2.1.1.

CHAPTER 2

DEFINITION:

LINE PASSING THROUGH x_1, x_2 IS

$$\{y \in \mathbb{R}^n \mid y = \theta x_1 + (1-\theta)x_2,\ \theta \in \mathbb{R}\}$$

$$\Leftrightarrow \{y \in \mathbb{R}^n \mid y = \underbrace{x_2}_{\text{BASE POINT}} + \theta \underbrace{(x_1 - x_2)}_{\text{DIRECTION}},\ \theta \in \mathbb{R}\}$$

2.1.2

DEFINITION: $C \subseteq \mathbb{R}^n$ IS AFFINE IF $\forall x_1, x_2 \in C$ THE LINE PASSING THROUGH x_1, x_2 BELONGS TO C.

EXAMPLES: $\emptyset$, $\{x_0\}$ (SINGLETONS), $\mathbb{R}^n$, LINES, PLANES IN $\mathbb{R}^3$

DEFINITION: AFFINE COMBINATION OF POINTS $x_1, \dots, x_k$ IS ANY POINT $\theta_1 x_1 + \theta_2 x_2 + \dots + \theta_k x_k$, WHERE $\theta_1 + \theta_2 + \dots + \theta_k = 1$

(NOTE: IF WE DIDN'T HAVE THE LAST CONSTRAINT, THE COMBINATION WOULD BE A LINEAR COMBINATION, AS USUAL)

PROPERTIES OF AFFINE SPACES:

1) IF C IS AFFINE, THEN $\forall x_1, x_2, \dots x_k \in C$,
$\theta_1 x_1 + \theta_2 x_2 + \dots + \theta_k x_k$, WHERE $\theta_1 + \theta_2 + \dots + \theta_k = 1$, BELONGS TO C.

PROOF: BY INDUCTION (*)

2) IF C IS AN AFFINE SET AND $x_0 \in C$, THEN
$V = C - x_0 = \{x - x_0 \mid x \in C\}$ IS A SUBSPACE

(*) $\theta_1 x_1 + \theta_2 x_2 + \dots + \theta_k x_k + \theta_{k+1} x_{k+1} = \left[\sum_{i=1}^{k} \theta_i\right] [\dots\dots] + \theta_{k+1} x_{k+1}$

PROOF:

LET $v_1, v_2 \in V$. THEN $v_1 = x_1 - x_0$, $v_2 = x_2 - x_0$,

$x_1, x_2 \in C$.

$$a v_1 + b v_2 + x_0 = a x_1 - a x_0 + b x_2 - b x_0 + x_0 =$$

$$a x_1 + b x_2 + [1 - a - b] x_0$$

AND SINCE $a + b + 1 - a - b = 1$, IT BELONGS TO C,

THEREFORE $a v_1 + b v_2 + x_0 - x_0 \in V$, SO V IS CLOSED UNDER LINEAR COMBINATIONS

THEREFORE, $C = V + x_0$ WHERE V IS LINEAR SPACE

3) THE VECTOR SPACE OF THE PREVIOUS PROPERTY IS THE SAME IRRESPECTIVE OF THE CHOICE OF x_0

PROOF: LET $C = V_0 + x_0$, $C = V_1 + x_1$.

$$x_1 \in C \Rightarrow x_1 - x_0 \in V_0 \underset{\text{LINEAR SPACE}}{\Rightarrow} x_0 - x_1 \in V_0$$

$$\text{LET } a \in V_0 \underset{\text{LINEAR SPACE}}{\Rightarrow} a + (x_0 - x_1) \in V_0 \Rightarrow$$

<u>DEFINITION</u>: THE DIMENSION OF AN AFFINE SPACE C IS THE DIMENSION OF THE VECTOR SPACE V:

$$C = V + x_0$$

<u>DEFINITION</u>: THE <u>AFFINE HULL</u> OF $C \in \mathbb{R}^n$ IS THE SET OF ALL AFFINE COMBINATIONS OF POINTS IN C:

$$\text{aff } C = \{ \theta_1 x_1 + \dots + \theta_k x_k \mid x_1, x_2, \dots x_k \in C_k,\ \theta_1 + \dots + \theta_k = 1 \}$$

