6.4

Wednesday, May 1, 2024 3:51 PM

ΑΝΕΞΑΡΤΗΤΕΣ Τ.Μ.

$P(AB) = P(A)P(B)$

ορισμος 6.3 X, Y S_X S_Y

Τ.Μ ΑΝΕΞΑΡΤΗΤΕΣ ΟΤΑΝ:

$\forall A \subseteq S_X, \forall B \subseteq S_Y$

$P(X \in A, Y \in B) = P(X \in A) P(Y \in B)$

10 10

$|X| = n \quad |\mathcal{P}_X| = 2^n$

ΛΗΜΜΑ 6.4

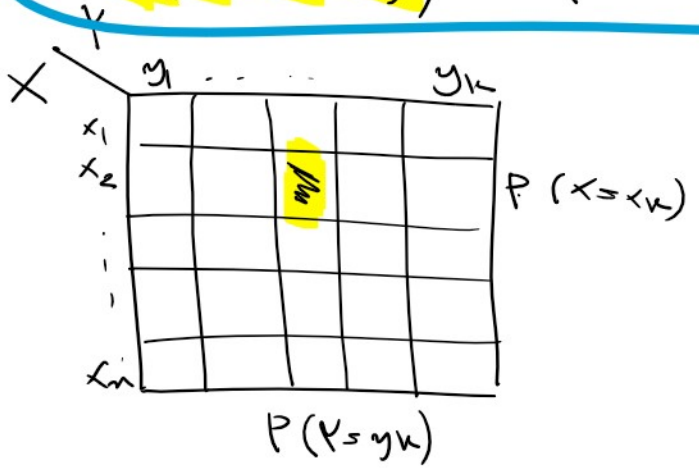
ΔΥΟ ΔΙΑΚΡΙΤΕΣ Τ.Μ. X, Y ΑΝΕΞΑΡΤΗΤΕΣ

$\Leftrightarrow$

$P_{XY}(x,y) = P_X(x) P_Y(y) \quad \forall x \in S_X, \forall y \in S_Y$

$\Leftrightarrow$

$P(X=x, Y=y) = P(X=x) P(Y=y)$



x / y	1	2	3	4	5	6	$p_Y(y)$

y \ x	1	2	3	4	5	6	$p_Y(y)$
1	1/36	1/36	1/36	1/36	1/36	1/36	1/6
2	1/36	1/36	1/36	1/36	1/36	1/36	1/6
3	1/36	1/36	1/36	1/36	1/36	1/36	1/6
4	1/36	1/36	1/36	1/36	1/36	1/36	1/6
5	1/36	1/36	1/36	1/36	1/36	1/36	1/6
6	1/36	1/36	1/36	1/36	1/36	1/36	1/6
$p_X(x)$	1/6	1/6	1/6	1/6	1/6	1/6	

y \ x	1	2	3	4	5	6	$p_Y(y)$
1	1/6	0	0	0	0	0	1/6
2	0	1/6	0	0	0	0	1/6
3	0	0	1/6	0	0	0	1/6
4	0	0	0	1/6	0	0	1/6
5	0	0	0	0	1/6	0	1/6
6	0	0	0	0	0	1/6	1/6
$p_X(x)$	1/6	1/6	1/6	1/6	1/6	1/6	

$P(X \in \{2,3\}, Y \in \{4,5\})$

y \ x	1	2	3	4	5	6	$p_Y(y)$
1	1/36	1/36	1/36	1/36	1/36	1/36	1/6
2	1/36	1/36	[illegible]	2/36	1/36	1/36	1/6
3	1/36	1/36	2/36	[illegible]	1/36	1/36	1/6
4	1/36	1/36	1/36	1/36	1/36	1/36	1/6
5	1/36	1/36	1/36	1/36	1/36	1/36	1/6
6	1/36	1/36	1/36	1/36	1/36	1/36	1/6
$p_X(x)$	1/6	1/6	1/6	1/6	1/6	1/6	

