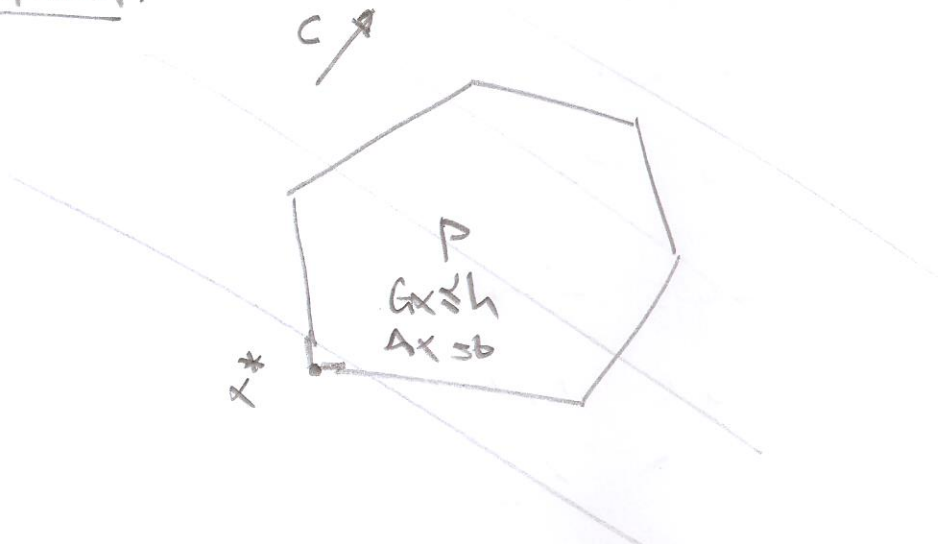
4.3 LINEAR OPTIMIZATION PROBLEMS

$$\left\{\begin{array}{ll} \text{minimize} & c^T x + d \\ \text{subject to:} & Gx \preceq h \\ & Ax = b \end{array}\right\} \quad (1) \qquad \begin{array}{l} G \in \mathbb{R}^{m \times n} \\ A \in \mathbb{R}^{p \times n} \end{array}$$

LINEAR PROGRAM

GEOMETRICALLY:



STANDARD FORM:

$$\left\{\begin{array}{ll} \text{minimize:} & c^T x \\ \text{subject to:} & Ax = b \\ & x \succeq 0 \end{array}\right\} \quad (2)$$

INEQUALITY FORM

$$\left\{\begin{array}{ll} \text{minimize:} & c^T x \\ \text{subject to:} & Ax \preceq b \end{array}\right\} \quad (3)$$

WE CAN BRING (1) TO THE FORM OF (2) EASILY:

$$(1) \iff \left\{\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Gx + s = h \\ & Ax = b \\ & s \succeq 0 \end{array}\right\} \overset{x = x^+ - x^-}{\iff}$$

$$\left\{\begin{array}{ll} \text{minimize} & c^T x^+ - c^T x^-, \\ \text{subject to:} & Gx^+ - Gx^- + s = h \\ & Ax^+ - Ax^- = b \\ & x^+ \succeq 0, \; x^- \succeq 0, \; s \succeq 0 \end{array}\right\}$$

WE CAN ALSO CONVERT (1) TO (3), (3) TO (2), ETC.

EXAMPLES 4.3.1

DIET PROBLEM: A DIET NEEDS m NUTRIENTS

WE HAVE n FOODS AVAILABLE. ONE UNIT OF FOOD j CONTAINS AMOUNT OF NUTRIENT i EQUAL TO a_{ij} AND COSTS c_j. WE WANT THE CHEAPEST DIET THAT ENSURES THAT WE GET AT LEAST b_j OF NUTRIENT j.