<u>PROPERTY</u>: THE AFFINE HULL of C IS THE SMALLEST AFFINE SET THAT CONTAINS C:

$$\left. S \text{ IS AFFINE},\ C \subseteq S \right\} \Rightarrow \text{aff } C \subseteq S$$

PROOF: LET ANY $x \in \text{aff } C$. THEN

$$x = \theta_1 x_1 + \dots + \theta_k x_k \quad \text{WITH } x_1, x_2, \dots x_k \in C$$

$\Rightarrow$ $x_1, x_2, \dots, x_k \in S$. BECAUSE S IS AFFINE, $x \in S$

<u>DEFINITION</u>: AFFINE DIMENSION OF A SET IS DIMENSION OF ITS AFFINE HULL

EXAMPLES:

1) AFFINE HULL OF LINE SEGMENT IS LINE
2) AFFINE HULL OF CIRCLE  IS WHOLE PLANE
3) AFFINE HULL OF SINGLE POINT IS SINGLE POINT
4) AFFINE HULL OF ANY SOLID (3D) SET IS WHOLE $\mathbb{R}^3$

2.1.4 CONVEX SET

DEFINITION: A SET C IS CONVEX IF

$$\forall x_1, x_2 \in C, \quad \forall \theta \in [0,1], \quad \theta x_1 + (1-\theta) x_2 \in C$$

DEFINITION: THE ABOVE SET IS A LINE SEGMENT

EXAMPLES:

, ETC.

(WHAT IS THE INTUITION?)

DEFINITION: $\theta_1 x_1 + \theta_2 + \dots + \theta_k x_k$, WHERE $\theta_i \geq 0$, $\sum \theta_i = 1$

IS A CONVEX COMBINATION OF THE POINTS $x_1, \dots, x_k$

DEFINITION: THE CONVEX HULL $\mathrm{conv}\, C$ OF $C \subseteq \mathbb{R}^n$ IS

$$\mathrm{conv}\, C = \{\theta_1 x_1 + \dots + \theta_k x_k \mid x_i \in C, \ \theta_i \geq 0, \ i = 1, \dots, k, \ \theta_1 + \dots + \theta_k = 1\}.$$

PROPERTY: $\mathrm{conv}\, C$ IS THE SMALLEST CONVEX SET THAT CONTAINS C.

~~PROOF~~: LET ANY CONVEX $S \supseteq C$

LET $x \in \mathrm{conv}\, C$. THEN x IS A CONVEX COMBINATION OF SOME $x_i \in C \Rightarrow x_i \in S \overset{S \text{ IS CONVEX}}{\Rightarrow} x \in S$

PROPERTY:
(GENERALIZATIONS OF CONVEX HULLS WITH CONVEX COMBINATIONS OF AN INFINITE NUMBER OF POINTS)

1) LET $\theta_i \geq 0$, $i = 1, 2, \ldots$ $\sum_{i=1}^{\infty} \theta_i = 1$

LET $x_1, x_2, \ldots \in C$, $C \subseteq \mathbb{R}^n$ CONVEX.

THEN $\sum_{i=1}^{\infty} \theta_i x_i \in C$

2) LET $p : \mathbb{R}^n \rightarrow \mathbb{R}$ SUCH THAT $p(x) \geq 0$, $\int_C p(x) = 1$.

THEN $\int_C p(x) x \, dx \in C$

3) LET $C \subseteq \mathbb{R}^n$ CONVEX, x IS RANDOM WITH PROBABILITY 1. THEN $\boxed{E(x) \in C.}$

WHEN VIEWED AS A DEFINITION, THIS IS THE MOST GENERAL ONE.

2.1.5. CONES

DEFINITION: C IS A CONE IF $\forall x \in C$, $\theta \geq 0$, $\theta x \in C$.