ΑΠΟΔΕΙΞΗ ΑΝΑΜΕΣΑΣ Θ. 4

$(\Rightarrow)$ X, Y ΑΝΕΞΑΡΤΗΤΕΣ

$A \subseteq S_X \quad B \subseteq S_Y$

$P(X \in A, Y \in B) = P(X \in A) P(Y \in B) \rightarrow (*)$

ΕΣΤΩ $x \in S_X$ $y \in S_Y$. ΟΠΟΤΕ $A = \{x\}$

$B = \{y\}$

$(*)$ $P(X \in \{x\}, Y \in \{y\}) = P(X \in \{x\}) P(Y \in \{y\})$

$P(X = x, Y = y) = P(X = x) P(Y = y)$

$(\Leftarrow)$

$(\Leftarrow)$ ΕΣΤΩ $A \subseteq S_X$, $B \subseteq S_Y$

$$P(X \in A, Y \in B) =$$

$$\sum_{x \in A, y \in B} P_{XY}(x,y) =$$

$$\sum_{x \in A, y \in B} P_X(x) P_Y(y) =$$

$$\sum_{x \in A} \left(\sum_{y \in B} P_X(x) P_Y(y) \right)$$

$$\sum_{x \in A} \left(P_X(x) \left(\sum_{y \in B} P_Y(y) \right) \right)$$

$$\left(\sum_{y \in B} P_Y(y) \right) \left(\sum_{x \in A} P_X(x) \right)$$

$$= P(Y \in B)\, P(X \in A)$$

ΠΑΡΑΤΗΡΗΣΗ:

$$P(Y = y \mid X = x) = \frac{P(X = x, Y = y)}{P(X = x)} = \frac{P_{XY}(x,y)}{P_X(x)}$$

$$\frac{P_X(x) P_Y(y)}{P_X(x)} = P_Y(y) = P(Y = y)$$

ΠΑΡΑΔΕΙΓΜΑ 6.1 β

$X \sim \text{Geom}(p_1)$

$Z = \min(X, Y$

$X \sim \text{GEOM}(p_1)$

$Y \sim \text{GEOM}(p_2)$

$Z = \min(X, Y)$

$P_Z(z) = ?$ $\quad P(Z = z)$

$(k \in \mathbb{N})$

$$P(Z \geq k) = P(\min(X,Y) \geq k) =$$

$$= P(X \geq k, Y \geq k) \overset{\text{INDEPENDENCIA}}{=} \circledast$$

$\min(a,b) \geq k \iff a \geq k, b \geq k$ ←

$\min(a,b) \leq k \iff a \leq k, b \leq k$ OX!

$$\circledast \; P(X \geq k) P(Y \geq k)$$

$$(1-p_1)^{k-1} (1-p_2)^{k-1} \Rightarrow$$

$$P(Z \geq k) = (1-p_1)^{k-1} (1-p_2)^{k-1} = \left[(1-p_1)(1-p_2)\right]^{k-1}$$

$$P(Z \geq k) = P(Z = k) + P(Z \geq k+1)$$

$$\Rightarrow P(Z = k) = P(Z \geq k) - P(Z \geq k+1)$$

$$= \left[(1-p_1)(1-p_2)\right]^{k-1} - \left[(1-p_1)(1-p_2)\right]^{k}$$

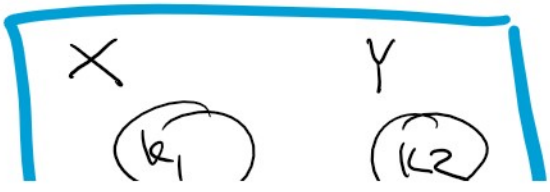
$$= \left[1 - (1-p_1)(1-p_2)\right] \left((1-p_1)(1-p_2)\right)^{k-1}$$

