Notes:

1. Duration: 2.5 hours
2. Explain everything carefully. You will be graded on the clarity of your arguments.

Exercises:

1. *(1 point)* Prove that the convex hull conv$A$ of a set A is the smallest convex set that contains A, in the sense that it is a subset of any convex set C that contains A.

   Solution: Let any convex set C containing A. Let any point $a \in$ conv$A$. Then $a = \sum_{i=1}^{n} \theta_i a_i$, where $a_i \in A$, and so also to C, therefore, since C is convex, a will also belong to C. Therefore, $a \in A \Rightarrow a \in C$, and the result follows.

2. *(1.5 points)* Let $y, x_2, x_2, \ldots, x_p \in \mathbb{R}^n$. Prove that $y \in$ conv$\{(x_1, x_2, \ldots, x_p\})$ if and only if

   $$\text{conv}(\{x_1, x_2, \ldots, x_p\}) = \text{conv}\{(y, x_1, x_2, \ldots, x_p)\}.$$

   With conv$\{S\}$ we denote the convex hull of the set $S \subseteq \mathbb{R}^n$.

   Solution: First, let us assume that $y \in$ conv$\{(x_1, x_2, \ldots, x_p\})$. To prove that the two sets are equal, we will prove that one is the subset of the other. To this effect, first observe that conv$(\{x_1, x_2, \ldots, x_p\}) \subseteq$ conv$\{(y, x_1, x_2, \ldots, x_p)\}$, because any point that can be written as a convex combination of the set $\{x_1, x_2, \ldots, x_p\}$ can also be written as a convex combination of the points in $\{y, x_1, x_2, \ldots, x_p\}$). To prove that conv$\{(y, x_1, x_2, \ldots, x_p)\} \subseteq$ conv$\{(x_1, x_2, \ldots, x_p)\}$, let some $z \in \{(y, x_1, x_2, \ldots, x_p)\}$. We have

   $$\begin{aligned} z &= \sum_{i=1}^{n} a_i x_i + a_{n+1} y, \\ y &= \sum_{i=1}^{n} b_i x_i. \end{aligned}$$

   Combining the above,

   $$z = \sum_{i=1}^{n} (a_i + a_{n+1} b_i) x_i,$$

   and the result follows.

   Now let us assume that the equality of the two sets holds. Then, $y \in$ conv$(\{x_1, x_2, \ldots, x_p\}) =$ conv$\{(y, x_1, x_2, \ldots, x_p)\}$, and so the results follows.

3. *(2.5 points)* Consider the problem

   $$\begin{aligned} \text{minimize:} \quad & x_1^2 + x_2^2 + x_3^2 + x_4^2 \\ \text{subject to:} \quad & x_1 + x_2 + x_3 + x_4 = 1, \qquad x_4 \leq K, \end{aligned}$$

   where K is a parameter.

   (α′) Bring the problem in the standard form of an optimization problem.

   (β′) Is the problem convex? Explain?

   (γ′) Write the Lagrangian.

   (δ′) Write the KKT conditions for this problem.

   (ε′) Find the solution of the problem as a function of the parameter K.

   Solution:

   (α′)

   $$\begin{aligned} \text{minimize:} \quad & f_0(x) = x_1^2 + x_2^2 + x_3^2 + x_4^2 \\ \text{subject to:} \quad & h_1(x) = x_1 + x_2 + x_3 + x_4 - 1 = 0, \qquad f_1(x) = x_4 - K \leq 0, \end{aligned}$$

   (β′) Yes it is.