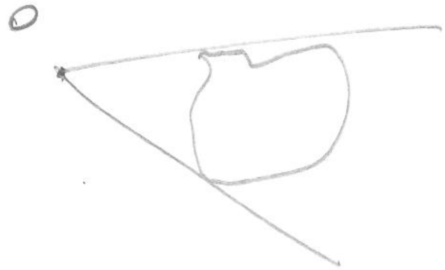
DEFINITION: C IS A CONVEX CONE IF IT IS A CONE AND IT IS CONVEX. THEREFORE $\forall \theta_1, \theta_2 \geq 0$, $\theta_1 x_1 + \theta_2 x_2 \in C$.

DEFINITION: A POINT $\theta_1 x_1 + \theta_2 x_2 + \ldots + \theta_k x_k$, $\theta_1, \theta_2 \ldots \theta_k \geq 0$ IS A CONIC COMBINATION OF $x_1, \ldots x_k$.

PROPERTY: C IS A CONVEX CONE IFF IT CONTAINS ALL CONIC COMBINATIONS OF ITS ELEMENTS

DEFINITION: CONIC HULL OF C IS

$$\{\theta_1 x_1 + \dots + \theta_n x_n \mid x_i \in C,\ \theta_i \geq 0,\ i=1,\dots,n\}$$



PROPERTY: THE CONIC HULL OF C IS THE SMALLEST CONVEX CONE THAT CONTAINS C

2.2.1 HYPERPLANES AND HALFSPACES

DEFINITION: A HYPERPLANE IS A SET OF THE FORM

$$H = \{x \mid a^T x = b\}, \quad a \in \mathbb{R}^n,\ a \neq 0,\ b \in \mathbb{R}.$$

PROPERTY: H IS AFFINE, WITH AFFINE DIMENSION $n-1$

INTUITION: OBSERVE THAT $a^T x$ IS THE INNER PRODUCT OF THE VECTORS a, x.

IN THE FOLLOWING, WE TREAT VECTORS AS COLUMN MATRICES.

NOTE THAT $H = \{x : a^T (x - x_0) = 0\} = x_0 + a^\perp$

WHERE THE ORTHOGONAL COMPLEMENT $a^\perp = \{v \mid a^T v = 0\}$

IN 2D, H IS A LINE

IN 3D, H IS A PLANE

DEFINITION: 1) (CLOSED) HALFSPACE: $H = \{x \mid a^T x \leq b\}$, $a \neq 0$.

2) OPEN HALFSPACE: $\{x \mid a^T x < b$.

PROPERTY: HALFSPACES ARE CONVEX BUT NOT AFFINE

INTUITION: $H = \{x \mid a^T (x - x_0) \leq 0\}$, WHERE $a^T x_0 = b$,

SO IT IS THE SET OF POINTS THAT FORM AN OBTUSE ANGLE WITH a

2.2.2

DEFINITION: (EUCLIDEAN) BALL

$$B(x_c, r) = \{x \mid \|x - x_c\|_2 \leq r\}$$

$$= \{x \mid (x - x_c)^T (x - x_c) \leq r^2\}$$

$$= \{x \mid x = x_c + r u \mid \|u\|_2 \leq 1\}$$

WHEN $x_c \in \mathbb{R}^n$, $r \in \mathbb{R}$. AND

$$\|(x_1, x_2, \ldots x_n)\|_2 = (x_1^2 + x_2^2 + \ldots + x_n^2)^{\frac{1}{2}}$$

PROPERTY: EUCLIDEAN BALLS ARE CONVEX.

