CHAPTER 3

3.1.1.

DEFINITION: $f : \mathbb{R}^n \to \mathbb{R}$ IS CONVEX IF

① $\operatorname{dom} f$ IS A CONVEX SET

② $\forall x, y \in \operatorname{dom} f$, $\theta \in [0,1]$,

$$f(\theta x + (1-\theta) y) \leq \theta f(x) + (1-\theta) f(y)$$

GEOMETRICALLY: THE CHORD IS BELOW THE GRAPH OF THE FUNCTION

DEFINITION: 1) f IS STRICTLY CONVEX IF THE INEQUALITY IS STRICT FOR $\theta \neq 0, 1$

2) f IS CONCAVE IF $-f$ IS CONVEX

3) f IS STRICTLY CONCAVE IF $-f$ IS STRICTLY CONVEX

PROPERTY: A FUNCTION IS AFFINE IFF IT IS BOTH CONVEX AND CONCAVE

PROPERTY: A FUNCTION IS CONVEX IFF IT IS CONVEX WHEN RESTRICTED TO ANY LINE THAT INTERSECTS ITS DOMAIN

THIS MEANS $\forall x \in \mathbb{R}^n$, $\forall v$, $g(t) = f(x + tv)$ IS CONVEX ON ITS DOMAIN $\{t \mid x + tv \in \operatorname{dom} f\}$

3.1.2

DEFINITION: IF f IS CONVEX WE DEFINE ITS EXTENDED-VALUE EXTENSION $\tilde{f} : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ BY

$$\tilde{f}(x) = \begin{cases} f(x), & x \in \operatorname{dom} f \\ \infty, & x \in \operatorname{dom} f \end{cases}$$

THIS HELPS WITH NOTATION. FOR EXAMPLE,

1. IF f IS CONVEX, WE CAN WRITE

$$\tilde{f}(\theta x + (1-\theta)y) \le \theta \tilde{f}(x) + (1-\theta)\tilde{f}(y) \quad \forall x, y \in \mathbb{R}^n,$$

AND NOT ONLY FOR $x, y \in \operatorname{dom} f$. INDEED, IF x OR y NOT IN $\operatorname{dom} f$, THEN R.H.S $= \infty$

2. WE CAN WRITE THAT

$$f(x) = f_1(x) + f_2(x) \quad \forall x \in \operatorname{dom} f_1 \cap \operatorname{dom} f_2$$

ALTERNATIVELY, WE WRITE

$$\tilde{f}(x) = \tilde{f}_1(x) + \tilde{f}_2(x)$$ AND THE DOMAIN IS IMPLIED

DEFINITION: IF f IS CONCAVE, THEN

$$\tilde{f}(x) = \begin{cases} f(x), & x \in \operatorname{dom} f \\ -\infty, & x \notin \operatorname{dom} f \end{cases}$$

3.1.3 FIRST-ORDER CONDITION:

f IS CONVEX IFF $\operatorname{dom} f$ IS CONVEX AND

$$f(y) \ge f(x) + \nabla f(x)^T (y-x) \quad \forall x, y \in \operatorname{dom} f$$

$\leftarrow$ PLANE $f(x) + \nabla f(x)^T (y-x)$

THEREFORE THE TANGENT PLANE IS A GLOBAL UNDERESTIMATOR.

PROPERTY:

AS A SPECIAL APPLICATION: IF f IS CONVEX AND $\nabla f(x) = 0$ THEN $f(y) \ge f(x) \quad \forall y \in \operatorname{dom} f$,

I.E. WE HAVE A GLOBAL INFIMUM

PROOF OF FIRST-ORDER CONDITION:

FIRST, LET $n=1$

WE WILL SHOW THAT A DIFFERENTIABLE f IS CONVEX IFF

$$f(y) \geq f(x) + f'(x)(y-x) \quad (1)$$

($\Rightarrow$)

FIRST, LET f BE CONVEX, AND LET $x, y \in \operatorname{dom} f$

$\forall t \in [0,1]$,

$$f(x + t(y-x)) \leq (1-t) f(x) + t f(y)$$

$$\Rightarrow \quad f(y) \geq \frac{1}{t}\left[f(x+t(y-x)) - f(x)\right] + f(x)$$

$$\xRightarrow{t \to 0} \quad f(y) \geq f'(x)(y-x) + f(x)$$

($\Leftarrow$) NOW LET (1) HOLD. TAKE ANY $x, y \in \operatorname{dom} f$ AND ANY $\theta \in [0,1]$. LET $z = \theta x + (1-\theta) y$.

FROM (1), FOR x, z, WE HAVE

$$\left.\begin{aligned} f(x) &\geq f(z) + f'(z)(x-z) \\ \text{LIKEWISE:}& \\ f(y) &\geq f(z) + f'(z)(y-z) \end{aligned}\right\} \Rightarrow$$

$$\theta f(x) + (1-\theta) f(y) \geq f(z) + \theta f'(z)(x-z) + (1-\theta) f'(z)(y-z)$$

$$\Rightarrow \quad \theta f(x) + (1-\theta) f(y) \geq f(z) + f'(z)\left[\theta x - \theta z + (1-\theta) y + (\theta - 1) z\right]$$

$$= f'(z)\Big[\underbrace{\theta x + (1-\theta) y}_{z} - z\Big] + f(z) = f(z)$$

SO WE HAVE PROVED THE PROPERTY FOR $n=1$

GENERAL CASE (FOR ANY n).

