ΟΙΚΟΝΟΜΙΚΟ ΠΑΝΕΠΙΣΤΗΜΙΟ ΑΘΗΝΩΝ

ATHENS UNIVERSITY OF ECONOMICS AND BUSINESS

Special Topics on Algorithms
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The Traveling Salesman Problem (TSP)

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Traveling Salesman Problem (TSP)

TSP

I: A complete directed weighted graph G=(V,E), integer B

Q (Decision): Is there a permutation of V, $<v_1,v_2,\ldots,v_n>$

such that $\Sigma_{i=1\ldots n}\, w(v_i, v_{i \bmod n + 1}) \leq B$, i.e is there a TSP tour of cost at most B ?

(Note: this is equivalent with asking if there is a Hamiltonian Cycle in G (a tour) of cost $\leq$ B ?)

Optimization: Find a tour of minimum cost

One of the most well studied problems in Computer Science, Operations Research, ...

Brute force approach: O(n!) – No way!

Traveling Salesman Problem (TSP)

Some related problems:

HAMILTON CYCLE (HC) [or **RUDRATA CYCLE**]

I: A (possibly directed) graph G=(V,E)

Q: Is there a Hamiltonian cycle in G? (i.e., a cycle that goes through all the vertices)

HAMILTON PATH (HP)

I: A (possibly directed) graph G=(V,E)

Q: Is there a Hamiltonian path in G?

Both HC and HP are NP-complete

NP-hardness

HC $\leq_p$ TSP

G=(V,E)

G has a HC
All its edges have cost 1 in G'
G' has a tour of cost B

G' = (V, E')
E' = V × V

$$w(u,v) = w(v,u) = \begin{cases} 1, & \text{if } (u,v) \in E \\ 2, & \text{otherwise} \end{cases}$$

B= |V|

G' has a tour of cost ≤ B
It uses only edges of cost 1 (cost = B)
G has a HC

Some interesting special cases:

- Δ-TSP: A special case of TSP where the triangle inequality holds, i.e., $w(i,k) \leq w(i,j) + w(j,k)$ $1 \leq i, j, k \leq n$
- TSP(1,2): all weights equal to 1 or 2
- And many others...

Most interesting cases turn out to be NP-complete as well

Coping with NP-complete problems

Recall:

1. Small instances
2. Special cases
3. **Exponential algorithms (Dynamic Programming, Branch and Bound,...)**
4. Approximation algorithms
5. Randomized algorithms
6. Heuristic algorithms

DP for TSP

We need to identify first the subproblems we will solve

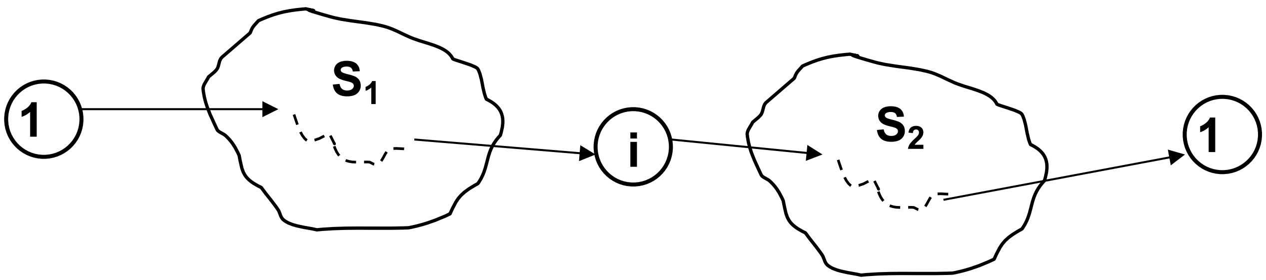
We will also make use of the TSP path problem, i.e., find a permutation of V, $<v_1,v_2,\ldots,v_n>$ such that $\Sigma_{i=1\ldots n-1} w(v_i, v_{i+1}) \leq B$.

