CHAPTER 4 - CONVEX OPTIMIZATION PROBLEMS

$$\begin{aligned} \text{minimize} \quad & f_0(x) \\ \text{subject to:} \quad & f_i(x) \leq 0, \quad i=1,\dots,m \\ & h_i(x) = 0, \quad i=1,\dots,p \end{aligned} \qquad (1)$$

- $x \in \mathbb{R}^m$: OPTIMIZATION VARIABLE
- $f_0: \mathbb{R}^n \to \mathbb{R}$: OBJECTIVE FUNCTION OR COST FUNCTION
- $f_i(x) \leq 0$ INEQUALITY CONSTRAINTS
- $f_i(x): \mathbb{R}^n \to \mathbb{R}^m$ INEQUALITY CONSTRAINT FUNCTIONS
- $h_i(x)$: EQUALITY CONSTRAINTS
- $h_i(x): \mathbb{R}^n \to \mathbb{R}$ EQUALITY CONSTRAINT FUNCTIONS
- $m = p = 0 \Rightarrow$ PROBLEM IS UNCONSTRAINED
- $\mathcal{D} = \bigcap_{i=0}^{m} \operatorname{dom} f_i \cap \bigcap_{i=1}^{p} \operatorname{dom} h_i$ IS THE DOMAIN OF THE OPTIMIZATION PROBLEM
- $x \in \mathcal{D}$ IS FEASIBLE IF IT SATISFIES ALL CONSTRAINTS.
- THE PROBLEM IS FEASIBLE IF IT HAS AT LEAST ONE FEASIBLE POINT, AND NOT FEASIBLE OTHERWISE
- OPTIMAL VALUE p^* IS

$$p^* = \inf \{ f_0(x) \mid f_i(x) \leq 0, \ i=1,\dots,m, \\ h_i(x) = 0, \ i=1,\dots,p \}$$

IF THE PROBLEM IS INFEASIBLE, $p^* = +\infty$

IF $\exists$ FEASIBLE x_k WITH $f_0(x_k) \to -\infty$ WITH $k \to \infty$, THEN $p^* = -\infty$ AND THE PROBLEM IS UNBOUNDED BELOW

- x^* IS AN OPTIMAL POINT, OR SOLVES THE PROBLEM, IF x^* IS FEASIBLE AND $f_0(x^*) = p^*$
- THE OPTIMAL SET IS

$$X_{opt} = \{ x \mid f_i(x) \leq 0,\ i=1,\dots,m,\ h_i(x)=0,\ i=1,\dots,p,\ f_0(x) = p^* \}$$

- IF THERE IS AN OPTIMAL POINT, THE OPTIMAL VALUE IS ATTAINED OR ACHIEVED, AND THE PROBLEM IS SOLVABLE. OTHERWISE, THE PROBLEM IS NOT ACHIEVED.
- IF $f_0(x) \leq p^* + \epsilon$, WITH $\epsilon > 0$, THEN x IS ϵ-SUBOPTIMAL, AND ALL SUCH POINTS FORM THE ϵ-SUBOPTIMAL SET
- A FEASIBLE POINT x IS LOCALLY OPTIMAL IF IT SOLVES THE OPTIMIZATION PROBLEM

$$\begin{array}{lll} \text{minimize} & f_0(z) & \\ \text{subject to} & f_i(z) \leq 0 & i=1,\dots,m \\ & h_i(z) = 0 & i=1,\dots,p \\ & \|z - x\|_2 \leq R & \end{array}$$

- TO DISTINGUISH OPTIMAL AND LOCALLY OPTIMAL POINTS, WE SOMETIMES CALL THE FORMER <u>GLOBALLY</u> OPTIMAL.
- IF x IS FEASIBLE AND $f_i(x) = 0$, WE SAY THE CONSTRAINT $f_i(x) \leq 0$ IS ACTIVE. OTHERWISE, WE SAY IT IS INACTIVE.
- IF A CONSTRAINT IS DELETED AND THE FEASIBLE SET

DOES NOT CHANGE, WE CALL IT REDUNDANT.

