Z}$$

X	...	-3	-2	-1	0	1	2	3	4	5	...	x
	...	$1/16$	$1/8$	$1/4$	$1/4$	$1/8$	$1/16$	...				$P(X=x)$

Y	...	-3	-2	-1	0	1	2	3	4	5	6	7	...
0	0	0	0	0	0	$1/6$	$1/6$	$1/6$	$1/6$	$1/6$	$1/6$	0	...

$S_X = \mathbb{Z}$

$X + Y = m$

$\vdots$

$X=-2 \quad Y=m+2$

$X=-1 \quad Y=m+1$

$X=0 \quad Y=m$

$X=1 \quad Y=m-1$

$X=2 \quad Y=m-2$

$\vdots$

$$P(X+Y=m) = \dots + P(X=-2, Y=m+2) + P(X=-1, Y=m+1) + P(X=0, Y=m) + P(X=+1, Y=m-1) + \dots$$

$$= \dots + P(X=-2)P(Y=m+2) + P(X=-1)P(Y=m+1) + P(X=0)P(Y=m) + P(X=1)P(Y=m-1) + \dots$$

$$= \dots + P_X(-2)P_Y(m+2) + P_X(-1)P_Y(m+1) + P_X(0)P_Y(m) + P_X(1)P_Y(m-1) + \dots$$

$$p_X(0)\,p_Y(m) + p_X(1)\,p_Y(m-1) + \dots$$

$$= \sum_{k=-\infty}^{\infty} p_X(k)\,p_Y(m-k)$$

$$= \lim_{k \to -\infty} \sum_{k}^{0} p_X(k)\,p_Y(m-k) + \lim_{k \to \infty} \sum_{1}^{k} p_X(k)\,p_Y(m-k)$$

$X \quad p_X(m)$

$Y \quad p_Y(m)$

ΣΥΝΕΛΙΞΗ
CONVOLUTION

$X+Y \qquad \sum_{k=-\infty}^{\infty} p_X(k)\,p_Y(m-k)$

$\mathrm{conv}(x, y)$

ΠΑΡΑΔΕΙΓΜΑ 6.23

X, Y ΑΝΕΞΑΡΤΗΤΕΣ

$X \sim \mathrm{POIS}(\lambda) \qquad Y \sim \mathrm{POIS}(\mu)$

$$P(X=k) = e^{-\lambda}\frac{\lambda^k}{k!} \qquad k=0,1,\dots$$

$$P(Y=k) = e^{-\mu}\frac{\mu^k}{k!} \qquad k=0,1,\dots$$

$Z = X+Y$

$$P(Z=m) = \sum_{\cancel{k=-\infty}\ k=0}^{\cancel{\infty}\ m} p_X(k)\,p_Y(m-k)$$

$$\sum^{m} p_X(k)\,p_Y(m-k)$$

$$\sum_{k=0}^{m} P_X(k)\, P_Y(m-k)$$

$$= \sum_{k=0}^{m} e^{-\lambda} \frac{\lambda^k}{k!} e^{-\mu} \frac{\mu^{m-k}}{(m-k)!} =$$

$$e^{-\lambda-\mu} \sum_{k=0}^{m} \frac{m!}{k!\,(m-k)!} \frac{\lambda^k \mu^{m-k}}{m!} =$$

$$\frac{e^{-(\lambda+\mu)}}{m!} \boxed{\sum_{k=0}^{m} \binom{m}{k} \lambda^k \mu^{m-k}} \quad \boxed{= (\lambda+\mu)^m}$$

ΔΙΩΝΥΜΙΚΟ ΘΕΩΡΗΜΑ $\Rightarrow$

$m=2$ $\quad (\lambda+\mu)^2 = \sum_{k=0}^{2} \binom{2}{k} \lambda^k \mu^{2-k}$

$$= \mu^2 + 2\lambda\cdot\mu + \lambda^2$$

$m=3$ $\quad (\lambda+\mu)^3 = \sum_{k=0}^{3} \binom{3}{k} \lambda^k \mu^{3-k}$

$$\mu^3 + 3\lambda\mu^2 + 3\lambda^2\mu + \lambda^3$$

$$\boxed{P(X+Y=m) = e^{-(\lambda+\mu)} \frac{(\lambda+\mu)^m}{m!}} \quad \lambda+\mu$$

$$\left.\begin{array}{l} X \sim \text{POISSON}(\lambda) \\ Y \sim \text{POISSON}(\mu) \end{array}\right\} \Rightarrow X+Y \sim \text{POISSON}(\lambda+\mu)$$

100000 λ

100000 λ

120.000 μ

1
2
3
4

ΠΑΡΑΔΕΙΓΜΑ 6.24

$X \sim \text{ΔΙΩΝ}(n_1, p)$

$Y \sim \text{ΔΙΩΝ}(n_2, p)$

$Z = X + Y \sim \text{ΔΙΩΝ}(n_1 + n_2, p)$

$$P_{X+Y}(m) = \sum_{k=-\infty}^{\infty} P_X(k)\, P_Y(m-k)$$

$\min\{n_1, m\}$

$k = \max\{0, m-n_2\}$

1) $k \geq 0$

2) $k \leq n_1$

3) $m-k \geq 0 \Leftrightarrow k \leq m$

4) $m-k \leq n_2 \Leftrightarrow k \geq m-n_2$

$$\Rightarrow \quad P_{X+Y}(m) = \sum_{k=\max\{0,m-n_2\}}^{\min(n_1,m)} \binom{n_1}{k} p^k (1-p)^{n_1-k} \binom{n_2}{m-k} p^{m-k} (1-p)^{n_2-(m-k)}$$

$$p^m (1-p)^{n_1+n_2-m} \sum_{k=\max\{0,m-n_2\}}^{\min(n_1,m)} \binom{n_1}{k}\binom{n_2}{m-k} = \binom{n_1+n_2}{m}$$

ΤΑΥΤΟΤΗΤΑ VANDERMONDE

$$\Rightarrow \quad P_{X+Y}(m) = p^m (1-p)^{n_1+n_2-m} \binom{n_1+n_2}{m}$$

$$\Rightarrow \quad X+Y \sim \text{ΔΙΩΝ}(n_1+n_2, p)$$

ΑΠΟΔΕΙΞΗ VANDERMONDE

ΑΠΟΔΕΙΞΗ VANDERMONDE

$$\sum_{k=\max\{0,\,m-n_2\}}^{\min(n_1,m)} \binom{n_1}{k}\binom{n_2}{m-k} = \binom{n_1+n_2}{m}$$

ΕΣΤΩ n_1 ΑΝΤΡΕΣ

n_2 ΓΥΝΑΙΚΕΣ

(ΣΥΝΟΛΟ n_1+n_2 ΑΤΟΜΑ)

ΕΠΙΛΕΓΩ m ΑΠΟ ΑΥΤΑ. $\binom{n_1+n_2}{m}$

k ΑΝΤΡΕΣ

$$\binom{n_1}{k}\binom{n_2}{m-k}$$

$$\boxed{\sum_{k=\max\{0,\,m-n_2\}}^{\min\{n_1,m\}} \binom{n_1}{k}\binom{n_2}{m-k} = \binom{n_1+n_2}{m}}$$

$k \le n_1$

$k \ge 0$

$m-k \ge 0 \Leftrightarrow k \le m$

$n_2 \ge m-k \Leftrightarrow k \ge m-n_2$