THEREFORE, WE NEED TO SOLVE FOLLOWING LP:

$$\text{minimize: } c^T x = c_1 x_1 + c_2 x_2 + \dots + c_n x_n$$

$$\text{subject to: } Ax \succeq b \quad (\Leftrightarrow a_{i1} x_1 + a_{i2} x_2 + \dots a_{in} x_n \geq b_i)$$

$$x \succeq 0$$

CHEBYCHEV CENTER OF A POLYHEDRON:

PROBLEM: FIND LARGEST EUCLIDEAN BALL THAT LIES IN POLYHEDRON $\mathcal{P} = \{ x \in \mathbb{R}^n \mid a_i^T x \leq b_i,\ i = 1, \dots, m \}$

THE CENTER x_c IS CALLED CHEBYCHEV CENTER. THE BALL IS

$$\mathcal{B} = \{ x_c + u \mid \|u\|_2 \leq r \}$$

$\underset{\in \mathbb{R}^n}{x_c}, \underset{\in \mathbb{R}}{r}$ ARE THE UNKNOWN PARAMETERS

AT FIRST GLANCE, IT IS NOT OBVIOUS THAT THIS IS AN LP, BUT IT IS. INDEED:

WE NEED $\|u\|_2 \leq r \Rightarrow a_i^T (x_c + u) \leq b_i$

BUT $a_i^T u$ IS MAXIMUM WHEN $u = \dfrac{a_i}{\|a_i\|_2} r$

$\Rightarrow a_i^T u = r \|a_i\|_2$

THEREFORE, WE NEED $a_i^T x_c + r\|a_i\|_2 \leq b_i$ FOR ALL HALFSPACES i. THEREFORE WE ARRIVE AT:

$$\left\{ \begin{array}{ll} \text{maximize}: & r \\ \text{subject to}: & a_i^T x_c + r\|a_i\|_2 \leq b_i, \quad i=1,\dots,m \end{array} \right\}$$

DYNAMIC ACTIVITY PLANNING

LET $x_j(t) \geq 0$, $t=1,\dots,N$ THE ACTIVITY DURING PERIOD t OF SECTOR j, $j=1,\dots,n$.

ACTIVITIES PRODUCE GOODS AS FOLLOWS: AMOUNT x_j ACTIVITY PRODUCES $a_{ij} x_j$ GOODS OF TYPE i.

LIKEWISE, ACTIVITIES CONSUME GOODS, AS FOLLOWS: AMOUNT x_j OF ACTIVITY CONSUMES $b_{ij} x_j$ GOODS. WE MUST HAVE:

$$\underset{\substack{\uparrow \\ \text{INITIAL AMOUNT OF GOODS}}}{Bx(1) \preceq g_0}, \qquad \underset{\substack{\uparrow \\ \text{GOODS CONSUMED IN PERIOD } t+1}}{Bx(t+1)} \preceq \underbrace{Ax(t)}_{\text{GOODS PRODUCED IN PERIOD } t}$$

EXCESS GOODS:

$$s(0) = g_0 - Bx(1)$$

$$s(t) = Ax(t) - Bx(t+1), \quad t=1,\dots,N-1$$

$$s(N) = Ax(N)$$

WE WANT TO MAXIMIZE:

$$c^T x(0) + \gamma c^T s(1) + \dots + \gamma^N c^T s(N)$$

SUBJECT TO:

$$x(t) \succeq 0, \quad t=1,\dots,N$$

$$s(t) \succeq 0, \quad t=0,\dots,N$$

$$s(0) = g_0 - Bx(1)$$

$$s(t) = Ax(t) - Bx(t+1), \quad t=1,\dots,N-1, \quad s(N) = Ax(N)$$

CHEBYCHEV INEQUALITIES

PROBABILITY DISTRIBUTIONS ARE VECTORS $p \in \mathbb{R}^n$,

$$p_i = P(X = u_i), \quad S = \{u_1, u_2, \ldots, u_n\}.$$

WITH $p \succeq 0$, $\sum p_i = 1 \iff \mathbb{1}^T p = 1$

THEN: 1) $E f(x) = \sum_{i=1}^{n} p_i f(u_i)$, 2) $P(X \in S) = \sum_{u_i \in S} p_i$