($\Rightarrow$)

LET f BE CONVEX. LET ANY $x, y \in \text{dom} f$

$$\boxed{g(t) = f(ty + (1-t)x)}$$ IS CONVEX

THEREFORE, BY FIRST PART $g(1) \geq g(0) + g'(0)$

BUT $g'(t) = [\nabla f(ty + (1-t)x)]^T (y-x) \Rightarrow$

(FROM CHAIN RULE)

$$f(y) \geq f(x) + [\nabla f(x)]^T (y-x)$$

($\Leftarrow$) LET $f(y) \geq f(x) + [\nabla f(x)]^T (y-x)$ HOLD

FOR ANY x, y, AND $\text{dom} f$ IS CONVEX. WE WILL SHOW THAT $g(t)$ ABOVE IS CONVEX

LET $x, y \in \text{dom} f$ AND LET TWO POINTS

$$ty + (1-t)x, \quad \tilde{t}y + (1-\tilde{t})x \in \text{dom} f$$

APPLYING THE ABOVE INEQUALITY,

$$f(ty + (1-t)x) \geq f(\tilde{t}y + (1-\tilde{t})x) + [\nabla f(\tilde{t}y + (1-\tilde{t})x)]^T \begin{bmatrix} ty + (1-t)x \\ -\tilde{t}y - (1-\tilde{t})\tilde{x} \end{bmatrix}$$

$$\Rightarrow \quad g(t) \geq g(\tilde{t}) + g'(\tilde{t})(t-\tilde{t}) \quad \overline{(t-\tilde{t})(y-x)}$$

$\Rightarrow$ g IS CONVEX, BY FIRST PART

PROPERTY: (SECOND-ORDER CONDITION)

LET f BE TWICE DIFFERENTIABLE, I.E. $\nabla^2 f = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}$

EXIST AT EACH POINT OF $\mathrm{dom} f$, WHICH IS OPEN.

THEN:

1) f IS CONVEX $\iff$ $\mathrm{dom} f$ CONVEX AND $\nabla^2 f \succeq 0 \ \forall x \in \mathrm{dom} f$
2) f IS CONCAVE $\iff$ $\mathrm{dom} f$ CONVEX AND $\nabla^2 f \preceq 0 \ \forall x \in \mathrm{dom} f$
3) $\nabla^2 f \succ 0 \ \forall x \in \mathrm{dom} f \Rightarrow f$ IS STRICTLY CONVEX
4) $\nabla^2 f \prec 0 \ \forall x \in \mathrm{dom} f \Rightarrow f$ IS STRICTLY CONCAVE

NOTES: 1) 1D VERSION IS THAT $\frac{\partial^2 f}{\partial x^2} \geq 0$.

2) $f(x) = x^4$ IS STRICTLY CONVEX, BUT $\nabla^2 f = 0$ AT $x = 0$, SO INVERSE OF 3, 4 DOES NOT HOLD.

3) TAYLOR'S THEOREM:

$$f(x) = f(x_0) + \nabla f\big|_{x=x_0} (x - x_0) + \frac{1}{2} (x - x_0)^T \nabla^2 f\big|_{x=x_0} (x - x_0) + o\left((x - x_0)^2\right)$$

Proof: EITHER BY TAYLOR'S THEOREM, OR SIMILAR TO 1D-CASE

EXAMPLES IN 1D

1) EXPONENTIAL e^{ax} IS CONVEX IN $\mathbb{R}$ $\forall a \in \mathbb{R}$
2) x^a IS CONVEX IN $\mathbb{R}_{++}$ FOR $a \geq 1$ OR $a \leq 0$, AND CONCAVE FOR $0 \leq a \leq 1$
3) $|x|^p$ IS CONVEX IN $\mathbb{R}$ FOR $p \geq 1$
4) $\log x$ IS CONCAVE IN $\mathbb{R}_{++}$

NORMS ARE CONVEX:

If $f: \mathbb{R}^n \to \mathbb{R}$ is a norm,

$$f(\theta x + (1-\theta)y) \underset{\substack{\uparrow \\ \text{TRIANGLE INEQUALITY}}}{\leq} f(\theta x) + f((1-\theta)y) \underset{\substack{\uparrow \\ \text{HOMOGENEITY}}}{=} \theta f(x) + (1-\theta) f(y)$$

THE MAX FUNCTION IS CONVEX:

$$f(x) = \max_i x_i$$

$$f(\theta x + (1-\theta)y) = \max_i (\theta x_i + (1-\theta) y_i) \leq$$

$$\theta \max_i x_i + (1-\theta) \max_i y_i$$

$$= \theta f(x) + (1-\theta) f(y)$$

THE QUADRATIC OVER LINEAR FUNCTION IS CONVEX:

LET $f(x,y): \mathbb{R} \times \mathbb{R}_{++} \to \mathbb{R}$ $\quad f(x,y) = \dfrac{x^2}{y}$

WITH

$$\nabla^2 f(x,y) = \frac{2}{y^3} \begin{bmatrix} y^2 & -xy \\ -xy & x^2 \end{bmatrix} = \frac{2}{y^3} \begin{bmatrix} y \\ -x \end{bmatrix} \begin{bmatrix} y \\ -x \end{bmatrix}^T \succeq 0$$

OTHER CONVEX FUNCTIONS (WITH HARDER PROOFS)

