1. Identify the locations of saddle points and extrema of the following function:

$$f(x, y) = 4xy - x^4 - y^4.$$

2. Find the location on the curve $xy^2 = 54$ that is closest to the origin.

3. Compute the double integral of the function $f(x, y) = \cos(x + y)$ on the region R specified by the lines $x = 0$, $y = 0$, and $x + y = 1$.

4. Use the Gram-Schmidt process to calculate a set of orthonormal vectors using the following vectors *in the given order*:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

5. Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{bmatrix}.$$

   Then, find matrices S, Λ such that $A = S\Lambda S^{-1}$, and compute S^{-1}.

1. Identify the locations of saddle points and extrema of the following function:

$$f(x,y) = 4xy - x^4 - y^4.$$

Solution: Observe that

$$\nabla f(x,y) = (4y - 4x^3, 4x - 4y^3).$$

Therefore,

$$\nabla f(x,y) = (0,0) \Leftrightarrow y = x^3,\ x = y^3.$$

The above equations give

$$x^9 = x \Rightarrow x(x^8-1) = 0 \Rightarrow (x^4-1)(x^4+1)x = 0 \Rightarrow (x^2-1)(x^2+1)x = 0 \Rightarrow (x-1)(x+1)x = 0.$$

It follows that there are three locations where we may have saddle points and extrema, i.e.,

$$(0,0), \quad (1,1), \quad (-1,-1).$$

Next, observe that

$$f_{xx}(x,y) = -12x^2, \quad f_{yy}(x,y) = -12y^2, \quad f_{xy}(x,y) = 4.$$

Regarding the point $(0,0)$, we have

$$D(0,0) = 0 - 4^2 < 0,$$

so this point is a saddle point. Regarding the point $(1,1)$, we have

$$D(1,1) = 144 - 4^2 > 0, \quad f_{xx} < 0,$$

so we have a maximum at $(1,1)$. Likewise, at point $(-1,-1)$, again we have

$$D(1,1) = 144 - 4^2 > 0, \quad f_{xx} < 0,$$

so this point is a maximum as well. ■

2. Find the location on the curve $xy^2 = 54$ that is closest to the origin.

Solution: The problem is equivalent to minimizing the function $x^2 + y^2$ subject to the constraint

$$g(x,y) \triangleq xy^2 - 54 = 0.$$

Note that

$$\nabla f = (2x, 2y), \quad \nabla g = (y^2, 2xy).$$

Observe that $\nabla g = (0,0)$ at the origin, that is not a solution. Therefore, if there is a minimum, it will have to be at a location where, for some λ, we have

$$\nabla f = \lambda \nabla g \Leftrightarrow (2x, 2y) = (\lambda y^2, 2\lambda xy) \Leftrightarrow 2x = \lambda y^2, \quad 2y = 2\lambda xy \Leftrightarrow 2x = \lambda y^2, \quad \lambda = \frac{1}{x}.$$

From the last two equations, together with equation $xy^2 = 54$, it follows that

$$2x^2 = y^2, \quad xy^2 = 54 \Rightarrow 2x^3 = 54 \Rightarrow x = 3 \Rightarrow y = \pm 3\sqrt{2},$$

therefore there are two locations, $(3, 3\sqrt{2})$ and $(3, -3\sqrt{2})$, where the curve is closest to the origin. ■

3. Compute the double integral of the function $f(x, y) = \cos(x + y)$ on the region R specified by the lines $x = 0$, $y = 0$, and $x + y = 1$.

Solution: Using Fubini's theorem, we have

$$\begin{aligned}
\iint_R f(x,y)\,dA &= \int_0^1 \left(\int_0^{1-x} \cos(x+y)\,dy \right) dx \\
&= \int_0^1 \left(\int_0^{1-x} (\sin(x+y))'\,dy \right) dx = \int_0^1 (\sin(x+1-x) - \sin x)\,dx \\
&= \int_0^1 (\cos x + x \sin 1)'\,dx = \cos 1 + \sin 1 - 1.
\end{aligned}$$

■

4. Use the Gram-Schmidt process to calculate a set of orthonormal vectors using the following vectors *in the given order*:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

Solution: The first vector is found by normalization:

$$\mathbf{e}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}.$$

For the second vector, we have

$$\mathbf{q}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix} \Rightarrow \mathbf{e}_2 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}.$$

Observe that

$$\mathbf{e}_1 \cdot \mathbf{v}_3 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}}, \quad \mathbf{e}_2 \cdot \mathbf{v}_3 = \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}},$$

therefore,

$$\mathbf{q}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} - \frac{1}{\sqrt{2}} \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

■

5. Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{bmatrix}.$$

Then, find matrices S, Λ such that $A = S\Lambda S^{-1}$, and compute S^{-1}.