BOTH ARE LINEAR. SO, FOR EXAMPLE:

$$\begin{aligned} &\text{minimize} \quad a_0^T p \\ &\text{subject to} \quad p \succeq 0, \quad \mathbb{1}^T p = 1 \\ &\qquad\qquad \alpha_i \le a_i^T p \le \beta_i \quad i = 1, \ldots, m \end{aligned}$$

THIS MEANS FINDING THE MINIMUM EXPECTATION / PROBABILITY CONSISTENT WITH BOUNDS ON OTHER EXPECTATIONS / PROBABILITIES

PIECEWISE-LINEAR MINIMIZATION

$$\text{minimize} \quad f(x) = \max_{i=1,\ldots,m} (a_i^T x + b_i) \iff$$

$$\left\{ \begin{aligned} &\text{minimize} \quad t \\ &\text{subject to} \quad \max_{i=1,\ldots,m} (a_i^T x + b_i) \le t \end{aligned} \right\} \iff$$

$$\left\{ \begin{aligned} &\text{minimize} \quad t \\ &\text{subject to} \quad a_i^T x + b_i \le t, \quad i = 1, \ldots, m \end{aligned} \right\}$$

LINEAR-FRACTIONAL PROGRAMMING

$$\left\{\begin{array}{lll} \text{minimize:} & f_0(x) = \dfrac{c^T x + d}{e^T x + f} \\ \text{subject to:} & Gx \preceq h \\ & Ax = b \end{array}\right\} \quad (4.32)$$

WITH $\mathrm{dom} f_0 = \{x \mid e^T x + f > 0\}$.

FEASIBLE SET $X = \{x \mid Gx \preceq h,\ Ax = b,\ e^T x + f > 0\}$

THIS PROGRAM IS QUASICONVEX, BECAUSE $f_0(x)$ IS QUASILINEAR.

INDEED:

$$\frac{c^T x + d}{e^T x + f} \leq k \Leftrightarrow c^T x + d \leq k e^T x + k f \Leftrightarrow$$

$$\left[c^T - k e^T\right] x \leq kf - d,$$ SO SUBLEVEL SETS ARE HALF SPACES.

(4.32) IS EQUIVALENT TO:

$$\left\{\begin{array}{ll} \text{minimize} & c^T y + dz \\ \text{subject to} & Gy - hz \preceq 0 \\ & Ay - bz = 0 \\ & e^T y + fz = 1 \\ & z \geq 0 \end{array}\right\} \quad (4.33)$$

($y \in \mathbb{R}^n$, $z \in \mathbb{R}$, ARE THE NEW OPTIMIZATION VARIABLES)

PROOF: LET x BE FEASIBLE IN (4.32). LET

$$y = \frac{x}{e^T x + f}, \quad z = \frac{1}{e^T x + f}$$

THEN (y, z) IS FEASIBLE IN (4.33), WITH SAME VALUE FOR THE OBJECTIVE FUNCTION

INDEED:

1) $Gx \preceq h \Leftrightarrow$

$$\frac{Gx}{e^T x+f} \preceq \frac{h}{e^T x+f} \Leftrightarrow Gy - hz \preceq 0$$

2) $Ax = b \Leftrightarrow \dfrac{Ax}{e^T x+f} = \dfrac{b}{e^T x+f} \Leftrightarrow Ay - bz = 0.$

3) $e^T y + fz = 1$ IS ALWAYS SATISFIED AUTOMATICALLY

4) $c^T y + dz = \dfrac{cx+d}{e^T x+f} = f_0(x).$

SO IT FOLLOWS THAT OPTIMUM OF (4.32) IS GREATER OF EQUAL THAN OPTIMUM OF (4.32). NOW WE SHOW THE INVERSE.