1) LOG-SUM-EXP: $f(x) = \log\left[e^{x_1} + \dots + e^{x_n}\right]$ IS CONVEX ON $\mathbb{R}^n$

2) THE GEOMETRIC MEAN $f(x) = \left(\prod_{i=1}^{n} x_i\right)^{\frac{1}{n}}$ IS CONCAVE IN $\mathbb{R}^n_{++}$

3) $f(X) = \log\det X$ IS CONCAVE ON $\operatorname{dom} f = S^n_{++}$

3.1.6 SUBLEVEL SETS

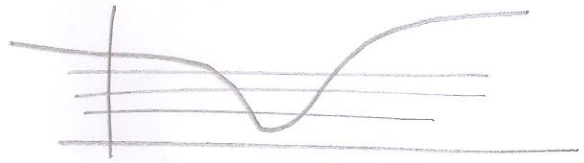
DEFINITION: α-SUBLEVEL SET OF A FUNCTION $f: \mathbb{R}^n \to \mathbb{R}$ IS $C_\alpha = \{x \in \mathrm{dom} f \mid f(x) \leq \alpha\}$

PROPERTY: C_α IS CONVEX IF $f: \mathbb{R}^n \to \mathbb{R}$ IS CONVEX.

PROOF IS EASY: IF $x \in C_\alpha$, $y \in C_\alpha$ THEN

$$f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta) f(1-x)$$
$$\leq \theta \alpha + (1-\theta)\alpha = \alpha.$$

INVERSE IS NOT TRUE, IE:



DEFINITION: α-SUPERLEVEL SET $\{x \in \mathrm{dom} f \mid f(x) \geq \alpha\}$

PROPERTY: IF f IS CONCAVE, THE α-SUPERLEVEL SET IS CONCAVE

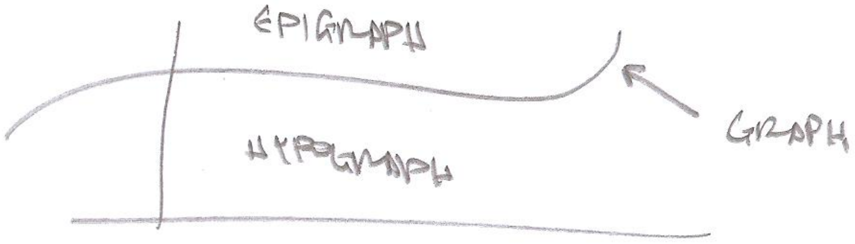
3.1.7.

DEFINITION: LET $f: \mathbb{R}^n \to \mathbb{R}$

GRAPH: $\{(x, f(x)) \mid x \in \mathrm{dom} f\} \in \mathbb{R}^{n+1}$

EPIGRAPH: $\mathrm{epi}\, f = \{(x, t) \mid x \in \mathrm{dom} f,\ t \geq f(x)\} \in \mathbb{R}^{n+1}$

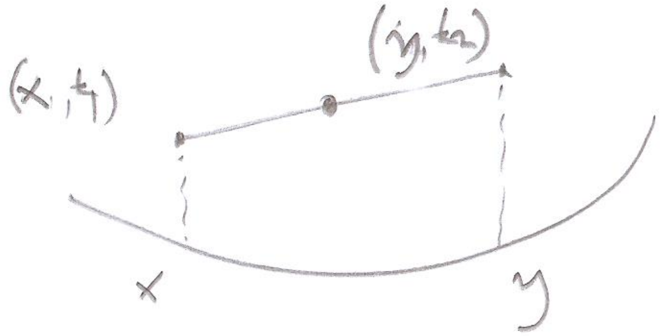
HYPOGRAPH: $\mathrm{hypo} f = \{(x, t) \mid x \in \mathrm{dom} f,\ t \leq f(x)\} \in \mathbb{R}^{n+1}$



<u>PROPERTY</u>: 1) f IS CONVEX $\Leftrightarrow$ EPIGRAPH IS CONVEX

2) f IS CONCAVE $\Leftrightarrow$ HYPOGRAPH IS CONVEX

<u>PROOF</u> of 1): ($\Rightarrow$) LET f CONVEX. LET $(x, t_1), (y, t_2) \in \text{epi} f$



$$\theta (x, t_1) + (1-\theta)(y, t_2) = \Big(\underbrace{\theta x + (1-\theta) y}_{\in \text{dom} f}, \theta t_1 + (1-\theta) t_2\Big)$$

ALSO $\theta t_1 + (1-\theta) t_2 \geq \theta f(x) + (1-\theta) f(y) \geq$
$f(\theta x + (1-\theta) y)$

SO EPIGRAPH IS CONVEX.

($\Leftarrow$) NOW LET EPIGRAPH BE CONVEX

LET $x, y \in \text{dom} f$. LET $\theta x + (1-\theta) y$.