Solution: The characteristic polynomial is

$$\begin{vmatrix} -\lambda & 0 & 2 \\ 0 & 2-\lambda & 0 \\ 2 & 0 & -\lambda \end{vmatrix} = 0 \Leftrightarrow -\lambda(2-\lambda)(-\lambda) - 2(2-\lambda)2 = 0$$

$$\Leftrightarrow \lambda^2(2-\lambda) + 4(\lambda-2) = (\lambda-2)(2-\lambda)(2+\lambda) = -(\lambda-2)^2(\lambda+2) = 0,$$

therefore the eigenvalues are $\lambda_1 = 2$, $\lambda_2 = 2$, and $\lambda_3 = -2$, and

$$\Lambda = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

To find the eigenvectors corresponding to the double eigenvalue $\lambda_1 = \lambda_2 = 2$, we write

$$\begin{bmatrix} -2 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \Leftrightarrow -x_1 + x_3 = 0, \quad x_1 - x_3 = 0 \Leftrightarrow x_1 = x_3,$$

whereas x_2 can take any value. We select the vectors $\begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$.
To find the eigenvector corresponding to the single eigenvalue $\lambda_3 = -2$, likewise, we write

$$\begin{bmatrix} 2 & 0 & 2 \\ 0 & 4 & 0 \\ 2 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Leftrightarrow x_1 + x_3 = 0, \quad x_2 = 0.$$

We select the vector $\begin{bmatrix} 1 & 0 & -1 \end{bmatrix}$.
Therefore,

$$S = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix}.$$

It remains to find S^{-1}. To this effect, we can perform Gauss-Jordan elimination

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & -2 & -1 & 0 & 1 \end{array}\right] \qquad (R2 = R2 - R1, \;\; R3 = R3 - R1)$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & 0 & -\frac{1}{2} \end{array}\right] \qquad (R3 = -\frac{1}{2}R3)$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 & -\frac{1}{2} & 1 & -\frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} & 0 & -\frac{1}{2} \end{array}\right] \qquad (R2 = R2 + R3, \; R1 = R1 - R3)$$

Therefore,

$$S^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{bmatrix}.$$

■

1. Identify the locations of saddle points and extrema of the following function:

$$f(x, y) = x^4 + y^4 + 4xy.$$

2. Find the location on the curve $x^2 + xy + y^2 = 1$ that is closest to the origin.

3. Compute the double integral of the function $f(x, y) = (x + y)^2$ on the region R specified by the lines $x = 0$, $y = 1$, and $y = x$.

4. Use the Gram-Schmidt process to calculate a set of orthonormal vectors using the following vectors *in the given order*:

$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

5. Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 3 & 4 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

1. Identify the locations of saddle points and extrema of the following function:

$$f(x, y) = x^4 + y^4 + 4xy.$$

Solution: Observe that

$$\nabla f(x, y) = (4x^3 + 4y, 4y^3 + 4x).$$

Therefore,

$$\nabla f(x, y) = (0, 0) \Leftrightarrow x^3 + y = 0, \quad y^3 + x = 0.$$

The above equations give

$$x^9 = x \Rightarrow x = -1, 0, 1.$$

It follows that there are three locations where we may have saddle points and extrema, i.e.,