$q = 1 - (1-p_1)(1-p_2)$

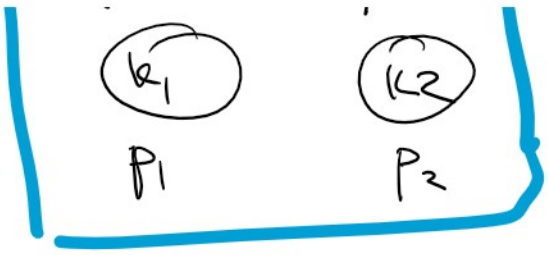
$$\boxed{P(Z = k) = q(1-q)^{k-1}}$$

$$\Rightarrow Z \sim \text{GEOM}(q)$$

←



$Z = \min(X, Y)$



$t = \min(X, Y)$

$P(\Phi) = 1 - P(\Phi') = 1 - (1-p_1)(1-p_2)$

$q = 1-(1-p_1)(1-p_2)$

$= 1 - 1 + p_1 + p_2 - p_1 p_2$

$p_1 + p_2 - p_1 p_2$

HAT PROBLEM

100 A → ΠΑΡΤΥ → 100 A
100 K → 100 K (ΑΛΛΑ)

$Y_1 = \begin{cases} 1 \\ 0 \end{cases}$ ΑΤΟΜΟ 1 ΜΕ ΚΑΠΟΙΟ ΤΥΠ
ΑΛΛΙΩΣ

$Y_2 \quad Y_3 \quad \dots$

$E\left[\sum_{i=1}^{100} Y_i\right], \quad \sum_{i=1}^{100} E(Y_i) = 100\,E(Y_1) = 100 \cdot \frac{1}{100} = 1$

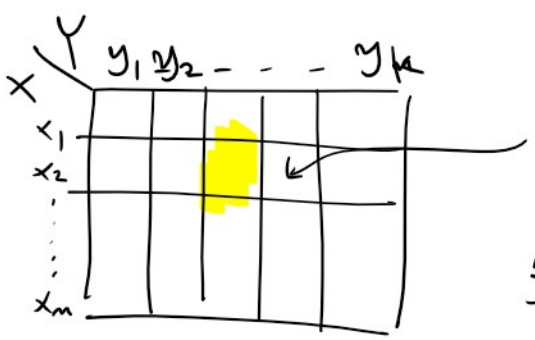
$E(Y_1) = 1 \cdot \frac{1}{100} + 0 \cdot \frac{99}{100} = \frac{1}{100}$

HAT PROBLEM

ΞΕΚΙΝΑΜΕ 11:15

ΕΠΑΝΑΛΗΨΗ

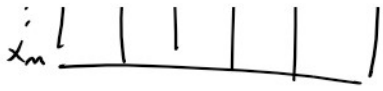
X, Y



(i,j)

$P(X = x_i, Y = y_j)$

$\triangleq P_{XY}(x_i, y_i)$



$\Rightarrow P_{XY}(x_i, y_i)$

ΑΣΚΗΣΗ 44

ΜΕ ΤΡ ΓΡΑΦΗ → ΕΠΙΤΥΧΗΜΕΝΗ, Μ.Π. $\frac{1}{2}$ (X)

ΙΚΑΝΟΠΟΙΗΤΙΚΗ, Μ.Π. $\frac{1}{3}$ (Y)

3 ΜΕΤΑΓΡΑΦΕΣ

ΑΠΟΤΥΧΗΜΕΝΗ, Μ.Π. $\frac{1}{6}$ (Z)

$$\boxed{X+Y+Z=3}$$

(α) $P(Y=y)$ $\quad y=0,1,2,3$

$Y \sim Bin(N,p) \quad N=3 \quad p=\frac{1}{3}$

$$P(Y=y)=\binom{3}{y}\left(\frac{1}{3}\right)^y\left(1-\frac{1}{3}\right)^{3-y}$$

(b) $P_{XY}(x,y)$

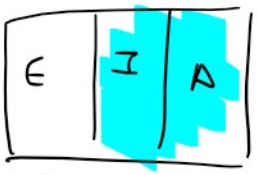
$X=0,1,2,3$

$Y=0,1,2,3$

X \ Y	0	1	2	3
0				
1		1/6		0
2		1/4	0	0
3	1/8	0	0	0

$$P(X=1,Y=1)=P(X=1)\,P(Y=1\mid X=1)$$

$$P(X=x,Y=y) \quad \binom{3}{1}\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^{3-1}$$



$$P(I\mid E')=\frac{P(I\cap E')}{P(E')}=\frac{P(I)}{1-P(\bar{I})-P(A)}=\frac{1/3}{1/2}=\frac{2}{3}$$