FEASIBILITY PROBLEM

WHEN THE OBJECTIVE FUNCTION IS CONSTANT, THE PROBLEM IS EQUIVALENT TO:

$$\left\{\begin{array}{lll} \text{FIND} & x & \\ \text{SUBJECT TO} & f_i(x) \leq 0, & i=1,\ldots,m \\ & h_i(x) = 0, & i=1,\ldots,p \end{array}\right\}$$

4.1.2 EXPRESSING PROBLEMS IN STANDARD FORM

PROBLEM (1) HAS THE STANDARD FORM. OTHER PROBLEMS CAN BE BROUGHT TO THAT FORM, FOR EXAMPLE,

$g_i(x) = \tilde{g}_i(x) \Leftrightarrow \quad h_i(x) \triangleq g_i(x) - \tilde{g}_i(x) = 0.$

EXAMPLE:

$$\left\{\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & l_i \leq x_i \leq u_i \end{array}\right. \leftarrow \text{VARIABLE BOUNDS / BOX CONSTRAINTS}$$

$\Longleftrightarrow$

$$\begin{array}{lll} \text{minimize} & f_0(x) & \\ \text{subject to} & f_i(x) = l_i - x_i \leq 0, & i=1,\ldots,n \\ & f_i(x) = x_{i-n} - u_{i-n} \leq 0, & i=n+1,\ldots,2n \end{array}$$

MAXIMIZATION PROBLEMS

$$\begin{array}{lll} \text{maximize} & f_0(x) & \\ \text{subject to} & f_i(x) \leq 0, & i=1,\ldots,m \\ & h_i(x) = 0 & i=1,\ldots,p \end{array} \quad (2)$$

THIS IS EQUIVALENT TO MINIMIZING THE FUNCTION $-f_0$. f_0 IS CALLED THE UTILITY, OR SATISFACTION FUNCTION/LEVEL.

SOME OF THE PREVIOUS TERMINOLOGY AND DEFINITIONS MUST BE ADAPTED, E.G.:

$$p^* = \sup \left\{ f_0(x) \mid f_i(x) \leq 0,\ i=1,\dots,m \quad h_i(x) = 0,\ i=1,\dots,p \right\}$$

4.1.3 EQUIVALENT PROBLEMS

TWO PROBLEMS ARE EQUIVALENT IF SOLVING ONE MEANS WE CAN SOLVE THE OTHER. AS A SIMPLE EXAMPLE:

$$\left\{ \begin{array}{lll} \min & f_0(x) & \\ \text{subject to} & f_i(x) \leq 0 & i=1,\dots,m \\ & h_i(x) = 0 & i=1,\dots,p \end{array} \right\} \Leftrightarrow$$

$$\left\{ \begin{array}{ll} \min & \tilde{f}(x) = \alpha_0 f_0(x) \\ \text{subject to} & \tilde{f}_i(x) = \alpha_i f_i(x) \leq 0,\ i=1,\dots,m \\ & \tilde{h}_i(x) = \beta_i h_i(x) = 0,\ i=1,\dots,p \end{array} \right\}$$

WHERE $\alpha_i > 0,\ i=1,\dots,m$, $\beta_i \neq 0,\ i=1,\dots,p$, ARE EQUIVALENT.

INDEED, THE FEASIBLE SET OF ONE IS EQUAL TO THE FEASIBLE SET OF THE OTHER, AND THE OPTIMAL POINT OF THE ONE EQUALS THE OPTIMAL POINT OF THE OTHER.

IN THE FOLLOWING, WE EXPLORE MANNERS WITH WHICH TWO PROBLEMS CAN BE EQUIVALENT

1 CHANGE of VARIABLES

LET $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ 1-1, $\varphi(\mathrm{dom}\,\varphi) \supseteq D$

LET $\tilde{f}_i(z) = f_i(\varphi(z))$, $i = 0, \ldots, m$

$\tilde{h}_i(z) = h_i(\varphi(z))$, $i = 1, \ldots, p$

THEN, THE FOLLOWING PROBLEM IS EQUIVALENT TO THE STANDARD ONE (1):

$$\left\{ \begin{array}{lll} \text{minimize:} & \tilde{f}_0(z) & \\ \text{subject to:} & \tilde{f}_i(z) \le 0, & i = 1, \ldots, m \\ & \tilde{h}_i(z) = 0, & i = 1, \ldots, p \end{array} \right\} \quad (4)$$

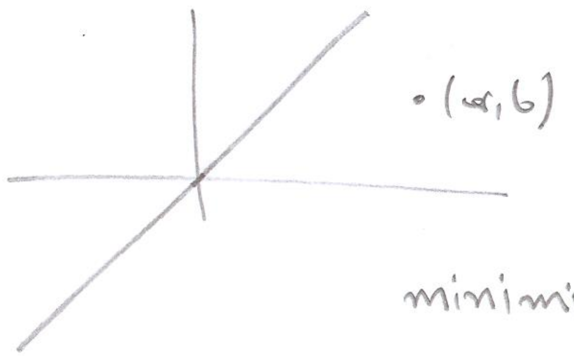
INDEED: A) x SOLVES (1) $\Rightarrow$ $z = \varphi^{-1}(x)$ SOLVES (4)

B) z SOLVES (4) $\Rightarrow$ $x = \varphi(z)$ SOLVES (1).