$$(0, 0), \quad (1, -1), \quad (-1, 1).$$

Next, observe that

$$f_{xx}(x, y) = 12x^2, \quad f_{yy}(x, y) = 12y^2, \quad f_{xy}(x, y) = 4.$$

Regarding the point $(0, 0)$, we have

$$D(0, 0) = 0 - 4^2 < 0,$$

so this point is a saddle point. Regarding the point $(1, -1)$, we have

$$D(1, -1) = 144 - 4^2 > 0, \quad f_{xx} > 0,$$

so we have a minimum at $(1, 1)$. Likewise, at point $(-1, 1)$, again we have

$$D(-1, 1) = 144 - 4^2 > 0, \quad f_{xx} > 0,$$

so this point is a minimum as well. ■

2. Find the location on the curve $x^2 + xy + y^2 = 1$ that is closest to the origin.

Solution: The problem is equivalent to minimizing the function $f(x, y) = x^2 + y^2$ subject to the constraint

$$g(x, y) \triangleq x^2 + xy + y^2 - 1 = 0.$$

Note that

$$\nabla f = (2x, 2y), \quad \nabla g = (2x + y, 2y + x).$$

Observe that $\nabla g = (0, 0)$ at the location where

$$2x + y = 0, \quad 2y + x = 0,$$

i.e., the location $(0, 0)$, which does not belong to the curve, therefore that location is excluded, and the extrema must all belong to locations where, for some λ, we have

$$\nabla f = \lambda \nabla g \Leftrightarrow 2x = \lambda(2x + y), \quad 2y = \lambda(2y + x) \Leftrightarrow 2x(1 - \lambda) = \lambda y, \quad 2y(1 - \lambda) = \lambda x.$$

Multiplying the two equations we arrive at $4xy(1 - \lambda)^2 = \lambda^2 xy$. Now observe that if $x = 0$, when we must either have $y = 0$, which is impossible because $x = y = 0$ does not satisfy the constraint, or

$\lambda = 1$, which then leads to $y = 0$ and again the constraint is not satisfied. So we must have $x \neq 0$. By a similar argument, $y \neq 0$. It then follows, dividing by xy, that

$$4(1-\lambda)^2 = \lambda^2 \Leftrightarrow 4 + 4\lambda^2 - 8\lambda = \lambda^2 \Leftrightarrow 3\lambda^2 - 8\lambda + 4 = 0 \Leftrightarrow \lambda_1 = 2, \lambda_2 = \frac{2}{3}.$$

Regarding the case $\lambda = 2$, it leads to

$$-2x = 2y, \quad -2y = 2x \Leftrightarrow x = -y,$$

and plugging this to the constraint we arrive at

$$x^2 = 1 \Rightarrow x = \pm 1 \Rightarrow y = \mp 1,$$

therefore we find the locations $(1,-1)$ and $(-1,1)$, which are both at a distance $\sqrt{2}$ from the origin.

Regarding the case $\lambda = \frac{2}{3}$, it leads to $x = y$, and plugging this to the constraint we arrive at

$$3x^2 = 1 \Rightarrow x = y = \pm\frac{1}{\sqrt{3}},$$

therefore we find the locations $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ and $\left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$ which are both at a distance $\sqrt{\frac{2}{3}}$ from the origin.

Therefore, the first two locations we found are maxima, and the last two locations we found are minima.

■

3. Compute the double integral of the function $f(x,y) = (x+y)^2$ on the region R specified by the lines $x = 0$, $y = 1$, and $y = x$.

Solution: Using Fubini's theorem, we have

$$\begin{aligned}\iint_R f(x,y)\,dA &= \int_0^1 \left(\int_x^1 (x+y)^2\,dy\right)dx = \int_0^1 \left(\int_x^1 \left(\frac{(x+y)^3}{3}\right)' dy\right)dx \\ &= \int_0^1 \left(\frac{(x+1)^3}{3} - \frac{8x^3}{3}\right)dx = \int_0^1 \left(\frac{(x+1)^4}{12} - \frac{8x^4}{12}\right)' dx \\ &= \frac{16}{12} - \frac{8}{12} - \frac{1}{12} + 0 = \frac{7}{12}.\end{aligned}$$

■

4. Use the Gram-Schmidt process to calculate a set of orthonormal vectors using the following vectors *in the given order*:

$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

Solution: The first vector is found by normalization:

$$\mathbf{e}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}.$$

For the second vector, we have, since $\mathbf{v}_2 \cdot \mathbf{e}_1 = \frac{1}{\sqrt{2}}$, that

$$\mathbf{q}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}.$$

Normalizing,

$$\mathbf{e}_2 = \frac{\mathbf{q}_2}{\|\mathbf{q}_2\|} = \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}.$$