LET (y,z) FEASIBLE IN (4.33), WITH $z \neq 0$

THEN $x = \dfrac{y}{z}$ IS FEASIBLE IN (4.32). INDEED, WITH SAME VALUE FOR OBJECTIVE FUNCTION

1) $Gy - hz \preceq 0 \Leftrightarrow G\left(\dfrac{y}{z}\right) - h \preceq 0$

2) $Ay - bz = 0 \Leftrightarrow A\left(\dfrac{y}{z}\right) - b = 0$

3) $e^T y + fz = 1 \Rightarrow e^T\left(\dfrac{y}{z}\right) + f = \dfrac{1}{z} > 0$

4) $\dfrac{c^T x + d}{e^T x+f} = \dfrac{c^T y + dz}{e^T y + fz} = c^T y + dz$

FINAL CASE: LET (y,z) FEASIBLE, BUT WITH $z = 0$

LET x_0 BE FEASIBLE IN (4.32). THEN

$x = x_0 + ty$ IS FEASIBLE IN (4.32) FOR ALL $t \geq 0$

INDEED: 1) $Gx = \underbrace{Gx_0}_{\leq h} + \underbrace{Gty}_{\leq 0} \leq h$.

2) $Ax - b = Ax_0 - b + Aty = 0$.

3) $e^T x + f = e^T x_0 + e^T ty + f = \underbrace{e^T x_0 + f}_{\geq 0} + t \underbrace{e^T y}_{= t} \geq 0$

ALSO: $\lim_{t \to \infty} f_0(x_0 + ty) =$

$$\lim_{t \to \infty} \frac{c^T x_0 + c^T ty + d}{e^T x_0 + e^T ty + f} = \lim_{t \to \infty} \frac{c^T x_0 + c^T ty + d}{e^T x_0 + t + f}$$

$= c^T y = c^T y + dz$, SO THERE ARE FEASIBLE POINTS OF (4.32) ARBITRARILY CLOSE TO THE OBJECTIVE VALUE OF (y, z). THEREFORE, OPTIMAL VALUE OF (4.32) IS LESS THAN OR EQUAL TO OPTIMAL VALUE OF (4.33)

AND NOW SOME STUFF FROM BERTSEKAS, NETWORK OPTIMIZATION, CHAPTER #1

LET A GRAPH $G = (N, A)$

NODES $1, 2, \ldots, N$

ARCS (i, j), $1 \leq i, j \leq N$ FOR SOME VALUES i, j

LET x_{ij} THE FLOW THROUGH EACH ARC ij.

LET

$$y_i = \sum_{j:(i,j)\in A} x_{ij} - \sum_{j:(j,i)\in A} x_{ji}$$

THE DIVERGENCE OF NODE i.

WHEN IT IS POSITIVE, WE HAVE FLOW COMING IN THE NETWORK FROM OUTSIDE.

LET ALSO $b_{ij} \le x_{ij} \le c_{ij}$ LOWER AND UPPER BOUNDS ON THE FLOWS

MINIMUM COST FLOW PROBLEM

minimize $\sum_{(i,j)\in A} a_{ij} x_{ij}$ COST, GIVEN

SUPPLY, GIVEN

subject to

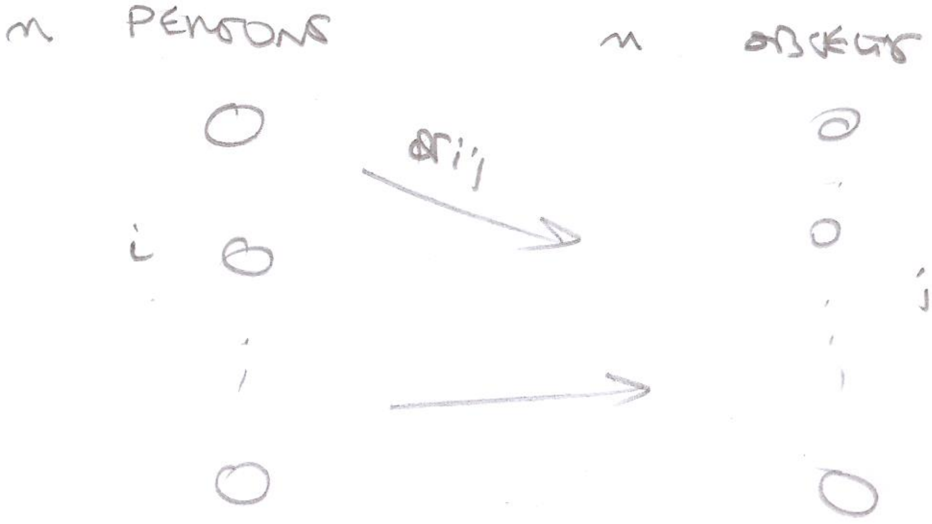
$$\sum_{\{j \mid (i,j)\in A\}} x_{ij} - \sum_{\{j \mid (j,i)\in A\}} x_{ji} = s_i \quad \forall i \in N$$