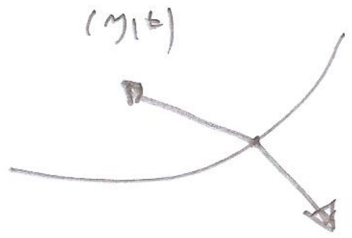
IT BELONGS TO $\text{dom} f$ BECAUSE

$$\theta (x, f(x)) + (1-\theta)(y, f(y)) \in \text{epi} f$$

ALSO $\theta f(x) + (1-\theta) f(y) \geq f(\theta x + (1-\theta) y)$, FOR SAME REASON

MANY RESULTS CAN BE EXPLAINED IN TERMS OF EPIGRAPHS. FOR EXAMPLE:

$$f(y) \geq f(x) + \nabla f^T(x)(y-x).$$



INDEED, LET $(y,t) \in \text{epi} f$

$$t \geq f(y) \geq f(x) + \nabla f^T(x)(y-x) \Leftrightarrow$$

$$\begin{bmatrix} \nabla f(x) \\ -1 \end{bmatrix}^T \left(\begin{bmatrix} y \\ t \end{bmatrix} - \begin{bmatrix} x \\ f(x) \end{bmatrix} \right) \leq 0$$

OBSERVE THAT THE VECTOR $\begin{bmatrix} \nabla f(x) \\ -1 \end{bmatrix}^T$ IS VERTICAL

TO TANGENT PLANE

$$y = f(x) + \nabla f^T(x)(x - x_0)$$

INDEED LET (x,y) BELONG TO PLANE. THEN

$$(x - x_0, y - y_0) \cdot (\nabla f(x) \;\; -1) =$$

$$\nabla f(x)(x - x_0) = y - y_0$$

3.1.8 JENSEN'S INEQUALITY

$$f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta) f(y)$$

PROPERTY: IF f IS CONVEX, THEN:

1) IF $x_1, x_2, \ldots, x_n \in \text{dom} f$, $\theta_1, \ldots, \theta_k \geq 0$, $\theta_1 + \theta_2 + \ldots + \theta_k = 1$

THEN $f(\theta_1 x_1 + \ldots + \theta_k x_k) \leq \theta_1 f(x_1) + \ldots + \theta_k f(x_k)$

2) IF $p(x) \geq 0$ ON $S \subseteq \mathrm{dom} f$, $\int_S p(x)\,dx = 1$,

THEN $$f\left(\int_S p(x)\,x\,dx\right) \leq \int_S f(x)\,p(x)\,dx$$

3) IF p IS ANY PROBABILITY MEASURE,

$$f(Ex) \leq Ef(x)$$

ALL THESE ARE CALLED JENSEN'S INEQUALITY

JENSEN'S INEQUALITY IS VERY USEFUL FOR PROVING INEQUALITIES.

<u>EXAMPLE 1:</u> $-\log x$ IS CONVEX $\Rightarrow$

$$-\log\left(\frac{a+b}{2}\right) \leq \frac{-\log a - \log b}{2} \Rightarrow$$

$$\log\left(\frac{a+b}{2}\right) \geq \frac{\log a + \log b}{2} \Rightarrow \frac{a+b}{2} \geq \exp\left[\log a^{\frac{1}{2}} + \log b^{\frac{1}{2}}\right]$$

$$= \sqrt{ab} \Rightarrow$$

$$\boxed{\frac{a+b}{2} \geq \sqrt{ab}}$$

<u>EXAMPLE 2:</u> HÖLDER'S INEQUALITY. FOR $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $x \in \mathbb{R}^n$, $y \in \mathbb{R}^n$,

$$\sum_{i=1}^{n} x_i y_i \leq \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} |y_i|^q\right)^{\frac{1}{q}}$$

PROOF:

$$-\log(\theta a + (1-\theta)b) \leq -\theta \log a - (1-\theta)\log b$$

$$\Rightarrow \quad \theta a + (1-\theta)b \geq \exp\left[\log a^{\theta} + \log b^{1-\theta}\right] \Leftrightarrow$$

$$\boxed{\theta a + (1-\theta)b \geq a^{\theta} b^{1-\theta}}$$

LET $\theta = \frac{1}{p}$, $a = \dfrac{|x_i|^p}{\underbrace{\boxed{\sum_{j=1}^{n}|x_j|^p}}_{X}}$, $b = \dfrac{|y_i|^q}{\underbrace{\boxed{\sum_{j=1}^{n}|y_j|^q}}_{Y}}$

$$\Rightarrow$$

$$\frac{|x_i|}{X^{\frac{1}{p}}}\ \frac{|y_i|}{Y^{\frac{1}{q}}} \leq \frac{1}{p}\ \frac{|x_i|^p}{X} + \frac{1}{q}\ \frac{|y_i|^q}{Y}$$

$$(+) \Rightarrow \quad \sum_{i=1}^{n} \frac{|x_i||y_i|}{X^{\frac{1}{p}} Y^{\frac{1}{q}}} \leq \frac{1}{p} \underbrace{\frac{\sum |x_i|^p}{X}}_{=1} + \frac{1}{q} \underbrace{\frac{\sum |y_i|^q}{Y}}_{=1}$$

$$\Rightarrow \quad \sum_{i=1}^{n} |x_i||y_i| \leq X^{\frac{1}{p}} Y^{\frac{1}{q}} \Rightarrow$$

$$\sum_{i=1}^{n} x_i y_i \leq X^{\frac{1}{p}} Y^{\frac{1}{q}}$$

WHICH IS HÖLDER'S INEQUALITY

3.2.1

PROPERTY: CONVEX FUNCTIONS ARE A CONVEX CONE, I.E.