Finally, noting that $\mathbf{v}_3 \cdot \mathbf{e}_1 = \frac{1}{\sqrt{2}}$ and $\mathbf{v}_3 \cdot \mathbf{e}_2 = \frac{1}{\sqrt{2}}$, we have

$$\mathbf{q}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} - \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

from which it follows that

$$\mathbf{e}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

■

5. Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 3 & 4 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then, find matrices S, Λ such that $A = S\Lambda S^{-1}$, and compute S^{-1}.

Solution: The characteristic polynomial is

$$|A - \lambda I| = 0 \Leftrightarrow \begin{vmatrix} 3-\lambda & 4 & 2 \\ 0 & 1-\lambda & 2 \\ 0 & 0 & -\lambda \end{vmatrix} = 0 \Leftrightarrow -(3-\lambda)(1-\lambda)\lambda = 0 \Leftrightarrow (\lambda-3)(\lambda-1)\lambda = 0,$$

therefore the three eigenvalues are the

$$\lambda_1 = 0, \quad \lambda_2 = 3, \quad \lambda_3 = 1.$$

Regarding the first eigenvector, we have

$$\begin{bmatrix} 3 & 4 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Leftrightarrow 3x_1 + 4x_2 + 2x_3 = 0, \quad x_2 + 2x_3 = 0$$
$$\Leftrightarrow x_2 = -2x_3, \quad 3x_1 = -2x_3 + 8x_3 \Leftrightarrow x_2 = -2x_3, \quad x_1 = 2x_3 \Leftrightarrow x_1 = 2x_3, \quad x_2 = -2x_3.$$

So, for example, one candidate eigenvector is $\begin{bmatrix} 2 & -2 & 1 \end{bmatrix}^T$.

Regarding the second eigenvector, we have

$$\begin{bmatrix} 0 & 4 & 2 \\ 0 & -2 & 2 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Leftrightarrow 4x_2 + 2x_3 = 0, \quad -2x_2 + 2x_3 = 0, \quad x_3 = 0 \Leftrightarrow x_2 = x_3 = 0.$$

So, for example, one candidate eigenvector is $\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$.

Regarding the third eigenvector, we have

$$\begin{bmatrix} 2 & 4 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Leftrightarrow 2x_1 + 4x_2 + 2x_3 = 0, \quad x_3 = 0, \quad x_3 = 0$$

$$\Leftrightarrow x_1 = -2x_2, \quad x_3 = 0.$$

So, for example, one candidate eigenvector is $\begin{bmatrix} -2 & 1 & 0 \end{bmatrix}^T$.

Therefore,

$$\Lambda = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad S = \begin{bmatrix} 2 & 1 & -2 \\ -2 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

It remains to find S^{-1}. To this effect, we can perform Gauss-Jordan elimination

$$\left[\begin{array}{ccc|ccc} 2 & 1 & -2 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & \frac{1}{2} & -1 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & -1 & 1 & 1 & 0 \\ 0 & -\frac{1}{2} & 1 & -\frac{1}{2} & 0 & 1 \end{array}\right] \qquad (R1 = \frac{1}{2}R1,\ R2 = R2 + R1,\ R3 = R3 - \frac{1}{2}R1)$$

$$\left[\begin{array}{ccc|ccc} 1 & \frac{1}{2} & -1 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & -1 & 1 & 1 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 1 \end{array}\right] \qquad (R3 = R3 + \frac{1}{2}R2)$$

$$\left[\begin{array}{ccc|ccc} 1 & \frac{1}{2} & 0 & \frac{1}{2} & 1 & 2 \\ 0 & 1 & 0 & 1 & 2 & 2 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 1 \end{array}\right] \qquad (R2 = R2 + 2R3,\ R1 = R1 + 2R3)$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 0 & 1 & 2 \end{array}\right] \qquad (R1 = R1 - \frac{1}{2}R2)$$

Therefore,

$$S^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 2 & 2 \\ 0 & 1 & 2 \end{bmatrix}.$$

■