   (γ′)

   $$L(x, \lambda, \mu) = x_1^2 + x_2^2 + x_3^2 + x_4^2 + \lambda(x_4 - K) + \mu(x_1 + x_2 + x_3 + x_4 - 1)$$

(δ΄)

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 1, \\ x_4 - K &\leq 0, \\ \lambda &\geq 0, \\ \lambda(x_4 - K) &= 0, \\ 2\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} &= 0. \end{aligned}$$

(ε΄) We take cases: assuming $\lambda = 0$, we have $x_1 = x_2 = x_3 = x_4 = -\frac{\mu}{2} = \frac{1}{4}$, so we need $K \geq \frac{1}{4}$ and the minimum value is $f(x) = 4\left(\frac{1}{4}\right)^2 = \frac{1}{4}$.
Assuming $\lambda > 0$, we need $x_4 = K$, and we have $x_1 = x_2 = x_3 = \frac{1-K}{3} = -\frac{\mu}{2}$ and the minimum value for $f(x)$ is $f(x) = 3\left(\frac{1-K}{3}\right)^2 + K^2$.

4. *(2.5 points)* Find the Lagrangian, the dual function $g(\lambda)$ and the dual problem of the problem

$$\begin{aligned} \text{minimize:} \quad & f_0(x) = \tfrac{1}{2}x^T Q x + c^T x, \\ \text{subject to:} \quad & Ax \geq b, \end{aligned}$$

where Q is a positive definite $n \times n$ matrix.

Solution: The Lagrangian is

$$L(x, \lambda) = \frac{1}{2}x^T Q x + c^T x + \lambda(b - Ax) = \frac{1}{2}x^T Q x + (c - A^T\lambda)^T x + \lambda b.$$

Regarding its minimization:

$$\nabla L(x, \lambda) = Qx + (c - A^T\lambda) = 0 \Leftrightarrow x = Q^{-1}(A^T\lambda - c),$$

therefore

$$\begin{aligned} g(\lambda) &= \frac{1}{2}(A^T\lambda - c)^T Q^{-1} Q Q^{-1}(A^T\lambda - c) + (c - A^T\lambda)^T Q^{-1}(A^T\lambda - c) + \lambda^T b \\ &= -\frac{1}{2}(A^T\lambda - c)^T Q^{-1}(A^T\lambda - c) + \lambda^T b \\ &= -\frac{1}{2}(\lambda^T A - c^T) Q^{-1}(A^T\lambda - c) + \lambda^T b \\ &= -\frac{1}{2}\lambda^T A Q^{-1} A^T \lambda - \frac{1}{2}c^T Q^{-1} c + \frac{1}{2}\lambda^T A Q^{-1} c + \frac{1}{2}c^T Q^{-1} A^T \lambda + \lambda^T b \\ &= \lambda^T(-\frac{1}{2}AQ^{-1}A^T)\lambda + (\frac{1}{2}c^T Q^{-1} A^T + b^T + \frac{1}{2}c^T Q^{-1} A^T)\lambda - \frac{1}{2}c^T Q^{-1} c. \end{aligned}$$

It follows that the resulting dual problem is

$$\begin{aligned} \text{maximize:} \quad & g(\lambda) = \lambda^T W \lambda + V^T \lambda - u, \\ \text{subject to:} \quad & \lambda \geq 0, \end{aligned}$$

where

$$W = -\frac{1}{2}AQ^{-1}A^T, \quad V = b + AQ^{-1}c, \quad u = -\frac{1}{2}c^T Q^{-1} c$$

5. *(2.5 points)* Let $y^1,\ y^2,\ \ldots, y^p$ be p points in $\mathbb{R}^n$. Show that the problem of finding the smallest possible ball that contains all these points is a convex optimization problem. Write the KKT conditions for that problem. In the special case $p = 3$, discuss different cases for the solution, without providing proofs, and with informal geometric arguments.

Solution:

(α΄) Let x be the center of the ball. We want to minimize the following function:

$$\max\{\|x - y^1\|^2, \|x - y^2\|^2, \ldots, \|x - y^p\|^2\}$$

which is a convex function, being the maximum of convex functions. The problem is equivalent to the following one:

$$\begin{aligned} \text{minimize:} \quad & t \\ \text{subject to:} \quad & \|x - y^1\|^2 \leq t, \\ & \|x - y^2\|^2 \leq t, \\ & \dots \\ & \|x - y^p\|^2 \leq t. \end{aligned}$$