$$b_{ij} \le x_{ij} \le c_{ij}$$

FIRST SPECIAL CASE: MINIMUM COST PROBLEM:

$$s_i = \begin{cases} 1, & \text{IF } i \text{ IS SOURCE} \\ -1, & \text{IF } i \text{ IS DESTINATION} \\ 0, & \text{ELSEWHERE} \end{cases}$$

SECOND SPECIAL CASE: ASSIGNMENT PROBLEM



EACH PERSON MUST GET AN OBJECT. THE VALUE WE GET IS a_{ij}

$$\text{maximize} \sum_{(i,j)\in A} a_{ij} x_{ij}$$

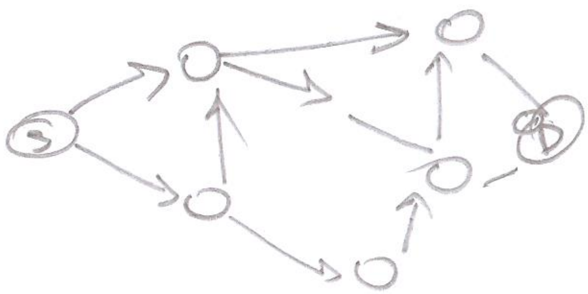
$$\text{subject to} \sum_{\{j \mid (i,j)\in A\}} x_{ij} = 1, \quad \forall i = 1,\dots,n$$

$$\sum_{\{i \mid (i,j)\in A\}} x_{ij} = 1 \quad \forall j = 1,\dots,n$$

$$0 \le x_{ij} \le 1$$

ACTUALLY, WE NEED $x_{ij} \in \{0,1\}$, BUT RELAXATION DOES NOT HURT. CLEARLY, THIS IS A SPECIAL CASE OF THE MINIMUM COST PROBLEM

THIRD SPECIAL CASE: MAX FLOW PROBLEM:



TRICK: CREATE AN ARTIFICIAL LINK, FROM DESTINATION TO SOURCE, WITH NEGATIVE COST, AND SET COST OF REST EQUAL TO ZERO.

MANY EXTENSIONS EXIST! $\left\{\begin{array}{l}\text{MULTICOMMODITY PROBLEM} \\ \text{NON-LINEAR COSTS} \\ \text{MORE COMPLICATED CAPACITY CONSTRAINTS, ETC.}\end{array}\right.$

4.4 QUADRATIC OPTIMIZATION

$$\left\{\begin{array}{ll}\text{minimize} & \frac{1}{2}x^TPx + q^Tx + r \\ \text{subject to:} & Gx \preceq h \\ & Ax = b\end{array}\right\} \quad P \in S_+^n, \; G \in \mathbb{R}^{m \times n}, \; A \in \mathbb{R}^{p \times n}$$

$P = 0 \Rightarrow$ LINEAR PROGRAM

(IF INEQUALITY CONSTRAINTS ARE $\frac{1}{2}x^TP_ix + q_i^Tx + r_i \leq 0$, WITH $P_i \in S_+^n$, THEN WE HAVE QUADRATICALLY CONSTRAINED QUADRATIC PROGRAM (QCQP)

EXAMPLE 1: LEAST SQUARES

$$\text{minimize} \quad \|Ax - b\|_2^2 = x^TA^TAx - 2b^TAx + b^Tb$$

THIS COMES FROM TRYING TO SATISFY AN OVERCONSTRAINED SYSTEM:

$$\begin{bmatrix} a_1^T \\ a_2^T \\ \vdots \\ a_p^T \end{bmatrix} x - \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_p \end{bmatrix} = 0 \qquad a_i^Tx = b_i, \quad i = 1, \dots p$$