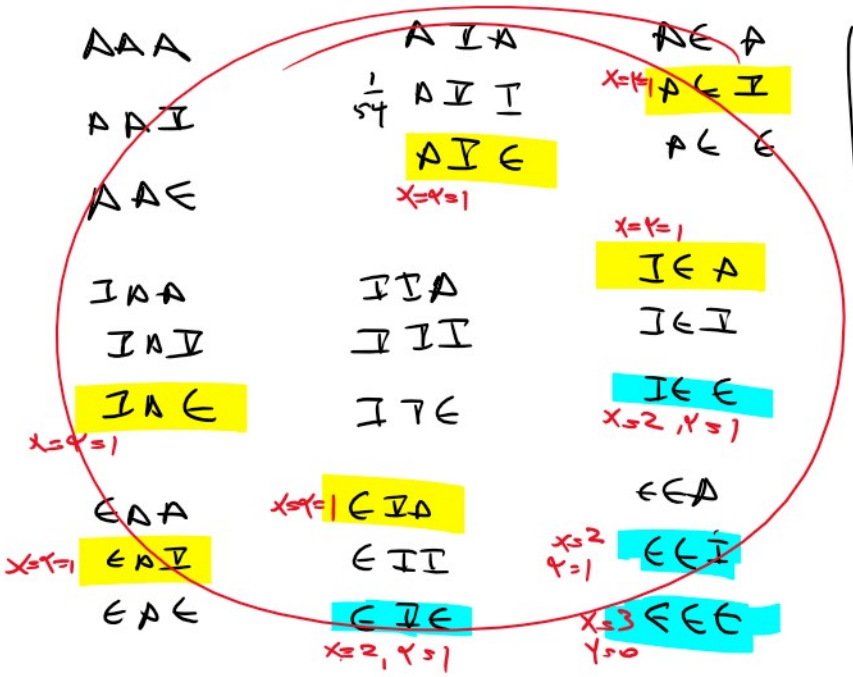
Ι = "ΙΚΑΝΟΠΟΙΗΤΙΚΗ ΜΕΤΑΓΡΑΦΗ"

Ε = " ΕΠΙΤΥΧΗΜΕΝΗ -"- "

Α = " ΑΠΟΤΥΧΗΜΕΝΗ -"- "

$N=2, \quad p=\frac{2}{3}$ $\qquad P(Y=1\mid X=1)=\binom{2}{1}\left(\frac{2}{3}\right)^1\left(1-\frac{2}{3}\right)^{2-1}$

$N=2, \quad p=\frac{1}{3}$



$P(AII) =$

$P(A)P(I)P(I) =$

$\frac{1}{6} \cdot \frac{1}{3} \quad \frac{1}{3}$

$P(X=1, Y=1) =$

$P(AEI) + P(AIE)$

$+ P(IEA) + P(IAE)$

$+ P(EIA) + P(EAI)$

$= 6 \cdot \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{6} = \frac{1}{6}$