Optimal Substructure Property:

Assume w.l.o.g. that we start the TSP Tour at node 1

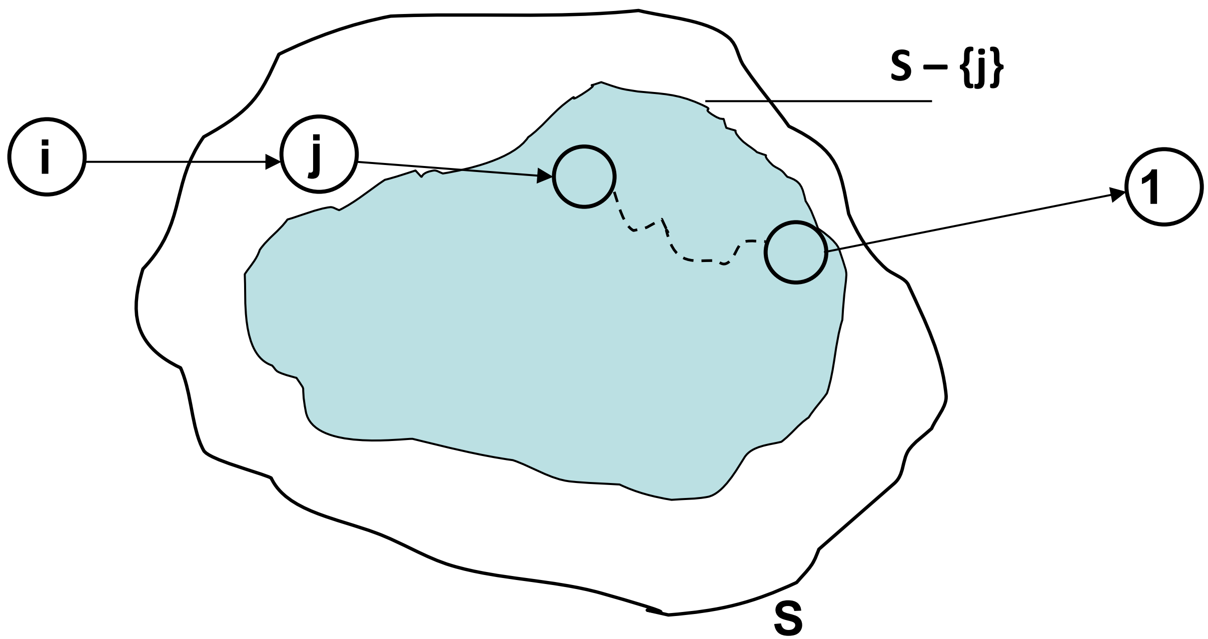
Assume that 1 ->…S_1 …->i->…S_2 …-> 1 is an optimal TSP tour

Then the path i->…S_2 …-> 1 must be an optimal TSP Path in $V \backslash S_1$



DP for TSP

Let g(i,S) = the cost of the shortest path i-> …….. -> 1, going from node i to node 1, using **all** the nodes of S (i.e., the minimum TSP path starting from i, in the graph induced by $S \cup \{i, 1\}$, $S \subset V$)