TO SHOW A), ASSERT FIRST THAT x FEASIBLE $\Rightarrow$ $z = \varphi^{-1}(x)$ FEASIBLE, AND THAT IF x MINIMIZES $f_0(x)$, THEN $z = \varphi^{-1}(x)$ MINIMIZES $\tilde{f}_0(z)$

EXAMPLE: MINIMIZE $\|(a,b) - (1,1)x\|$

SUBJECT TO $x \in \mathbb{R}$



THIS IS EQUIVALENT TO

minimize $\|(a,b) - (-10,-10)(z-1)\|$

SUBJECT TO $z \in \mathbb{R}$

$x = \dfrac{z-1}{-10}$, i.e. $\varphi(z) = \dfrac{1-z}{10}$

2 TRANSFORMATION OF OBJECTIVE AND CONSTRAINT FUNCTIONS:

EXAMPLE:

$$\text{minimize} \quad \|Ax-b\|_2$$

$$\Leftrightarrow \quad \text{minimize} \quad \|Ax-b\|_2^2$$

BUT: $\|Ax-b\|^2 = (Ax-b)^T(Ax-b) =$

$$x^TA^TAx - b^TAx - x^TA^Tb + b.b$$

$$= x^T \underbrace{(A^TA)}_{\text{SYMMETRIC}} x - 2b^TAx + \|b\|^2$$

, WHICH IS QUADRATIC AND MUCH EASIER TO SOLVE

ANOTHER EXAMPLE: minimize $f(x)$ $\Leftrightarrow$

minimize $g(f(x))$ WHERE $g\uparrow$.

THERE ARE CASES WHERE f IS NOT CONVEX, BUT $g(f(x))$ IS CONVEX

GENERAL CASE: $\psi_0 : \mathbb{R} \to \mathbb{R}$ INCREASING,

$\psi_1, \dots \psi_m : \mathbb{R} \to \mathbb{R}$ SATISFY $\psi_i(u) \leq 0 \Leftrightarrow u \leq 0$

$\psi_{m+1}, \dots, {m+p} : \mathbb{R} \to \mathbb{R}$ SATISFY $\psi_i(u) = 0 \Leftrightarrow u = 0$.

THEN THE STANDARD PROBLEM IS EQUIVALENT TO THE PROBLEM

$$\begin{aligned} \text{minimize:} \quad & \tilde{f}_0(x) \\ \text{subject to:} \quad & \tilde{f}_i(x) = \psi_i(f_i(x)) \leq 0, \quad i = 1, \dots, m \\ & \tilde{h}_i(x) = \psi_{m+i}(h_i(x)) = 0, \quad i = 1, \dots, p \end{aligned}$$

INDEED: 1) THE FEASIBLE SETS ARE IDENTICAL
2) THE OPTIMAL POINTS ARE IDENTICAL.

3 SLACK VARIABLES

OBSERVATION: $f_i(x) \le 0 \iff \exists s_i \geq 0 : f_i(x) + s = 0$.

THEREFORE, WE INTRODUCE SLACK VARIABLES:

$$\left\{ \begin{array}{ll} \text{minimize:} & f_0(z) \\ \text{subject to:} & s_i \geq 0 \quad i=1,\dots,m \\ & f_i(x) + s_i = 0, \quad i=1,\dots,m \\ & h_i(x) = 0, \quad i=1,\dots,p \end{array} \right\} (7)$$

s_i: SLACK VARIABLES

THIS PROBLEM IS EQUIVALENT TO (1)

THIS PROBLEM HAS NEW VARIABLES! THE m s_i.
OBSERVE THAT IF (x,s) IS FEASIBLE IN (7), x IS FEASIBLE IN (1). IF s IS FEASIBLE IN (1), WE TAKE $s_i = -f_i(x)$ AND FIND A FEASIBLE POINT (x,s) FOR (7)

ALSO, IF A VALUE OF f_0 IS ACHIEVED BY A FEASIBLE POINT IN (1), IT IS ACHIEVED BY A FEASIBLE POINT IN (7), SO THE OPTIMAL VALUES ARE THE SAME.