IF $f_1, f_2, \ldots, f_m$ ARE CONVEX AND $w_i \geq 0$, $i=1,\ldots,m$

THEN SO IS $f = w_1 f_1 + w_2 f_2 + \ldots + w_m f_m$

Proof:

$$f(\theta x + (1-\theta)y) = w_1 f_1(\theta x + (1-\theta)y) + \ldots + w_m f_m(\theta x + (1-\theta)y)$$

$$\leq w_1 \theta f_1(x) + w_1(1-\theta) f_1(y) + \ldots$$

$$+ w_m \theta f_m(x) + w_m(1-\theta) f_m(y)$$

$$= \theta f(x) + (1-\theta) f(y)$$

ALTERNATIVELY, WE CAN USE EPIGRAPHS

PROPERTY: ALSO HOLDS FOR CONCAVE, STRICTLY CONCAVE, AND STRICTLY CONVEX FUNCTIONS

PROPERTY: IF $f(x,y)$ IS CONVEX W.R.T. x FOR ALL y,

THEN $g(x) = \int_A w(y) f(x,y)\, dy$ IS CONVEX, IF

$w(y) \geq 0$ AND INTEGRALS EXIST.

3.2.2

PROPERTY: COMPOSITION WITH AFFINE MAPPING MAINTAINS CONVEXITY AND CONCAVITY: LET $f: \mathbb{R}^n \to \mathbb{R}$, $A \in \mathbb{R}^{n \times m}$, $b \in \mathbb{R}^n$. DEFINE $g: \mathbb{R}^m \to \mathbb{R}$ BY

$$g(x) = f(Ax+b), \text{ WITH } \mathbf{dom}\, g = \{x \mid Ax+b \in \mathbf{dom} f$$

3.2.3

PROPERTY: IF f_1, f_2 CONVEX, THEN $f(x) = \max\{f_1(x), f_2(x)\}$ ALSO CONVEX.

EASY PROOF:

$$f(\theta x + (1-\theta) y) = \max\{f_1(\theta x + (1-\theta)y), f_2(\theta x + (1-\theta)y)\}$$

$$\leq \max\{\theta f_1(x) + (1-\theta) f_1(y), \theta f_2(x) + (1-\theta) f_2(y)\}$$

$$\leq \theta \max\{f_1(x), f_2(x)\} + (1-\theta) \max\{f_1(y), f_2(y)\}.$$

PROPERTY: IF $f_1, f_2, \ldots, f_m$ ARE CONVEX, THEN SO IS $f(x) = \max\{f_1(x), f_2(x), \ldots, f_m(x)\}$

EXAMPLE: $f(x) = \max\{a_1^T x + b_1, \ldots, a_L^T x + b_L\}$ IS CONVEX.

EXAMPLE: LET $x[i]$ BE i-TH LARGEST COMPONENT OF $x \in \mathbb{R}$. THEREFORE $x[1] \geq x[2] \geq \ldots \geq x[n]$.

LET $f(x) = \sum_{i=1}^{r} x[i]$. THEN f IS CONVEX, BECAUSE

$$f(x) = \sum_{i=1}^{r} x[i] = \max\{x_{i_1} + \ldots + x_{i_r} \mid 1 \leq i_1 < i_2 < \ldots < i_r \leq n\}$$

THIS PROPERTY CAN BE EXTENDED WHEN WE HAVE INFINITE NUMBER OF AFFINE FUNCTIONS:

PROPERTY: IF for all $y \in \mathcal{A}$, $f(x,y)$ is convex in x,

$$g(x) = \sup_{y \in \mathcal{A}} f(x,y) \quad \text{is convex.}$$

DOMAIN IS $\mathrm{dom}\, g = \{x \mid (x,y) \in \mathrm{dom}\, f \ \forall y \in A, \ \sup_{y \in A} f(x,y) < \infty\}$.

INDEED, ONE PROOF COMES FROM THE FACT THAT

$$\mathrm{epi}\, g = \bigcap_{y \in A} \underbrace{\mathrm{epi}\, f(\cdot, y)}_{\text{CONVEX}}$$

PROPERTY: POINTWISE INFIMUM OF CONCAVE FUNCTIONS IS CONCAVE

EXAMPLE: LET $C \subseteq \mathbb{R}^n$. LET

$$f(x) = \sup_{y \in C} \|x - y\|.$$

f IS CONVEX, BECAUSE FOR ANY y, $\|x-y\|$ IS CONVEX IN x

EXAMPLE: LET $f(X) = \lambda_{max}(X)$. WE KNOW THAT

$$f(X) = \sup\{y^T X y \mid \|y\|_2 = 1\}$$ WITH $\mathrm{dom}\, f = S^m$

THEREFORE $f(X)$ IS ALSO CONVEX AS THE SUPREMUM OF LINEAR FUNCTIONS

PROPERTY: LET C, D CONVEX, WITH $C \cap D = \emptyset$.

THEN $\exists\, a \neq 0, b$ SUCH THAT: $\begin{cases} a^T x \le b & \forall x \in C \\ a^T x \ge b & \forall x \in D \end{cases}$.

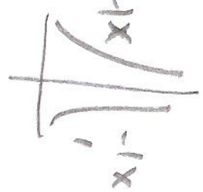
THEREFORE, THERE IS A PLANE SEPARATING THE TWO CONVEX SETS. THIS IS THE SEPARATING HYPERPLANE THEOREM.