PROOF: LET x_1, x_2: $\|x_1 - x_c\|_2 \leq r$, $\|x_2 - x_c\|_2 \leq r$

THEN

$$\|\theta x_1 + (1 - \theta_2) x_2 - x_c\|_2 =$$

$$\|\theta (x_1 - x_c) + (1 - \theta) (x_2 - x_c)\|_2 \overset{\text{(TRIANGE IDENTITY)}}{\leq}$$

$$\|\theta (x_1 - x_c)\|_2 + \|(1 - \theta)(x_2 - x_c)\|_2 \overset{\text{(HOMOGENEITY PROPERTY)}}{=}$$

$$\theta \|x_1 - x_c\|_2 + (1 - \theta) \|x_2 - x_c\|_2$$

DEFINITION: QUADRATIC FORM

ELLIPSOID:

$$\mathcal{E} = \{x \mid (x-x_c)^T P^{-1} (x-x_c) \le 1\}$$

$P = P^T \succ 0$, i.e. SYMMETRIC, POSITIVE DEFINITE.

LENGTHS of SEMI-AXES ARE $\sqrt{\lambda_i}$

BALL IS ELLIPSOID WITH $P = r^2 I$

PROPERTY: ELLIPSOIDS ARE CONVEX.

2.2.3

WE CAN DEFINE BALLS USING OTHER NORMS:

$$B(x_c, r) = \{x \mid \|x - x_c\|_1 \le r\}$$

$$\text{or} \quad \{x \mid \|x - x_c\|_\infty \le r\},$$

WHERE $\|(x_1, x_2, \dots, x_m)\|_1 = |x_1| + |x_2| + \dots + |x_m|$

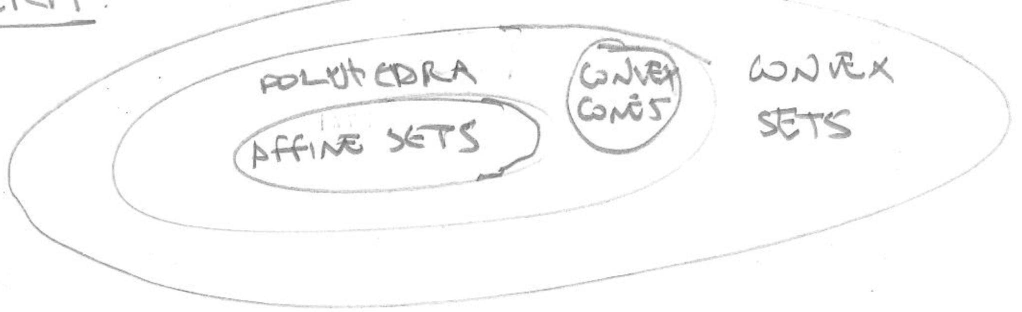
$$\|(x_1, x_2, \dots, x_m)\|_\infty = \max_{1 \le i \le m} |x_i|$$

2.2.4

DEFINITION: A POLYHEDRON (OR POLYTOPE)

$$\mathcal{P} = \{x \mid a_j^T x \le b_j,\ j = 1, \dots, m,\quad c_j^T x = d_j,\ j = 1, \dots, p\}$$

PROPERTY:



(NOTE: IT DOESN'T HAVE TO BE BOUNDED)

NOTATION:

$$P = \{x \mid Ax \preceq b,\ Cx = d\}$$

$$A = \begin{bmatrix} a_1^T \\ \vdots \\ a_m^T \end{bmatrix}, \quad C = \begin{bmatrix} c_1^T \\ \vdots \\ c_p^T \end{bmatrix}$$

AND $\preceq$ MEANS COMPONENTWISE INEQUALITY

EXAMPLE: NONNEGATIVE ORTHANT

$$\{x \in \mathbb{R}^n \mid x_i \geq 0,\ i=1,\dots,n\}$$
$$= \{x \in \mathbb{R}^n \mid x \succeq 0\}$$

IT IS BOTH A POLYHEDRON AND A CONE

DEFINITION: POINTS $v_0, v_1, \dots, v_k$ ARE AFFINELY INDEPENDENT IF THE AFFINE HULL OF ANY SUBSET OF THEM IS SMALLER THAN THEIR AFFINE HULL

PROPERTY: $v_0, v_1, \dots, v_k$ ARE AFFINELY INDEPENDENT IFF $v_1 - v_0, v_2 - v_0, \dots, v_k - v_0$ ARE LINEARLY INDEPENDENT