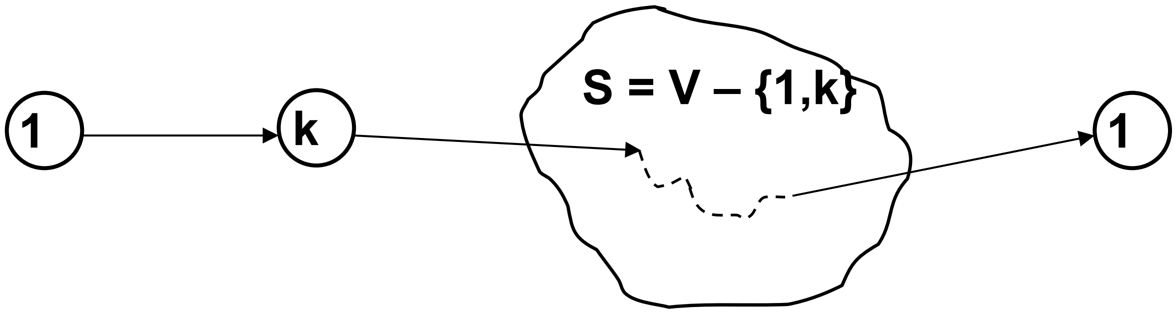


$$g(i,S) = \min_{j \in S}\{\, w(i,j) + g(j, S - \{j\})\,\}$$

DP for TSP

Our aim is to find

$$g(1, V-\{1\}) = \min_{2 \le k \le n}\{w(1,k) + g(k, V-\{1,k\})\}$$



How ?

By finding g(k, V-{1,k}) for all choices of k

This can be done by using the optimal substructure for g(i, S)

$$g(i,S) = \min_{j \in S}\{w(i,j) + g(j, S-\{j\})\}$$

DP for TSP

Obviously, $g(i, \emptyset) = w(i, 1)$

We can find $g(i, S)$ for all sets S, with $|S| = 1$

Then find $g(i, S)$ for all sets S, with $|S| = 2$

...

and then find $g(i, S)$ for all sets S, with $|S| = n-2$

Finally: $g(1, V-\{1\})$ --- $|S| = n-1$

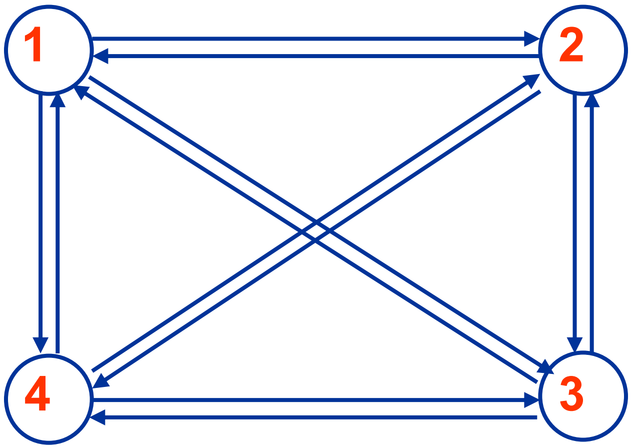
We need to compute g(i,S)

for EVERY set S of EACH possible size $|S| = 1,2,\ldots,n-2$,

and for all $i \in V - (S \cup \{1\})$

DP for TSP

Example



$$w: \begin{bmatrix} 0 & 10 & 15 & 20 \\ 5 & 0 & 9 & 10 \\ 6 & 13 & 0 & 12 \\ 8 & 8 & 9 & 0 \end{bmatrix}$$

DP for TSP

|S|=0: $g(2,\emptyset)=5$, $g(3,\emptyset)=6$, $g(4,\emptyset)=8$

|S|=1:

$g(2,\{3\}) = w_{23} + g(3,\emptyset) = 9 + 6 = 15$
$g(4,\{3\}) = 15$ } S = {3}

$g(2,\{4\}) = 18$
$g(3,\{4\}) = 20$ } S = {4}

$g(3,\{2\}) = 18$
$g(4,\{2\}) = 13$ } S = {2}

|S|=2:

$g(2,\{3,4\}) = \min\{ w_{23} + g(3,\{4\}), w_{24} + g(4,\{3\}) \} = 25$ S={3,4}
$g(3,\{2,4\}) = \min\{ w_{32} + g(2,\{4\}), w_{34} + g(4,\{2\}) \} = 25$ S={2,4}
$g(4,\{2,3\}) = \min\{ w_{42} + g(2,\{3\}), w_{43} + g(3,\{2\}) \} =23$ S={2,3}

$g(1,\{2,3,4\}) = \min\{ w_{12} + g(2,\{3,4\}),$ S={2,3,4}
$w_{13} + g(3,\{2,4\}),$
$w_{14} + g(4,\{2,3\}) \} =$
$= \min\{35, 40, 43\} = 35$