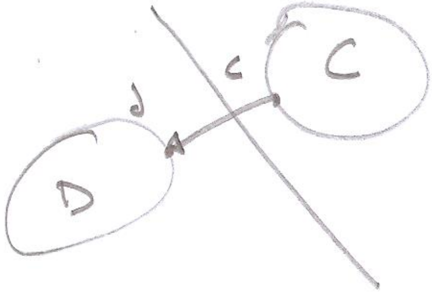
PROOF: WE CONSIDER SPECIAL CASE:

$$\operatorname{dist}(C,D) = \inf\left\{ \|u-v\|_2 \mid u \in C,\ v \in D \right\}$$

WE ASSUME THAT $\operatorname{dist}(C,D) > 0$ AND $\|c-d\|_2 = \operatorname{dist}(C,D)$, WHERE $c \in C$, $d \in D$.

THIS IS NOT ALWAYS TRUE, FOR EXAMPLE:





WE DEFINE

$$a = d - c, \quad b = \frac{\|d\|_2^2 - \|c\|_2^2}{2}$$

AND THE FUNCTION

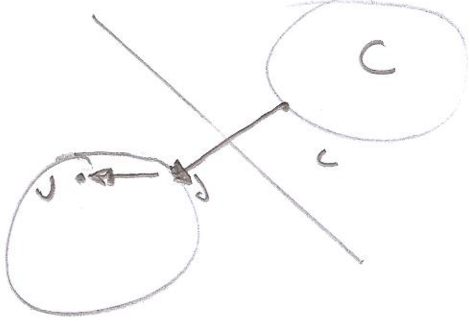
$$f(x) = a^T x - b = (d-c)^T \left(x - \frac{1}{2}(d+c) \right)$$

WE WILL SHOW THAT THIS IS THE AFFINE FUNCTION NEEDED, IF $f(x) \ge 0$ IN D AND $f(x) \le 0$ IN C.

WE WILL USE CONTRADICTION. LET $u \in D$ WITH $f(u) < 0$.

$$0 > f(u) = (d-c)^T \left(u - \frac{1}{2}(d+c) \right)$$

$$= (d-c)^T \left[u - d + \frac{1}{2}(d-c) \right] = \underbrace{(d-c)^T (u-d)}_{<0\text{, SINCE } f(u)<0} + \frac{1}{2}\|d-c\|_2^2$$



THE INEQUALITY MEANS THAT THESE TWO VECTORS HAVE AN OBTUSE ANGLE, WHICH IS NOT POSSIBLE, GEOMETRICALLY. INDEED:

LET THE FUNCTION $g(t) = \| d - c + t(u-d) \|_2$

(WHAT DOES IT MEAN?) THEN

$$g'(t) = \left[(d-c) + t(u-d) \right]^T \left[(d-c) + t(u-d) \right]'$$

$$= \left[\|d-c\|_2^2 + 2t\left[(u-d)^T (d-c) \right] + t^2 \|u-d\|_2^2 \right]'$$

$$= 2(u-d)^T (d-c) + 2t \|u-d\|_2^2 \quad \text{AND}$$

$g'(t) < 0$ AT $t = 0$, BY ABOVE INEQUALITY. THEREFORE, THERE IS SOME $t > 0$ $(t < 1)$ SUCH THAT

$$\| d + t(u-d) - c \|_2 < \|d - c\|_2$$

SO THE POINT $d + t(u-d)$ IS CLOSER TO c THAN d. BY CONVEXITY, THIS POINT BELONGS TO D, WHICH IS A CONTRADICTION

2.5.2

DEFINITION: LET $C \subseteq \mathbb{R}^n$ $x_0 \in \text{bd}\, C = \text{cl}\, C \setminus \text{int}\, C$

(bd: BOUNDARY; cl: CLOSURE)

IF $a \neq 0$ SATISFIES $a^T x \leq a^T x_0$ $\forall x \in C$, THEN $\{x \mid a^T x = a^T x_0\}$ IS A SUPPORTING HYPERPLANE

SUPPORTING HYPERPLANE THEOREM:

IF C IS CONVEX AND $x_0 \in \text{bd}\, C$, THEN THERE IS A SUPPORTING HYPERPLANE. PROOF: BY SEPARATING PLANE THEOREM

INVERSE: IF C IS CLOSED, HAS NONEMPTY INTERIOR AND HAS A SUPPORTING HYPERPLANE AT EVERY POINT IN BOUNDARY, IT'S CONVEX

BACK TO 3.2.3

THEOREM: IF $f: \mathbb{R}^n \to \mathbb{R}$ IS CONVEX WITH $\mathrm{dom} f = \mathbb{R}^n$

THEN

$$f(x) = \sup \{ g(x) \mid g \text{ AFFINE}, \ g(z) \le f(z) \ \forall z \}$$

PROOF: IT'S CLEAR THAT

$$f(x) \ge \sup \{ g(x) \mid g \text{ AFFINE}, \ g(z) \le f(z) \ \forall z \},$$

AS IT IS THE SUPREMUM OF UNDERESTIMATORS. SO WE JUST NEED TO PROVE EQUALITY. THE EPIGRAPH IS CONVEX, SO IT HAS A SUPPORTING PLANE, I.E. $w \in \mathbb{R}^n$, $b \in \mathbb{R}$ WITH $(w, b) \ne (0, 0)$, SUCH THAT

$$(w, b) \cdot (z, t) \ge k \quad \text{IF } (z, t) \in \text{EPIGRAPH}$$

INTUITION

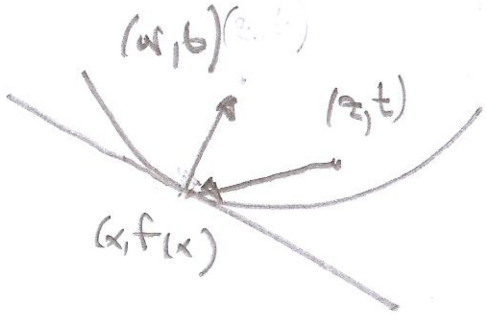
WE SELECT K SUCH THAT ABOVE INEQUALITY BECOMES