DEFINITION: LET $v_0, v_1, \dots, v_k$ BE AFFINELY INDEPENDENT THEN THE SIMPLEX DEFINED BY THEM IS

$$C = \mathbf{conv}\{v_0, \dots, v_k\} =$$
$$\{\theta_0 v_0 + \dots + \theta_k v_k \mid \theta \succeq 0,\ \mathbf{1}^T\theta = 1$$

IMPORTANT EXAMPLES:

1) 1D SIMPLEXES ARE LINE SEGMENTS
2) 2D SIMPLEXES ARE TRIANGLES AND LINESEGMENTS (INCLUDING INTERIOR) (BUT NOT SQUARES!)
3) 3D SIMPLEXES ARE TETRAHEDRA (INCLUDING THEIR INTERIOR), TRIANGLES AND LINE SEGMENTS

4) UNIT SIMPLEX: $\text{conv}\{0, e_1, e_2, \ldots e_m\}$

$$= \{x \mid x \succeq 0,\ \mathbf{1}^T x \leq 1\}$$

5) PROBABILITY SIMPLEX

$$\text{conv}\{e_1, e_2, \ldots, e_m\}$$

$$= \{x : x \succeq 0,\ \mathbf{1}^T x = 1\}$$

NOTE: MANY OPTIMIZATION PROBLEMS HAVE THESE EXACT CONSTRAINT SETS! (UNIT SIMPLEX AND PROBABILITY SIMPLEX)

PROPERTY: SIMPLEXES ARE POLYHEDRA

PROPERTY: CONVEX HULLS ARE FINITE SETS ARE BOUNDED POLYHEDRA HOWEVER IS IS HARD TO CONVERT EXPRESSIONS OF THE FORM

$$\{\theta_1 v_1 + \ldots + \theta_k v_k \mid \theta \succeq 0,\ \mathbf{1}^T \theta = 1\} \quad (1)$$

INTO EXPRESSIONS OF THE FORM

$$\{ x \mid a_j^T x \leq b_j,\ j = 1, \dots, m,\ G_j^T x = d_j,\ j = 1, \dots, p \} \quad (2)$$

A KIND OF INVERSE ALSO HOLDS:

PROPERTY: EVERY POLYHEDRON CAN BE EXPRESSED IN THE FORM

$$\{ \theta_1 v_1 + \dots + \theta_k v_k \mid \theta_1 + \dots + \underline{\underline{\theta_m}} = 1,\ \theta_i \geq 0,\ i = 1, \dots, k \}$$

NOT k!

IE. THE SET ADDITION OF THE CONVEX HULL OF $v_1, \dots, v_m$ AND THE CONIC HULL OF THE REST.

REMINDER: SET ADDITION OF A AND B =

$$A + B = \{ a + b \mid a \in A,\ b \in B \}$$

COMMENT: IT IS ALSO HARD TO GO FROM EXPRESSIONS OF THE FORM (2) TO "NICE" EXPRESSION OF THE FORM (1)

FOR EXAMPLE:

$$C = \{ x \mid |x_i| \leq 1,\ i = 1, \dots, n \} = \mathrm{conv}\{ v_1, v_2, \dots v_{2^n} \}$$

2n INEQUALITIES $\qquad$ 2^n POINTS,

WHERE v_i ARE ALL POINTS WITH COMPONENTS +1 OR −1

2.2.5

DEFINITION:

1) $$S^n = \{X \in \mathbb{R}^{n \times n} \mid X = X^T\}$$

(VECTOR SPACE OF DIMENSION $\frac{n(n+1)}{2}$

2) $$S^n_+ = \{X \in S^n \mid X \succeq 0\}$$

I.E., THE SET OF SYMMETRIC POSITIVE DEFINITE MATRICES

3) $$S^n_{++} = \{X \in S^n \mid X \succ 0\}$$, I.E. THE SET OF POSITIVE DEFINITE MATRICES

PROPERTY: S^n_+ AND S^n_{++} ARE CONVEX

INDEED, LET, E.G. $S_1, S_2 \in S^n_+$.

