Notes:

1. Duration: 3 hours
2. Explain everything carefully. You will be graded on the clarity of your arguments.

Exercises:

1. We define the sum of two sets $A, B \subseteq \mathbb{R}^n$ and the product of a real number λ with a set $A \subseteq \mathbb{R}^n$ as:

$$\begin{aligned} A + B &\triangleq \{x = a + b : a \in A,\ b \in B\}, \\ \lambda A &\triangleq \{x = \lambda a :\ a \in A\}. \end{aligned}$$

   (α′) *(0.5 point)* Prove that for all $\lambda \in \mathbb{R}$ and all $A, B \subseteq \mathbb{R}^2$ we have $\lambda(A + B) = \lambda A + \lambda B$.

   (β′) *(0.5 point)* Prove that, on the other hand, the following property does not hold, by providing a counterexample: for all $\lambda, \mu \in \mathbb{R}$, and $A \subseteq \mathbb{R}^n$, $(\lambda + \mu)A = \lambda A + \mu A$.

   (γ′) *(1 point)* Prove that if $\lambda,\ \mu \geq 0$ and $A \subset \mathbb{R}^n$ is convex, than $(\lambda + \mu)A = \lambda A + \mu A$.

   (δ′) *(0.5 point)* Prove that the previous property does not hold is A is not convex.

   Hint: two sets A and B are equal iff $A \subseteq B$ and $B \subseteq A$.

2. *(2 points)* Prove that the following function is convex:

$$f(x, y) = x^2 - 4xy + 5y^2 - \log(xy), \qquad x, y > 0$$

3. Consider the following optimization problem:

$$\begin{aligned} \text{minimize:} \quad & 3x_1 + 7x_2 + 10x_3, \\ \text{subject to:} \quad & x_1 + 3x_2 + 5x_3 \geq 7, \\ & x_1(1 - x_1) = 0, \quad x_2(1 - x_2) = 0, \quad x_3(1 - x_3) = 0. \end{aligned}$$

   (α′) *(0.5 point)* What is the solution of the above problem?

   (β′) *(0.5 point)* Write the Lagrangian for the above problem.

   (γ′) *(0.5 point)* What is the dual function when any $\mu_i > 0$?

   (δ′) *(1 point)* What is the dual function when all $\mu_i < 0$? (the other cases are trickier)

4. Consider the following optimization problem:

$$\begin{aligned} \text{minimize:} \quad & (x - 2)^2 + 2(y - 1)^2 \\ \text{subject to:} \quad & x + 4y \leq 3, \qquad x \geq y \end{aligned}$$

   (α′) *(0.5 point)* Explain why it is a convex optimization problem.

   (β′) *(0.5 point)* Draw a plot showing the constraints and the contours of the optimization function

   (γ′) *(0.5 point)* Write the Lagrangian.

   (δ′) *(0.5 point)* Write the KKT conditions.

   (ε′) *(1 point)* Solve the KKT conditions