$P(X=3) = P(EEE) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$

$P(X=3, Y=0) = P(X=3) = \frac{1}{8}$

$P(X=2, Y=1) = P(EEI) + P(EIE) + P(IEE)$

$= 3 \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{4}$

ΑΣΚΗΣΗ 33

Α ΨΗΦΟΣ

Β ΚΕΡΔΙΖΩΝ

$X \sim \text{Geom}(p)$

$P(X > k) = (1-p)^k$

$P(X \leq \alpha) \geq 0.8$

$(1-p)^k \leq 0.2$

$$(1-p)^k \leq 0.2$$

$\Leftrightarrow$ $k \log(1-p) \leq \log 0.2$ $\Leftrightarrow$

$$k \geq \frac{\log 0.2}{\log(1-p)}$$

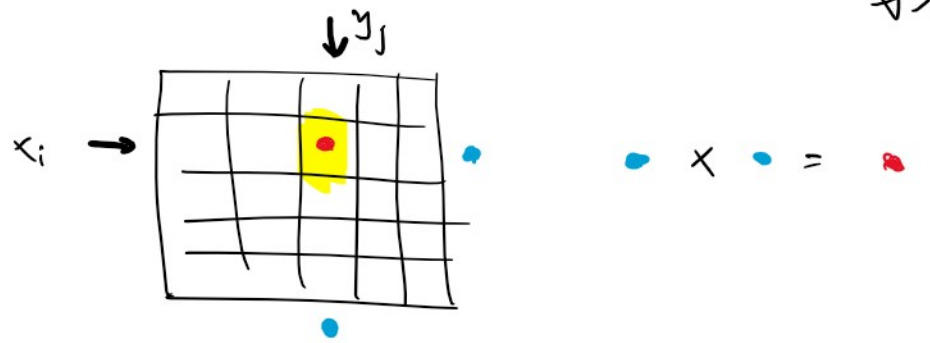
X, Y ΑΝΕΞΑΡΤΗΤΕΣ

$$P(X \in A,\ Y \in B) = P(X \in A)\, P(Y \in B) \quad \forall A \subseteq S_X,\ \forall B \subseteq S_Y$$

ΚΡΙΤΗΡΙΟ:

X, Y ΑΝΕΞΑΡΤΗΤΕΣ $\Leftrightarrow$

$$P_{XY}(x,y) = P_X(x)\, P_Y(y) \quad \forall x \in S_X,\ \forall y \in S_Y$$



ΛΗΜΜΑ 6.2 X, Y $Z = g(X,Y)$

$$E(Z) = E[g(X,Y)] = \sum_{\substack{x \in S_X \\ y \in S_Y}} g(x,y)\, P_{XY}(x,y)$$

$$E(XY) = \sum_{\substack{x \in S_X \\ y \in S_Y}} x \cdot y\, P_{XY}(x,y)$$

$$x \in S_X,\ y \in S_Y$$

$$Cov(X,Y) \triangleq E\left[(X - E(X))(Y - E(Y))\right]$$

$$= E[XY] - E(X)\,E(Y)$$

ΛΗΜΜΑ 6.5

X, Y $\quad E[g(X,Y)] = \sum_{x \in S_X,\ y \in S_Y} g(x,y)\, P_{XY}(x,y)$

X, Y ΑΝΕΞΑΡΤΗΤΕΣ

1) $E[g(X)h(Y)] = E[g(X)]\cdot E[h(Y)]$

($g(x,y) = e^{x+y} = e^x \cdot e^y$

$g(x,y) = \log(x+y) \neq f_1(x) f_2(y)$)

ΕΙΔΙΚΕΣ ΠΕΡΙΠΤΩΣΕΙΣ:

X, Y ΑΝΕΞΑΡΤΗΤΕΣ $\quad E(X \cdot Y) = E(X)\,E(Y)$

↳ $Cov(X,Y) = E(XY) - E(X)E(Y) = 0$

↳ $Var(X+Y) = Var(X) + Var(Y)$

ΑΠΟΔΕΙΞΗ: X, Y ΑΝΕΞΑΡΤΗΤΕΣ $\quad E[g(X)h(Y)] = E[g(X)]\cdot E[h(Y)]$

$$E[g(X)h(Y)] = \sum_{x \in S_X,\ y \in S_Y} P_{XY}(x,y)\, g(x)\, h(y) =$$

$P_{XY}(x,y) = P_X(x) P_Y(y)$

$$\sum_{x \in S_X} \left(\sum_{y \in S_Y} P_X(x) P_Y(y) g(x) h(y) \right)$$

$$\sum \sum$$

$$\sum_{x\in S_X}\sum_{y\in S_Y} \qquad \sum_{x\in S_X}\left(\sum_{y\in S_Y} p_X(x)g(x)p_Y(y)h(y)\right)$$

$$\sum_{x\in S_X}\left(p_X(x)g(x)\left(\sum_{y\in S_Y} p_Y(y)h(y)\right)\right)$$

$$\left(\sum_{y\in S_Y} p_Y(y)h(y)\right)\left(\sum_{x\in S_X} p_X(x)g(x)\right)$$