4 ELIMINATING EQUALITY CONSTRAINTS

THIS IS A VERY OLD TRICK.

EXAMPLE: FIND THE POINT WHERE THE PARABOLA $y = x^2 - 2x + 5$ PASSES THE CLOSEST FROM THE ORIGIN minimize: x^2+y^2

subject to: $y = x^2-2x+5$.

ONE SOLUTION IS TO SUBSTITUTE y IN THE OBJECTIVE FUNCTION:

minimize $x^2+(x^2-2x+5)^2$

THIS METHOD CAN BE GENERALIZED:

LET $\varphi: \mathbb{R}^k \to \mathbb{R}^n$ SUCH THAT

$$\{h_i(x)=0,\ i=1,\dots,p\} \iff \{\exists z\in\mathbb{R}^k : x=\varphi(z)\}$$

THEN THE STANDARD FORM PROBLEM IS EQUIVALENT TO

$$\left\{\begin{array}{l} \text{minimize: } \tilde{f}_0(z)=f_0(\varphi(z)) \\ \text{subject to: } \tilde{f}_i(z)=f_i(\varphi(z))\leq 0 \quad i=1,\dots,m \end{array}\right\}$$

WHICH HAS ONLY INEQUALITY CONSTRAINTS AND NO EQUALITY CONSTRAINTS. 1) LET x THE OPTIMAL OF THE ORIGINAL PROBLEM. THEN $\exists z: x=\varphi(z)$, SO z IS FEASIBLE WITH OBJECTIVE $f_0(\varphi(z))$ THE OPTIMAL.

2) LET z THE OPTIMAL OF THE NEW PROBLEM. LET $x=\varphi(z)$. THEN $f_i(x)\leq 0$ AND $h_i(x)$, SO x IS FEASIBLE IN THE ORIGINAL PROBLEM WITH OBJECTIVE SAME AS OPTIMAL OF MODIFIED PROBLEM

BY 1), 2) IT FOLLOWS THAT OPTIMA ARE ALSO EQUAL

5 ELIMINATING LINEAR INEQUALITY CONSTRAINTS.

THIS IS A SPECIAL CASE OF THE PREVIOUS ONE
LET THE CONSTRAINTS BE $Ax=b$, AND LET $Ax_0=b$.
THEN THE SET OF FEASIBLE POINTS IS x_0+ VECTOR SPACE, IN PARTICULAR THE NULL SPACE OF A.
LET $F \in \mathbb{R}^{n \times k}$ BE A MATRIX WITH $\mathcal{R}(F) = \mathcal{N}(A)$, SO THAT THE FEASIBLE x ARE $Fz + x_0$, $z \in \mathbb{R}^k$
THEN, WE CREATE THE NEW PROBLEM

$$\begin{cases} \text{minimize} & f_0(Fz + x_0) \\ \text{subject to} & f_i(Fz + x_0) \leq 0, \quad i=1,\dots,m \end{cases}$$

IS CLEARLY EQUIVALENT TO ORIGINAL

6 INTRODUCING EQUALITY CONSTRAINTS

CONSIDER THE EXAMPLE

$$\begin{array}{lll} \text{minimize} & f_0(A_0 x + b_0) & \\ \text{SUBJECT TO} & f_i(A_i x + b_i) \leq 0 & i=1,\dots,m \\ & h_i(x) = 0, & i=1,\dots,p \end{array}$$

WHERE $x \in \mathbb{R}^n$ $A_i \in \mathbb{R}^{k_i \times n}$, $f_i : \mathbb{R}^{k_i} \to \mathbb{R}$
WE INTRODUCE $y_i \in \mathbb{R}^{k_i}$ AND EQUALITY CONSTRAINTS $y_i = A_i x + b_i$, $i=0,\dots,m$, AND THE PROBLEM BECOMES:

$$\begin{cases} \text{minimize:} & f_0(y_0) \\ \text{subject to:} & f_i(y_i) \leq 0, \quad i=1,\dots,m \\ & y_i = A_i x + b_i, \quad i=0,\dots,m \\ & h_i(x) = 0, \quad i=1,\dots,p \end{cases}$$

THE TWO PROBLEMS ARE EQUIVALENT, AS CAN BE SHOWN BY STANDARD METHOD.