$\forall x \in \mathbb{R}^n$,

$$\left.\begin{array}{l} x^T S_1 x \geq 0 \\ x^T S_2 x \geq 0 \end{array}\right\} \Rightarrow x^T(\theta S_1 + (1-\theta)S_2)x \geq 0$$

2.3 OPERATIONS THAT PRESERVE CONVEXITY

2.3.1 INTERSECTION !

PROPERTY: IF S_1, S_2 ARE CONVEX, $S_1 \cap S_2$ IS CONVEX

IF FAMILY $\{S_\alpha : \alpha \in A\}$ IS CONVEX, SO IS $\bigcap_{\alpha \in A} S_\alpha$

PROOF: SIMPLE! LET $x_1, x_2 \in S_1 \cap S_2$, $\theta \in [0,1]$.

$$\left.\begin{aligned} &\theta x_1 + (1-\theta) x_2 \in S_1 \quad \text{BECAUSE IT IS CONVEX} \\ &\theta x_1 + (1-\theta) x_2 \in S_2 \quad \text{BECAUSE IT IS CONVEX} \end{aligned}\right\} \Rightarrow$$

$$\theta x_1 + (1-\theta) x_2 \in S_1 \cap S_2$$

EXAMPLE 1: POLYHEDRA ARE CONVEX.

EXAMPLE 2:

$$S_+^n = \bigcap_{z \neq 0} \underbrace{\{X \in S^n \mid z^T X z \geq 0\}}_{\text{HALFSPACE WITH RESPECT TO ELEMENTS OF } X}$$

PROPERTIES: EVERY CLOSED CONVEX SET IS THE INTERSECTION OF ALL HALFSPACES THAT CONTAIN IT.

$$S = \bigcap \{H \mid H \text{ HALFSPACE}, S \subseteq H\}$$

2.3.2 DEFINITION: $f: \mathbb{R}^n \to \mathbb{R}^m$ IS AFFINE

IF $f(x) = Ax + b$, $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$

PROPERTY: LET $S \subseteq \mathbb{R}^n$ CONVEX, $f: \mathbb{R}^n \to \mathbb{R}^m$ AFFINE

THEN $f(S) = \{ f(x) \mid x \in S \}$ IS AFFINE

PROOF: LET $y_1 \in f(S)$ $\quad y_1 = f(x_1) = Ax_1 + b$

$y_2 \in f(S)$ $\quad y_2 = f(x_2) = Ax_2 + b$

THEN

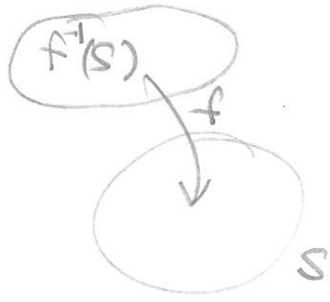
$$\theta y_1 + (1-\theta) y_2 = \theta A x_1 + \theta b + (1-\theta) A x_2 + (1-\theta) b$$
$$= \underbrace{A \left[\underbrace{\theta x_1 + (1-\theta) x_2}_{\in S} \right] + b}_{\in f(S)}$$

PROPERTY: LET $f: \mathbb{R}^k \to \mathbb{R}^n$ AFFINE AND $S \subseteq \mathbb{R}^n$ CONVEX.

THEN THE INVERSE IMAGE OF S UNDER f IS

CONVEX:

$$f^{-1}(S) = \{ y \mid f(y) \in S \}$$



PROOF: LET $y_1 \in f^{-1}(S) \Rightarrow f(y_1) \in S$

$y_2 \in f^{-1}(S) \Rightarrow f(y_2) \in S$

S CONVEX

$$\Rightarrow \quad \theta f(y_1) + (1-\theta) f(y_2) \in S \Rightarrow$$

$$f[\theta y_1 + (1-\theta) y_2] \in S \Rightarrow$$

$$\theta y_1 + (1-\theta) y_2 \in f^{-1}(S)$$

EXAMPLES

1) SCALING MAINTAINS CONVEXITY:

$$S \subseteq \mathbb{R}^n \Rightarrow \lambda S \text{ is convex} \quad (\lambda \in \mathbb{R})$$