$$= E[g(Y)] \cdot E(g(X))$$

ΠΑΡΑΔΕΙΓΜΑ 6.19

$$VAR(X+Y) = VAR(X) + VAR(Y) = \frac{35}{12} + \frac{35}{12} = \frac{35}{6}$$

ΠΑΡΑΔΕΙΓΜΑ 6.20

$X \sim BERNOULLI\left(\frac{1}{2}\right)$

$$X = \begin{cases} 1 & \text{M.Π. } \frac{1}{2} \\ 0 & \text{M.Π. } \frac{1}{2} \end{cases} \qquad Y = -X$$

$$Y = \begin{cases} -1 & \text{M.Π. } \frac{1}{2} \\ 0 & \text{M.Π. } \frac{1}{2} \end{cases} \Rightarrow X+Y = 0$$

$$VAR(X+Y) = VAR(X) + VAR(Y) + 2\,cov(X,Y)$$
$$0 \qquad 1/2 \qquad 1/2 \qquad -\frac{1}{4}$$

$$VAR(X) = p\cdot(1-p) = \frac{1}{4}$$

$$= E(X^2) - (E(X))^2$$

$$= \left(0^2\cdot\frac{1}{2} + 1^2\cdot\frac{1}{2}\right) - \left(\frac{1}{2}\right)^2 = \frac{1}{4}$$

$$\frac{1}{2} \qquad \frac{1}{4}$$

$VAR(aX) = a^2\,VAR(X)$

$VAR(Y)$ $\quad VAR(\quad)$ $\quad ^2$

$\frac{1}{2}$ $\frac{1}{4}$ or Var(X)

$$\mathrm{Var}(Y) = \mathrm{Var}(-X) = (-1)^2 \mathrm{Var}(X) = \frac{1}{4}$$

$$E(Y) = (-1)\frac{1}{2} + 0\,\frac{1}{2} = -\frac{1}{2}$$

$$E(Y^2) = (-1)^2\frac{1}{2} + 0^2\frac{1}{2} = \frac{1}{2}$$

$$\mathrm{Var}(Y) = E(Y^2) - (E(Y))^2 = \frac{1}{4}$$

$X+Y=0$ M.D. 1

$$E(X+Y) = 0 \cdot 1 = 0$$

$$E((X+Y)^2) = 0^2 \cdot 1 = 0$$

$$\mathrm{Var}(X+Y) = 0$$

$$\mathrm{Cov}(X,Y) = E(XY) - E(X)E(Y) = -\frac{1}{4}$$

($E(X)=\frac{1}{2}$, $E(Y)=-\frac{1}{2}$)

$$E(XY) = \sum_{\substack{x\in S_X \\ y\in S_Y}} x\cdot y\, P(x,y) = -\frac{1}{2}$$

X \ Y	0	-1
0	1/2	0
1	0	1/2

$$= \frac{1}{2}\cdot 0\cdot 0 + \frac{1}{2}(1)(-1)$$

$$= -\frac{1}{2}$$

Παράδειγμα 6.21

Y \ X	-1	0	1	$P_Y(y)$
-1	0	1/4	0	1/4
0	1/4	0	1/4	1/2
1	0	1/4	0	1/4
$P_X(x)$	1/4	1/2	1/4	

$x=0,\ y=0$

$$P_{XY}(x,y) = P_X(x)P_Y(y)$$

$$0 \qquad \frac{1}{2}\cdot\frac{1}{2}$$

$$\mathrm{Cov}(X,Y) = E(XY) - E(X)E(Y) = 0 - 0\cdot 0 = 0$$

$$\mathrm{Cov}(X,Y) = E(XY) - E(X)E(Y) = 0 - 0 \cdot 0 = 0$$

$$E(X) = -1 \cdot \frac{1}{4} + 0 \cdot \frac{1}{2} + 1 \cdot \frac{1}{4} = 0 = E(Y)$$

$$E(XY) = 0(-1)(-1) + \frac{1}{4} \cdot 0(-1) + 0(-1)(1) + \dots = 0$$

ΠΑΡΑΤΗΡΗΣΗ 6.4

$\mathrm{Cov}(X,Y) = 0$ ΤΟΤΕ X, Y ΔΕΝ ΕΙΝΑΙ ΑΝΕΞΑΡΤΗΤΕΣ