Athens University of Economics and Business **Department of Informatics**
M.Sc. Program in Data Science
Optimization Techniques
Instructor: G. Zois **Spring Semester 2024**

1st Homework Assignment **Tuesday, April 23, 2024**

Deadline: MAY 22, 2024

[The answers should be sent via email at georzois@aueb.gr]

Problem 1. (10 points) Suppose that the following constraints have been provided for an optimization problem

$$
\begin{aligned}
-x_1 + 3x_2 &\leq 30 \\
-3x_1 + x_2 &\leq 30 \\
x_1, x_2 &\geq 0
\end{aligned}
$$

(a) **(3 points)** Draw the feasible region and demonstrate that it is unbounded.

(b) **(7 points)** Let $Z = -\frac{1}{2}x_1 - x_2$ be the objective function that we want to maximize. Use the graphical method to establish whether or not this optimization problem admits an optimal solution (**suggestion:** to better illustrate the method, you could start by setting Z = -30 or some other value nearby).

Problem 2. (10 points) Run Simplex for the linear program below. Explain the steps that need to be taken within each iteration.

$$
\begin{aligned}
\text{max.} \quad & 3x_1 + 2x_2 + 4x_3 \\
\text{s. t.:} \quad & \\
& x_1 + x_2 + 2x_3 \leq 4 \\
& 2x_1 + 3x_3 \leq 5 \\
& 2x_1 + x_2 + 3x_3 \leq 7 \\
& x_1, x_2, x_3 \geq 0
\end{aligned}
$$

Problem 3. (14 points) Consider the following (non-linear) program

$$
\min\left\{ F(x) = \frac{c^T x}{d^T x} : \; Ax = b; \; x \geq 0 \right\}
$$

where for its feasible region $P = \{x \in \mathbb{R}^n : Ax = b; x \geq 0\}$ it holds that $d^T x > 0$. Suppose also that we are given an upper bound $U \geq \max_{x \in P} \frac{c^T x}{d^T x}$ and a lower bound $L \leq \min_{x \in P} \frac{c^T x}{d^T x}$ on the objective cost. Show how one can use linear programming as a subroutine in order to find a solution to the above problem which is at most $(1+\epsilon)F^*$ where F^* is the cost of the optimal solution and $\epsilon > 0$ is the accuracy degree of the proposed solution.

Problem 4. (10 points) Consider the following linear program

$$
\begin{aligned}
\text{max.}\quad & x_1 + 2x_2 + 4x_3 \\
\text{s. t.:}\quad & \\
& 3x_1 + x_2 + 5x_3 \leq 10 \\
& x_1 + 4x_2 + x_3 \leq 8 \\
& 2x_1 + 2x_3 \leq 7 \\
& x_1, x_2, x_3 \geq 0
\end{aligned}
$$

We are given that the optimal solution to this program is $x = (0, 30/19, 32/19)$. Construct the dual linear program and find its optimal solution by utilizing the complementary slackness conditions.

Problem 5. (18 points) The following questions refer to the Maximum Flow problem in a directed graph $G = (V, E)$, with a source node $s \in V$, and a sink node $t \in V$ (see eclass for the definition and an example).

(i) **(2 points)** Write the problem of finding the maximum flow from s to t as a linear program.

(ii) **(3 points)** Write the dual of the linear program in question (i), using a variable y_e for each edge and x_v for each vertex $v \neq s, t$.

(iii) **(5 points)** Show that any solution to the general dual LP must satisfy the following property: for any directed path from s to t in the network, the sum of the y_e values along the path must be at least 1.

(iv) **(8 points)** Consider the following four variations of the Maximum Flow problem in a graph G:

- *a.* There are many sources and many sinks in G, and we wish to maximize the total flow from all sources to all sinks.
- *b.* Each vertex $v \in V$ also has a capacity on the maximum flow that can enter v.
- *c.* Each edge $e \in E$ has not only a capacity, but also a lower bound on the flow it must carry.

d. The outgoing flow from each node $v \in V$ is not the same as the incoming flow, but is smaller by a factor of $(1 - \epsilon_v)$, where ϵ_v is a loss coefficient associated with node v.

Show that each of the above variations can be solved efficiently, by reducing a and b to the original Maximum Flow problem, and reducing c and d to linear programming.

The following Problems 6-7 are from the book of Boyd and Vandenberghe, "Convex Optimization"

Problem 6. (12 points)

Exercise 5.1, parts a,b,c, page 273.

Problem 7. (12 points)

Exercise 9.6, page 515.

Problem 8. (14 points)

(a) **(4 points)** Show that the function $f(x_1, x_2) = x_1^2 + x_2^2$ is convex by using the definition of convexity based on the Hessian.

(b) **(3 points)** Show how to generalize the previous proof so as to establish convexity for the function $f(x_1, x_2, ..., x_n) = x_1^2 + x_2^2 + ... + x_n^2$.

(c) **(7 points)** Consider the function $f(x_1, x_2) = 2x_1^2 + x_2^2 - 2x_1x_2 + 4x_2$. Run two steps of the gradient descent method with exact line search, starting from the point $x^{(0)} = (1, -1)$.