DP for TSP

```
for i = 2 to n do g(i,∅) = w(i,1) ;

for k = 1 to n-2 do  // for all sizes of S
    for each S ⊆ V-{1} s.t. |S|=k do  // for all possible sets of size k
        for each i ∈ V-(S ∪ {1})
            g(i,S):= min_{j∈S} { w(i,j) + g(j,S-{j}) } ;

find g(1, V-{1});
```

DP for TSP

Complexity:

N = # of g(i,S) computations

For each value of |S| there are ≤ n – 1 choices for i

The number of sets S with |S| = k not including 1 and i is $\binom{n-2}{k}$

$$N = \sum_{k=0}^{n-2} (n-1)\binom{n-2}{k} = (n-1)2^{n-2}$$

T(n) = N · [time to compute g(i,S) by taking the min over g(j,S-{j}) = N · O(n)

T(n) = O($n^2 2^n$), better than n!, but still, appropriate only for small instances

Coping with NP-complete problems

1. Small instances
2. Special cases
3. Exponential algorithms
4. **Approximation algorithms**
5. Randomized algorithms
6. Heuristic algorithms

Approximability of TSP

Is there any f(n)-approximation algorithm for TSP ?   NO !

Theorem: For any (polynomial time computable) function f(n) (with f(n) ≥ 1 for all n), TSP cannot be approximated within a factor of f(n), unless P=NP.

Proof:

Claim: If there is an f(n)-approximation algorithm A for TSP,

then, there is a poly-time algorithm for HC, i.e., we can decide the HC problem in polynomial time, and thus P=NP!

Reduction from Hamilton Cycle (HC) to TSP:

Consider an instance of HC, i.e., a graph G=(V,E), with |V| = n

Construct a complete weighted graph G' = (V, E'), E' = all possible edges with weights

$$w(u,v) = \begin{cases} 1, & \text{if } (u,v) \in E \\ n\,f(n), & \text{otherwise} \end{cases}$$

Approximability of TSP

Proof (cont.):

Running A on G' returns a tour of cost C

a) if the original graph G is Hamiltonian,

- Optimal TSP tour in G' has $C^* = n$,
- Algorithm A will return a tour with cost $C \leq nf(n)$ (because we assumed A is a f(n)-approximation algorithm)

b) if the original graph G is not Hamiltonian

- The optimal TSP tour in G' must contain at least one edge of cost nf(n):
  - Hence, $C^* \geq nf(n) + (n-1) > n\, f(n)$
- Algorithm A will return a tour $C \geq C^* > nf(n)$ (since C*=OPT should be less than the solution of A)

Hence: if we had a f(n)-approximation for TSP, we could solve the HC problem.

TSP with triangle inequality

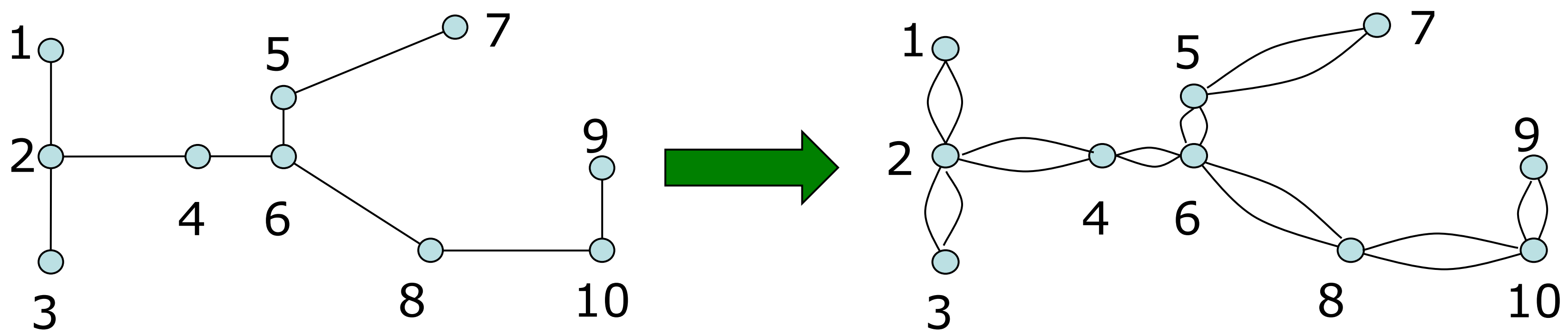
- Recall: Δ-TSP = special case of TSP where the triangle inequality holds,
  i.e., $w(i,k) \leq w(i,j) + w(j,k)$, $1 \leq i, j, k \leq n$
- A very natural special case, satisfied by many distance functions