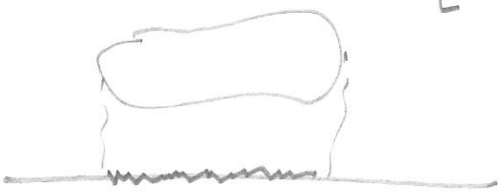
2) TRANSLATION MAINTAINS CONVEXITY

$$S \subseteq \mathbb{R}^n \Rightarrow S + a \text{ is convex} \quad (a \in \mathbb{R}^n)$$

3) THE PROJECTION OF A CONVEX SET INTO SOME OF ITS COORDINATES IS CONVEX:

IF $S \subseteq \mathbb{R}^m \times \mathbb{R}^n$ IS CONVEX, THEN

$$T = \{ x_1 \in \mathbb{R}^m \mid (x_1, x_2) \in S \text{ for some } x_2 \in \mathbb{R}^n \}$$



$$f(x) = \underbrace{\begin{bmatrix} I_{m\times m} & 0_{n\times m} \end{bmatrix}}_{m\times(m+m)} \underbrace{\begin{bmatrix} x_{11} \\ \vdots \\ x_{1m} \\ x_{21} \\ \vdots \\ x_{2m} \end{bmatrix}}_{(m+m)\times 1} = \begin{bmatrix} x_{11} \\ \vdots \\ x_{1m} \end{bmatrix}$$

(ASIDE: IF $S_1 \in \mathbb{R}^n$, $S_2 \in \mathbb{R}^m$ ARE CONVEX, THEN SO IS THE CARTESIAN PRODUCT $S_1 \times S_2$.

INDEED: LET $(x_1, y_1) \in S_1 \times S_2$ $\quad (x_2, y_2) \in S_1 \times S_2$

THEN $\theta(x_1, y_1) + (1-\theta)(x_2, y_2) =$

$$(\underbrace{\theta x_1 + (1-\theta)x_2}_{\in S_1}, \underbrace{\theta y_1 + (1-\theta) y_2}_{\in S_2}) \in S_1 \times S_2$$
)

4) SUM OF TWO SETS, $S_1 + S_2$ IS CONVEX, UNDER THE LINEAR FUNCTION $f(x_1, x_2) = x_1 + x_2 \in \mathbb{R}$.

IN MORE DETAIL: LET S_1, S_2 CONVEX. THEN $S_1 \times S_2$ CONVEX.

LET $f(x_1, x_2) = [I \ \ I] \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 + x_2$

5) PARTIAL SUM OF $S_1, S_2 \in \mathbb{R}^n \times \mathbb{R}^m$ IS

$$S = \{(x, y_1 + y_2) \mid (x, y_1) \in S_1, (x, y_2) \in S_2\}$$

WHERE $x \in \mathbb{R}^n$, $y_i \in \mathbb{R}^m$

NOTE THAT $\begin{cases} m = 0: & \text{INTERSECTION} \\ n = 0: & \text{SET ADDITION} \end{cases}$

PARTIAL SUM IS ALSO CONVEX.

6) THE POLYHEDRON $\{x \mid Ax \leq b, \ Cx = d\}$ IS AN INVERSE IMAGE, AS FOLLOWS:

$x \in \mathbb{R}^n$

$$\mathbb{R}^n \xrightarrow{f = (b - Ax, \ d - Cx)} \mathbb{R}^{m+k}$$

$A_{m \times n}$

$C_{k \times n}$

$d_{k \times 1}$

$b_{m \times 1}$

$$S = (\underset{\substack{\geq \\ 0}}{y_{m \times 1}}, \ \underset{\substack{\| \\ (0,0,\ldots 0)}}{z_{k \times 1}})$$

INVERSE IMAGE OF S

IS $\{x : b - Ax \geq 0, \ d - Cx = 0\}$