Notes:

1. Duration: 2.5 hours
2. Explain everything carefully. You will be graded also on the clarity of your arguments.

Exercises:

1. *(1 point)* Prove that the convex hull conv$A$ of a set A is the smallest convex set that contains A, in the sense that it is a subset of any convex set C that contains A.
2. *(1.5 points)* Let $y, x_1, x_2, \ldots, x_p \in \mathbb{R}^n$. Prove that $y \in \text{conv}\{(x_1, x_2, \ldots, x_p\})$ if and only if

$$\text{conv}(\{x_1, x_2, \ldots, x_p\}) = \text{conv}\{(y, x_1, x_2, \ldots, x_p)\}.$$

   With conv$\{S\}$ we denote the convex hull of the set $S \subseteq \mathbb{R}^n$.

3. *(2.5 points)* Consider the problem

$$\begin{array}{ll} \text{minimize:} & x_1^2 + x_2^2 + x_3^2 + x_4^2 \\ \text{subject to:} & x_1 + x_2 + x_3 + x_4 = 1, \qquad x_4 \leq K, \end{array}$$

   where K is a parameter.

   (α΄) Bring the problem in the standard form of an optimization problem.
   (β΄) Is the problem convex? Explain.
   (γ΄) Write the Lagrangian.
   (δ΄) Write the KKT conditions for this problem.
   (ε΄) Find the solution of the problem as a function of the parameter K.

4. *(2.5 points)* Find the Lagrangian, the dual function $g(\lambda)$ and the dual problem of the problem

$$\begin{array}{ll} \text{minimize:} & f_0(x) = \frac{1}{2}x^T Q x + c^T x, \\ \text{subject to:} & Ax \geq b, \end{array}$$

   where Q is a positive definite $n \times n$ matrix.

5. *(2.5 points)* Let $y^1,\ y^2,\ \ldots, y^p$ be p points in $\mathbb{R}^n$. Show that the problem of finding the smallest possible ball that contains all these points is a convex optimization problem. Write the KKT conditions for that problem. In the special case $p = 3$, discuss different cases for the solution, without providing proofs, and with informal geometric arguments.