Notes:

1. Duration: 2.5 hours
2. Explain everything carefully. You will be graded on the clarity of your arguments.

Exercises:

1. We define the sum of two sets $A, B \subseteq \mathbb{R}^n$ and the product of a real number λ with a set $A \subseteq \mathbb{R}^n$ as:

$$\begin{aligned} A + B &\triangleq \{x = a + b : a \in A,\ b \in B\}, \\ \lambda A &\triangleq \{x = \lambda a :\ a \in A\}. \end{aligned}$$

(α′) *(0.5 point)* Prove that for all $\lambda \in \mathbb{R}$ and all $A, B \subseteq \mathbb{R}^2$ we have $\lambda(A + B) = \lambda A + \lambda B$.

(β′) *(0.5 point)* Prove that, on the other hand, the following property does not hold, by providing a counterexample: for all $\lambda, \mu \in \mathbb{R}$, and $A \subseteq \mathbb{R}^n$, $(\lambda + \mu)A = \lambda A + \mu A$.

(γ′) *(1 point)* Prove that if $\lambda,\ \mu \geq 0$ and $A \subset \mathbb{R}^n$ is convex, than $(\lambda + \mu)A = \lambda A + \mu A$.

(δ′) *(0.5 point)* Prove that the previous property does not hold is A is not convex.

Hint: two sets A and B are equal iff $A \subseteq B$ and $B \subseteq A$.

Solution

(α′) Let $x \in \lambda(A + B)$. Then $x = \lambda(a + b) = \lambda a + \lambda b$, so $x \in \lambda A + \lambda B$. Now let $x \in \lambda A + \lambda B$. Then $x = \lambda a + \lambda b = \lambda(a + b)$ and so $x \in \lambda(A + B)$.

(β′) If $\lambda = -\mu$ and not zero, then the left-hand side is the set $\{0\}$, but the rigght hand side is in general not. For example, $[0, 1] - [0, 1] = [-1, 1]$.

(γ′) Let $x \in (\lambda + \mu)A$. Then $x = (\lambda + \mu)a = \lambda a + \mu a$ for some $a \in A$, therefore $x \in \lambda A + \mu A$. Now let $x \in \lambda A + \mu A$. We have

$$x = \lambda a_1 + \mu a_2 = (\lambda + \mu)\left(\frac{\lambda}{\lambda + \mu}a_1 + \frac{\mu}{\lambda + \mu}a_2\right),$$

and since A is convex, $\left(\frac{\lambda}{\lambda+\mu}a_1 + \frac{\mu}{\lambda+\mu}a_2\right)$ belongs to A, and so $x \in (\lambda + \mu)A$.

(δ′) As a counterexample, consider $\lambda = \mu = 1$ and $A = \{0, 1\}$. In this case, $2A = \{0, 2\}$ but $A + A = \{0, 1, 2\}$

2. *(2 points)* Prove that the following function is convex:

$$f(x, y) = x^2 - 4xy + 5y^2 - \log(xy), \qquad x, y > 0$$

Solution: We just compute the Hessian and show that it is positive. We have:

$$\begin{aligned} \frac{\partial f}{\partial x} &= 2x - 4y - \frac{1}{x}, \quad \frac{\partial^2 f}{\partial x^2} = 2 + \frac{1}{x^2}, \\ \frac{\partial f}{\partial y} &= -4x + 10y - \frac{1}{y}, \quad \frac{\partial^2 f}{\partial x^2} = 10 + \frac{1}{y^2}, \\ & \qquad\qquad\qquad\qquad \frac{\partial^2 f}{\partial x \partial y} = -4, \end{aligned}$$

therefore the Hessian is

$$H = \begin{bmatrix} 2 + \frac{1}{1}x^2 & -4 \\ -4 & 10 + \frac{1}{y^2} \end{bmatrix},$$

whose characteristic polynomial is, setting $A = 2 + \frac{1}{x^2}$ and $B = 2 + \frac{1}{y^2}$,

$$(A - s)(B - s) - 16 = s^2 - (A + B)s + AB - 16,$$

with determinant

$$\Delta = (A + B)^2 - 4(AB - 16) = (A - B)^2 + 48,$$

and solutions

$$s_{1,2} = \frac{A + B \pm \sqrt{(A - B)^2 + 48}}{2},$$

which can be shown to be always positive, observing that $A, B > 0$ and taking cases on whether $A \geq B$ or $B \geq A$.