WE HAVE SOLUTION IN CLOSED FORM: $x = A^\dagger b$, WHERE

$A^{+} = V\Sigma^{-1}U^T \in \mathbb{R}^{n\times m}$ THE PSEUDO-INVERSE,

WHERE $A = U\Sigma V^T$

EXAMPLE 2: DISTANCE BETWEEN POLYHEDRA

$$\mathcal{P}_1 = \{x : A_1 x \le b_1\}, \quad \mathcal{P}_2 = \{x \mid A_2 x \le b_2\}$$

$$\mathrm{dist}(\mathcal{P}_1, \mathcal{P}_2) = \inf\{\|x_1 - x_2\|_2 \mid x_1 \in \mathcal{P}_1, x_2 \in \mathcal{P}_2\}$$

TO FIND IT, WE SOLVE
$$\left\{\begin{array}{l} \text{minimize } \|x_1 - x_2\|_2^2 \\ \text{subject to } A_1 x_1 \le b_1,\ A_2 x_2 \le b_2 \end{array}\right\}$$

$$\|x_1 - x_2\|_2^2 = (x_1^T - x_2^T)(x_1 - x_2) = x_1^T x_1 + x_2^T x_2 - x_2^T x_1 - x_1^T x_2$$

$$= [x_1^T \ x_2^T] \begin{bmatrix} I_{n\times n} & -I_{n\times n} \\ -I_{n\times n} & I_{n\times n} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

THIS PROBLEM ALWAYS HAS SOLUTION, UNLESS $\mathcal{P}_1$ OR $\mathcal{P}_2$ ARE EMPTY.

EXAMPLE 3: BOUNDING VARIANCE. LET p BE A DISTRIBUTION, $p \in \mathbb{R}^n$. LET $f(x)$ BE A FUNCTION OF x. IF WE WANT TO MAXIMIZE ITS VARIANCE SUBJECT TO CONSTRAINTS ON THE MEAN:

$$\text{maximize} \sum_{i=1}^{n} f_i^2 p_i - \left(\sum_{i=1}^{n} f_i p_i\right)^2$$

subject to $p \succeq 0, \quad \mathbb{1}^T p = 1$

$$\alpha_i \le a_i^T p \le \beta_i \qquad i = 1, \dots, m$$

4.5 GEOMETRIC PROGRAMMING

DEFINITION: 1) $f: \mathbb{R}^n \to \mathbb{R}$, $\operatorname{dom} f = \mathbb{R}^n_{++}$,

$$f(x) = c x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n},$$

$c > 0$, $a_i \in \mathbb{R}$ IS A MONOMIAL

2) SUM OF MONOMIALS $f(x) = \sum_{k=1}^{K} c_k x_1^{a_{1k}} x_2^{a_{2k}} \cdots x_n^{a_{nk}}$

IS A POSYNOMIAL

GEOMETRIC PROGRAMMING

(4.43)
$$\begin{cases} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \le 1, \ i=1,\dots,m \\ & h_i(x) = 1, \ i=1,\dots,p \end{cases}$$

WHERE 1) $f_0(x)$, f_i ARE POSYNOMIALS.

2) h_i ARE MONOMIALS

REMEMBER: $x \succ 0$.

COMMENT: POSYNOMIALS AND MONOMIALS ARE CLOSED UNDER MANY OPERATIONS, SO IN MANY CASES IT IS EASY TO ARRIVE AT A GEOMETRIC PROGRAM FORM.

FOR EXAMPLE:

$$\begin{cases} \text{maximize} & \frac{x}{y} \\ \text{subject to} & 2 \le x \le 3 \\ & x^2 + 3\frac{y}{z} \le \sqrt{y} \\ & x, y, z > 0 \end{cases} \iff \begin{cases} \text{minimize} \ x^{-1} y \\ 2x^{-1} \le 1, \quad \frac{1}{3} x \le 1 \\ x^2 y^{-\frac{1}{2}} + 3 y^{\frac{1}{2}} z^{-1} \le 1 \\ x y^{-1} z^{-2} = 1 \end{cases}$$

4.5.3 GEOMETRIC PROGRAM IN CONVEX FORM

LET $f(x) = c x_1^{a_1} x_2^{a_2} \dots x_n^{a_n}$. A MONOMIAL

LET $y_i = \log x_i \iff x_i = e^{y_i}$.