$$\begin{bmatrix} w \\ b \end{bmatrix}^T \begin{bmatrix} x - z \\ f(x) - t \end{bmatrix} \le 0$$

$$(\forall t \ge f(z))$$

$$\Longleftrightarrow$$

$$w^T (x - z) + b [f(x) - t] \le 0 \quad (1)$$



FOR THE INEQUALITY TO HOLD FOR $t \to \infty$, WE NEED $b \ge 0$. IF $b = 0$, THEN $w^T (x - z) \le 0 \ \forall z \in \mathbb{R}^n$, $\Rightarrow w = 0$ SO WE HAVE CONTRADICTION $\Rightarrow \boxed{b > 0}$

THIS MEANS THAT THE SUPPORTING PLANE IS NOT VERTICAL

SO NOW DEFINE

$$g(z) = f(x) + \left(\frac{w}{b} \right)^T (x - z)$$

OBSERVE THAT 1 $\Leftrightarrow$

$g(z)$ — $f(z)$

$$g(z) = f(x) + \frac{w^T}{b}(x-z) \leq t \qquad \forall t \geq f(z) \Rightarrow$$

$$\Rightarrow \quad g(z) = f(x) + \frac{w^T}{b}(x-z) \leq f(z), \text{ so}$$

IT IS AN UNDERESTIMATOR OF $f(x)$, AND

$g(x) = f(x)$, SO QED

3.2.4 COMPOSITION

WE CONSIDER $h: \mathbb{R}^k \to \mathbb{R}$, $g: \mathbb{R}^n \to \mathbb{R}^k$, AND

$f = h \circ g : \mathbb{R}^n \to \mathbb{R}$ WITH

$$f(x) = h(g(x)), \quad \operatorname{dom} f = \{x \in \operatorname{dom} g \mid g(x) \in \operatorname{dom} h\}$$

WE WILL CONSIDER CONDITIONS UNDER WHICH f IS CONVEX.

SCALAR CASE: $k=1$, SO $h: \mathbb{R} \to \mathbb{R}$.

TO DISCOVER THE RULES, LET $n=1$ AND h'', g'' EXIST IN ALL OF $\mathbb{R}$.

THEN:

$$f'(x) = h'(g(x))\, g'(x) \Rightarrow$$

$$f''(x) = h''(g(x))(g'(x))^2 + h'(g(x))\, g''(x)$$

THEREFORE:

1) $\left.\begin{array}{l} h \text{ CONVEX, NONDECREASING} \\ g \text{ CONVEX} \end{array}\right\} \Rightarrow f$ IS CONVEX

2) $\left.\begin{array}{l} h \text{ CONVEX, NONINCREASING} \\ g \text{ CONCAVE} \end{array}\right\} \Rightarrow f$ IS CONVEX

3) h CONCAVE, NONDECREASING, g CONCAVE $\Rightarrow$ f CONCAVE

4) h CONCAVE, NONINCREASING, g CONVEX $\Rightarrow$ f CONCAVE

THESE STATEMENTS GENERALIZE WITHOUT ASSUMING DIFFERENTIABILITY OF h, g OR $n=1$ OR ANY ASSUMPTION ON THEIR DOMAIN, AS FOLLOWS:

1) h CONVEX, $\tilde{h}$ NONDECREASING, g CONVEX $\Rightarrow$ f CONVEX

2) h CONVEX, $\tilde{h}$ NONINCREASING, g CONCAVE $\Rightarrow$ f CONVEX

3) h CONCAVE, $\tilde{h}$ NONDECREASING, g CONCAVE $\Rightarrow$ f CONCAVE

4) h CONCAVE, $\tilde{h}$ NON-INCREASING, g CONVEX $\Rightarrow$ f CONCAVE

THEREFORE, CONDITIONS USE THE EXTENDED-VALUE FUNCTION $\tilde{h}$.

FOR EXAMPLE:

$\tilde{h}$ NONDECREASING $\Rightarrow$ ITS DOMAIN EXTENDS TO ∞

$\tilde{h}$ NONINCREASING $\Rightarrow$ ITS DOMAIN EXTENDS TO $-\infty$

OTHERWISE, WE HAVE CONTRADICTION

EXAMPLES

1) SO, FOR EXAMPLE, IF $h(x) = x^2$, $x > 0$, THEN $h(x)$ IS NONDECREASING, BUT $\tilde{h}(x)$ IS NOT NONDECREASING.

2) ON THE OTHER HAND, $h(x) = \log x$, $x > 0$ IS NONDECREASING, AND $\tilde{h}(x)$ IS ALSO NONDECREASING

WE WILL PROVE THE FIRST PROPERTY:

$$\left.\begin{array}{l} h \text{ CONVEX} \\ \tilde{h} \text{ NONDECREASING} \\ g \text{ CONVEX} \end{array}\right\} \Rightarrow f \text{ CONVEX}$$

FIRST, WE NEED TO SHOW THAT $\operatorname{dom} f$ IS CONVEX!

LET $x, y \in \operatorname{dom} f$. WE NEED TO SHOW THAT $\theta x + (1-\theta) y \in \operatorname{dom} f$, $\forall \theta \in [0,1]$.