3. Consider the following optimization problem:

$$\begin{array}{ll} \text{minimize:} & 3x_1 + 7x_2 + 10x_3, \\ \text{subject to:} & x_1 + 3x_2 + 5x_3 \geq 7, \\ & x_1(1-x_1) = 0, \quad x_2(1-x_2) = 0, \quad x_3(1-x_3) = 0. \end{array}$$

(α′) *(0.5 point)* What is the solution of the above problem?

(β′) *(0.5 point)* Write the Lagrangian for the above problem.

(γ′) *(0.5 point)* What is the dual function when any $\mu_i > 0$?

(δ′) *(1 point)* What is the dual function when all $\mu_i < 0$? (the other cases are trickier)

Solution

(α′) The variables $x_1,\ x_2,\ x_3$ can only take the values 0 and 1. There are 2^3 points that satisfy the equalities, and only two of them also satisfy the inequality, $(1,1,1)$ and $(0,1,1)$. The second one is clearly the optimum solution, for which the objective function becomes $f(x_1,x_2,x_3) = 17$.

(β′)

$$\begin{aligned} & L(x_1,x_2,x_3,\lambda,\mu_1,\mu_2,\mu_3) \\ = \;& 3x_1 + 7x_2 + 10x_3 + \lambda(7 - x_1 - 3x_3 - 5x_3) + \mu_1 x_1(1-x_2) + \mu_2 x_2(1-x_2) + \mu_3 x_3(1-x_3) \\ = \;& 7\lambda + (3x_1 - \lambda x_1 + \mu_1 x_1(1-x_1)) + (7x_2 - 3\lambda x_2 + \mu_2 x_2(1-x_1)) + (10x_3 - 5\lambda x_3 + \mu_3 x_3(1-x_3)) \\ = \;& 7\lambda + \left(-\mu_1 x_1^2 + (3 - \lambda + \mu_1)x_1\right) + \left(-\mu_2 x_2^2 + (7 - 3\lambda + \mu_2)x_2\right) + \left(-\mu_3 x_1^2 + (10 - 5\lambda + \mu_3)x_3\right) \end{aligned}$$

(γ′) When any of the μ_i is positive, by taking $x \to \pm\infty$ the Lagrangian goes to $-\infty$, so $g(\lambda,\mu_1,\mu_2,\mu_3) = -\infty$.

(δ′) In this case, each of the three trinomials has a minimum. Let $h(x) = ax^2 + bx$ a scalar function with $a > 0$. By taking the deriviative, its minimum is achieved for $x = -\frac{b}{2a}$, where

$$h\left(-\frac{b}{2a}\right) = a\frac{b^2}{4a^2} - \frac{b^2}{2a} = -\frac{b^2}{4a}$$

Using this result in the three functions of x_1, x_2, and x_3, we have

$$g(\lambda,\mu_1,\mu_2,\mu_3) = 7\lambda + \frac{3 - \lambda + \mu_1}{\mu_1} + \frac{7 - 3\lambda + \mu_2}{\mu_2} + \frac{10 - \lambda + \mu_3}{\mu_3}.$$

4. Consider the following optimization problem:

$$\begin{array}{ll} \text{minimize:} & (x-2)^2 + 2(y-1)^2 \\ \text{subject to:} & x + 4y \leq 3, \qquad x \geq y \end{array}$$

(α′) *(0.5 point)* Explain why it is a convex optimization problem.

(β′) *(0.5 point)* Draw a plot showing the constraints and the contours of the optimization function

(γ′) *(0.5 point)* Write the Lagrangian.

(δ′) *(0.5 point)* Write the KKT conditions.