(SIDE OBSERVATION: THE OBJECTIVE AND INEQUALITY CONSTRAINTS ARE INDEPENDENT, I.E., THEY INVOLVE DIFFERENT OPTIMIZATION VARIABLES. IF THE EQUALITY CONSTRAINTS DID NOT EXIST, THEN WE COULD INDEPENDENTLY OPTIMIZE THE OBJECTIVE FUNCTION, AND PROBLEM WOULD BREAK DOWN INTO SMALLER ONES. THIS IS WHAT WE DO WITH DUALIZATION)

7 OPTIMIZING OVER SOME VARIABLES

OBSERVE THAT $\inf\limits_{x,y} f(x,y) = \inf\limits_{x} \left(\inf\limits_{y} f(x,y) \right)$

BASED ON THIS OBSERVATION:

LET PROBLEM

$$\begin{cases} \text{minimize} & f_0(x_1, x_2) \\ \text{subject to:} & f_i(x_1) \leq 0, \quad i=1,\dots,m_1 \\ & \tilde{f}_i(x_2) \leq 0, \quad i=1,\dots,m_2 \end{cases}$$

DEFINE $\tilde{f}_0(x_1) = \inf \{ f_0(x_1, z) \mid \tilde{f}_i(z) \leq 0, \ i=1,\dots,m_2 \}$

THEN THE ABOVE PROBLEM IS EQUIVALENT TO THE PROBLEM

$$\begin{cases} \text{minimize} & \tilde{f}_0(x_1) \\ \text{SUBJECT TO} & f_i(x_1) \leq 0, \quad i=1,\dots,m_1 \end{cases}$$

8 EPIGRAPH PROBLEM FORM

$$
\left\{
\begin{array}{ll}
\text{minimize} & t \\
\text{subject to} & f_0(x) - t \leq 0 \\
& f_i(x) \leq 0, \quad i=1,\dots,m \\
& h_i(x) = 0, \quad i=1,\dots,p
\end{array}
\right\} \qquad (11)
$$

OBSERVE THAT (x,t) IS OPTIMAL FOR (11) $\Leftrightarrow$

$$\left\{ x \text{ is optimal for } (1), \ t = f_0(x) \right\}.$$

(BUT NOW, OBJECTIVE FUNCTION IS LINEAR!)

INDEED, LET (x,t) OPTIMAL FOR (11). THEN $t=f_0(x)$, OTHERWISE CONTRADICTION. THEN x IS FEASIBLE IN (1). IT MUST BE OPTIMAL TOO, OTHERWISE WE CAN FIND SMALLER $f_0(x)$, HENCE SMALLER t AND WE ARRIVE AT CONTRADICTION.

CONVERSELY, LET x OPTIMAL FOR (1) AND $t=f_0(x)$. THEN (x,t) IS OPTIMAL FOR (11) OTHERWISE WE ARRIVE AT EASY CONTRADICTION

9 IMPLICIT AND EXPLICIT CONSTRAINTS

WE CAN LUMP ANY KIND OF CONSTRAINT IN THE OBJECTIVE FUNCTION. FOR EXAMPLE, (1) IS EQUIVALENT TO

$$\text{minimize} \quad F(x)$$

WHERE
$$
F(x) = \begin{cases} f_0(x) & \text{IN } \{x \in \operatorname{dom} f_0 \mid f_i(x) \leq 0, \ i=1,\dots,m, \\ & \qquad h_i(x) = 0, \ i=1,\dots,p\} \\ \infty & \text{ELSEWHERE} \end{cases}
$$

THIS IS JUST A NOTATIONAL TRICK AND MIGHT HURT. FOR EXAMPLE, PERHAPS THE INITIAL DOMAIN IS THE COMPLETE $\mathbb{R}^n$ AND f_0 IS DIFFERENTIABLE, BUT THE NEW DOMAIN MIGHT NOT BE OPEN, SO NEW FUNCTION IS NOT DIFFERENTIABLE

WE CAN ALSO DO THE INVERSE. FOR EXAMPLE, LET THE PROBLEM

$$\text{minimize } f(x) = \begin{cases} x^T x, & Ax = b \\ \infty, & \text{OTHERWISE.} \end{cases}$$

IMPLICIT EQUALITY CONSTRAINT

THIS IS EQUIVALENT TO

$$\begin{aligned} &\text{minimize} \quad x^T x \\ &\text{subject to} \quad Ax = b \end{aligned}$$

NOW THE OBJECTIVE FUNCTION IS DIFFERENTIABLE!