Notes:

1. Duration: 2.5 hours
2. Explain everything carefully. You will be graded on the clarity of your arguments.

Exercises:

1. *(2 points)* Let a convex set $C \subset \mathbb{R}^n$ and a point $x_0 \notin C$. Show that

$$\text{conv}\,(C \cup \{x_0\}) = \{(1-\theta)x + \theta x_0 : x \in C, \theta \in [0,1]\}.$$

2. *(2 points)*

   (α′) Let $f : [a,b] \to \mathbb{R}$ convex function defined on a closed interval. Show that f is bounded above by $\max\{f(a), f(b)\}$.

   (β′) Generalize this property when $f : A \to \mathbb{R}$ and $A \subseteq \mathbb{R}^n$ and give its proof.

3. *(2 points)* Let the function

$$f(x,y,z) = xyz,$$

   defined for $x, y, z \geq 0$. Is this function convex?

4. *(2 points)* Consider the problem

$$\begin{aligned} \text{minimize:} \quad & xyz \\ \text{subject to:} \quad & x + y^2 + z^4 \leq 1, \qquad x + y + z = 1, \end{aligned}$$

   defined for $x, y, z \in \mathbb{R}^3$.

   (α′) Bring the problem in the standard form of an optimization problem.

   (β′) Is the problem convex? Explain?

   (γ′) Write the Lagrangian.

   (δ′) Write the KKT conditions for this problem, but do NOT solve them.

5. *(2 points)* Consider the problem

$$\begin{aligned} \text{minimize:} \quad & \tfrac{1}{2}\|x - b\|^2 + t, \\ \text{subject to:} \quad & t \geq a_i^T x + c_i, \quad i = 1, \ldots, m, \end{aligned}$$

   where we optimize with respect to both $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$, whereas $b, a_i \in \mathbb{R}^n$ and $c_i \in \mathbb{R}$ are parameters. Define the dual problem (but do not solve it).