(ε′) *(1 point)* Solve the KKT conditions

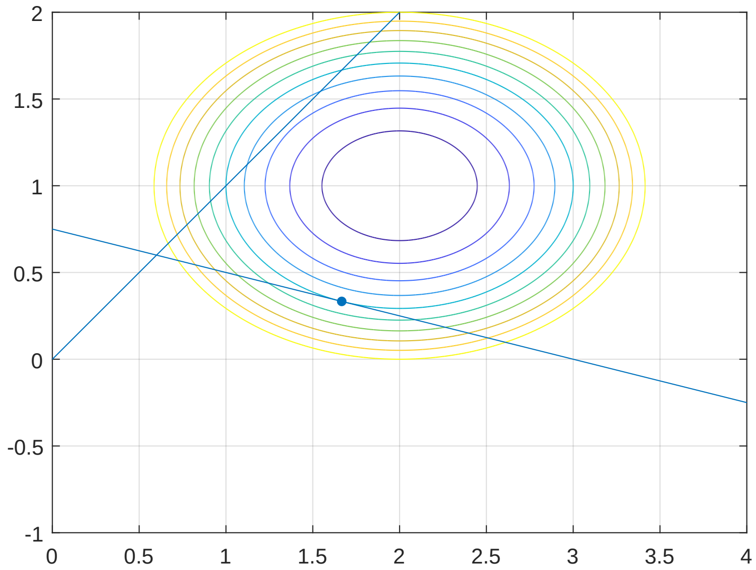
Solution:

(α′) The inequalities are linear and the Hessian of the quadratic optimization function is positive definite. Therefore, this is a convex optimization function.

(β′) See the plot

(γ′)

$$L(x,y,\lambda_1,\lambda_2) = (x-2)^2 + 2(y-1)^2 + \lambda_1(x + 4y - 3) + \lambda_2(y - x).$$



Σχήμα 1: Problem 1

(δ´) The KKT conditions are:

$$x + 4y \leq 3, \qquad y - x \leq 0 \qquad \lambda_1 \geq 0, \qquad \lambda_2 \geq 0,$$
$$\lambda_1(x + 4y - 3) = 0, \qquad \lambda_2(y - x) = 0, \quad 2(x-2) + \lambda_1 - \lambda_2 = 0, \quad 4(y-1) + 4\lambda_1 + \lambda_2 = 0.$$

(ε´) We consider four case using the two complementary slackness conditions.

i. If $\lambda_1 = 0$ and $\lambda_2 = 0$, then it follows that $y = 1$ and $x = 2$, which, however violates the first constraint. This is the location of the unconstrained minimum.

ii. If $\lambda_1 > 0$ and $\lambda_2 > 0$, then it follows that $y = x$ and $x + 4y - 3 = 0$, which means that $x = y = \frac{3}{5}$. In this case, the constraints on the gradient of the Lagrangian are:

$$\begin{aligned} 2\left(\tfrac{3}{5} - \tfrac{10}{5}\right) + \lambda_1 - \lambda_2 = 0, \\ 4\left(\tfrac{3}{5} - \tfrac{5}{5}\right) + 4\lambda_1 + \lambda_2 = 0, \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} \lambda_1 - \lambda_2 = \tfrac{14}{5}, \\ 4\lambda_1 + \lambda_2 = \tfrac{8}{5} \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} \lambda_1 = \tfrac{22}{25} \\ \lambda_2 = \tfrac{22}{25} - \tfrac{14}{5}, \end{aligned}$$

so $\lambda_2 < 0$ and again this is not permitted.

iii. If $\lambda_1 = 0$ but $\lambda_2 > 0$, then it follows that $x = y$, and then

$$\begin{aligned} 2(x-2) - \lambda_2 = 0, \\ 4(x-1) + \lambda_2 = 0 \end{aligned} \quad \Rightarrow 2x - 4 + 4x - 4 = 0 \Rightarrow x = \frac{4}{3} \Rightarrow y = \frac{4}{3},$$

which violates the first constraint, so this case is excluded.

iv. The last case is $\lambda_1 > 0$ and $\lambda_2 = 0$, in which case $x + 4y - 3 = 0 \Rightarrow x = 3 - 4y$, and so requiring the gradient of the Lagrangian to be zero we have

$$\begin{aligned} 2(3 - 4y - 2) + \lambda_1 = 0, \\ 4y - 4 + 4\lambda_1 = 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} 2 - 8y + \lambda_1 = 0, \\ 8y - 8 + 8\lambda_1 = 0 \end{aligned} \quad \Rightarrow -6 + 9\lambda_1 = 0 \Rightarrow \lambda_1 = \frac{2}{3},$$

and so

$$8y = 2 + \frac{2}{3} = \frac{8}{3} \Rightarrow y = \frac{1}{3} \Rightarrow x = 3 - \frac{4}{3} = \frac{5}{3},$$

so in this case the KKT are satisfied, and as the problem is convex, it follows that the location of the optimim is

$$(x_{\min}, y_{\min}) = \left(\frac{31}{17}, \frac{5}{17}\right).$$