THEN $\log f(x) = \log c + a_1 \log x_1 + a_2 \log x_2 + \dots + a_n \log x_n$

$\Rightarrow$ $f(x) = e^{a^T y + b}$, $b = \log c$.

LET $f(x) = \sum_{k=1}^{K} c_k x_1^{a_{1k}} x_2^{a_{2k}} \dots x_n^{a_{nk}}$ A POSYNOMIAL

LIKEWISE: $f(x) = \sum_{k=1}^{K} e^{a_k^T y + b_k}$, $b_k = \log c_k$.

(4.43) $\iff$
$$\begin{cases} \text{minimize} & \sum_{k=1}^{K_0} e^{a_{0k}^T y + b_{0k}} \\ \text{subject to} & \sum_{k=1}^{K_i} e^{a_{ik}^T y + b_{ik}} \le 1, \quad i = 1, \dots, m \\ & e^{g_i^T y + h_i} = 1, \quad i = 1, \dots, p \end{cases}$$

$\iff$

minimize: $\tilde{f}_0(y) = \log\left(\sum_{k=1}^{K_0} e^{a_{0k}^T y + b_{0k}} \right)$

SUBJECT TO: $\tilde{f}_i(y) = \log\left(\sum_{k=1}^{K_i} e^{a_{ik}^T y + b_{ik}} \right) \le 0$, $i = 1, \dots, m$

$\tilde{h}_i(y) = g_i^T y + h_i = 0$

CONVEX

AFFINE

NOTE THAT IF $K_0 = K_i = 1$, WE HAVE A LINEAR PROGRAM, SO GEOMETRIC PROGRAMMING IS AN EXTENSION OF LINEAR PROGRAMMING

TO SHOW THAT THE $\bar{f}_0(y)$, $\tilde{f}_i(y)$ ARE CONVEX, IT IS ENOUGH TO SHOW THAT THE FOLLOWING FUNCTION IS CONVEX:

LOG SUM EXP (pg. 74)

$$f(x) = \log(e^{x_1} + \dots + e^{x_n})$$

WE CALCULATE THE HESSIAN:

$$\frac{\partial f}{\partial x_i} = \frac{1}{\sum e} e^{x_i}, \quad \frac{\partial^2 f}{\partial x_i^2} = \frac{e^{x_i}(\sum e) - e^{x_i} e^{x_i}}{(\sum e)^2},$$

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = -\frac{e^{x_i} e^{x_j}}{(\sum e)^2} \Rightarrow$$

$$\nabla^2 f(x) = \frac{1}{(1^T z)^2}\left[(1^T z)\, \mathrm{diag}(z) - z z^T\right]$$

WHERE $z = (e^{x_1}, \dots e^{x_n})$. TO SHOW THAT $\nabla^2 f(x) \succeq 0$, WE SHOW THAT

$$v^T \nabla^2 f(x)\, v \geq 0 \quad \forall v \quad \text{INDEED}:$$

$$v^T \nabla f(x)\, v = \frac{1}{(1^T z)^2}\left[v^T (1^T z)\, \mathrm{diag}(z)\, v - v z z^T v^T\right]$$

$$= \frac{1}{(1^T z)^2}\left[(1^T z)\left[\sum_{i=1}^{n} z_i v_i^2\right] - (vz)^T (vz)\right] \geq 0$$

THE ABOVE FOLLOWS FROM THE CAUCHY-SCHWARZ INEQUALITY:

$$(a^T a)(b^T b) \geq (a^T b)^2$$

FOR $a_i = v_i \sqrt{z_i}$, $b_i = \sqrt{z_i}$