$x \in \operatorname{dom} f \Rightarrow x \in \operatorname{dom} g$, $g(x) \in \operatorname{dom} h$

$y \in \operatorname{dom} f \Rightarrow y \in \operatorname{dom} g$, $g(y) \in \operatorname{dom} h$.

h IS CONVEX $\Rightarrow$ $\theta g(x) + (1-\theta) g(y) \in \operatorname{dom} h$ $\Rightarrow$

ALSO $g(\theta x + (1-\theta) y) \leq \theta g(x) + (1-\theta) g(y)$

$\tilde{h}$ NONDECREASING $\Rightarrow$

$g(\theta x + (1-\theta) y) \in \operatorname{dom} h \Rightarrow \theta x + (1-\theta) y \in \operatorname{dom} h$

$\Rightarrow$ $\boxed{\operatorname{dom} f \text{ IS CONVEX}}$

NEXT, WE SHOW JENSEN'S INEQUALITY.

$$\underbrace{h(g(\theta x + (1-\theta) y)) \leq}_{\substack{g \text{ CONVEX} \\ h \text{ NONINCREASING}}} \underbrace{h(\theta g(x) + (1-\theta) g(y)) \leq}_{h \text{ CONVEX}}$$

$$\theta h(g(x)) + (1-\theta) h(g(y)) = \theta f(x) + (1-\theta) f(y)$$

EXAMPLES:

1) g CONVEX $\Rightarrow$ $\exp[g(x)]$ CONVEX.

2) g CONCAVE, POSITIVE $\Rightarrow$ $\log[g(x)]$ CONCAVE

3) g CONCAVE, POSITIVE $\Rightarrow$ $\dfrac{1}{g(x)}$ CONVEX

4) g CONVEX, NONNEGATIVE, $p \geq 1$, $g(x)^p$ CONVEX.

5) g CONVEX $\Rightarrow$ $-\log(-g(x))$ IS CONVEX ON $\{x \mid g(x)<0\}$

REMARK: MONOTONICITY OF $\tilde{h}$ IS ESSENTIAL.
FOR EXAMPLE, CONSIDER $g(x) = x^2$, $\mathrm{dom}\, g = \mathbb{R}$

$h(x) = 0$, $\mathrm{dom}\, h = [1,2]$

g CONVEX, $h(x)$ CONVEX, NONDECREASING.
BUT $f(x) = h(g(x))$ HAS $\mathrm{dom} f = [-\sqrt{2}, -1] \cup [1, \sqrt{2}]$
AND SO IS NOT CONVEX

3.4.1 QUASICONVEX FUNCTIONS

DEFINITION: 1) $f: \mathbb{R}^n \to \mathbb{R}$ IS CALLED QUASICONVEX (OR UNIMODAL) IF ITS DOMAIN AND ALL SUBLEVEL SET

$$S_\alpha = \{x \in \mathrm{dom} f \mid f(x) \leq \alpha\} \quad \forall \alpha \in \mathbb{R} \text{ ARE CONVEX.}$$

2) f IS QUASICONCAVE IF $-f$ IS QUASICONVEX $\Leftrightarrow$ ITS DOMAIN AND ALL SUPERLEVEL SETS ARE CONVEX

3) f IS QUASILINEAR IF IT IS QUASICONVEX AND QUASICONCAVE

EXAMPLES:

1) THE LOGARITHM $\log x$ IS CONCAVE AND QUASICONVEX
2) THE CEILING FUNCTION $\text{CEIL}(x) = \inf\{z \in \mathbb{Z} \mid z \geq x\}$
IS QUASILINEAR, AND NOT EVEN CONTINUOUS
3) $f : \mathbb{R}^2 \to \mathbb{R}$ WITH $\text{dom} f = \mathbb{R}_+^2$, $f(x_1, x_2) = x_1 x_2$.

$\nabla^2 f(x) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ WITH $\lambda_{1,2} = \pm 1 \Rightarrow$

f IS NEITHER CONVEX OR CONCAVE AND EACH POINT LOOKS LIKE A SADDLE POINT!
BUT IT IS QUASICONCAVE, SINCE $\{x \in \mathbb{R}_+^2 \mid x_1 x_2 \geq \alpha\}$
ARE CONVEX. IT IS NOT QUASICONVEX.

BASIC PROPERTIES

1) CONVEX FUNCTIONS ARE QUASICONVEX. LIKEWISE FOR CONCAVE FUNCTIONS. INVERSE DOES NOT HOLD.
2) JENSEN'S INEQUALITY FOR QUASICONVEX FUNCTIONS:

f IS QUASICONVEX IFF $\text{dom} f$ IS CONVEX
AND FOR ANY $x, y \in \text{dom} f$, $0 \leq \theta \leq 1$,
$$f(\theta x + (1-\theta) y) \leq \max\{f(x), f(y)\}$$
3) f IS QUASICONVEX IFF ITS RESTRICTION TO ANY LINE INTERSECTING ITS DOMAIN IS CONVEX
4) A CONTINUOUS FUNCTION $f : \mathbb{R} \to \mathbb{R}$ IS QUASICONVEX IFF ONE OF THE FOLLOWING HOLDS
A) $f \uparrow$ B) $\downarrow$
C) $\exists c \in \text{dom} f$: $f \downarrow$ FOR $t \leq c$ AND $f \uparrow$ FOR $t \geq c$