Theorem: There exists a 2-approximation algorithm for Δ-TSP

- How do we start with designing an approximation algorithm?
- First and most important step: we need a lower bound on the cost of the optimal solution
- Consider an instance I of TSP
- Claim: $OPT(I) \geq MST(I)$
- Proof: delete one edge e from an optimal solution, what remains is a spanning tree F

$$OPT(I) = w(e) + C(F) \geq w(e) + MST(I) \geq MST(I)$$

Δ-TSP: A 2-approximation



Step 1: Find a minimum spanning tree, T, of G, of cost C(T)

Step 2: Double the edges of T and let T' be the obtained (multi)graph

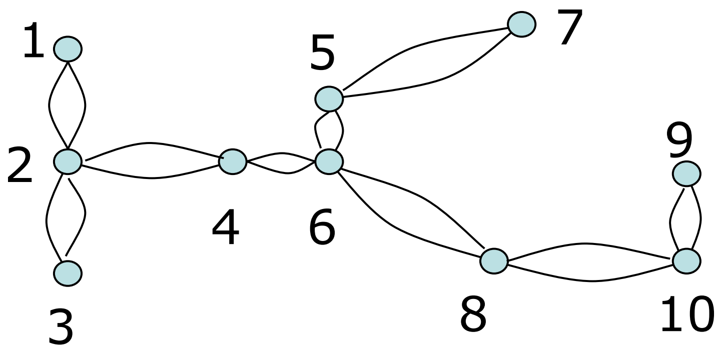
All vertices of T' are of even degree

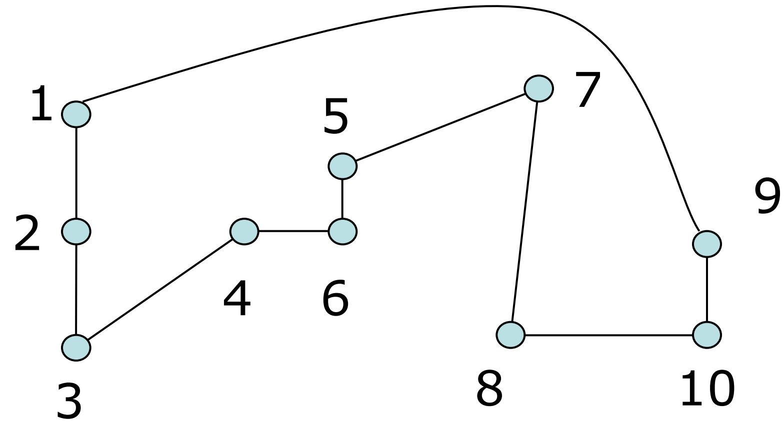
Recall from graph theory:

- Euler cycle: A tour that visits all the edges exactly once
- A graph is Eulerian (i.e., has an Euler cycle) iff every vertex has an even degree

In the example: Euler cycle W: 1, 2, 3, 2, 4, 6, 5, 7, 5, 6, 8, 10, 9, 10, 8, 6, 4, 2, 1

Δ-TSP: A 2-approximation





Step 3: Find an Euler cycle W in T'

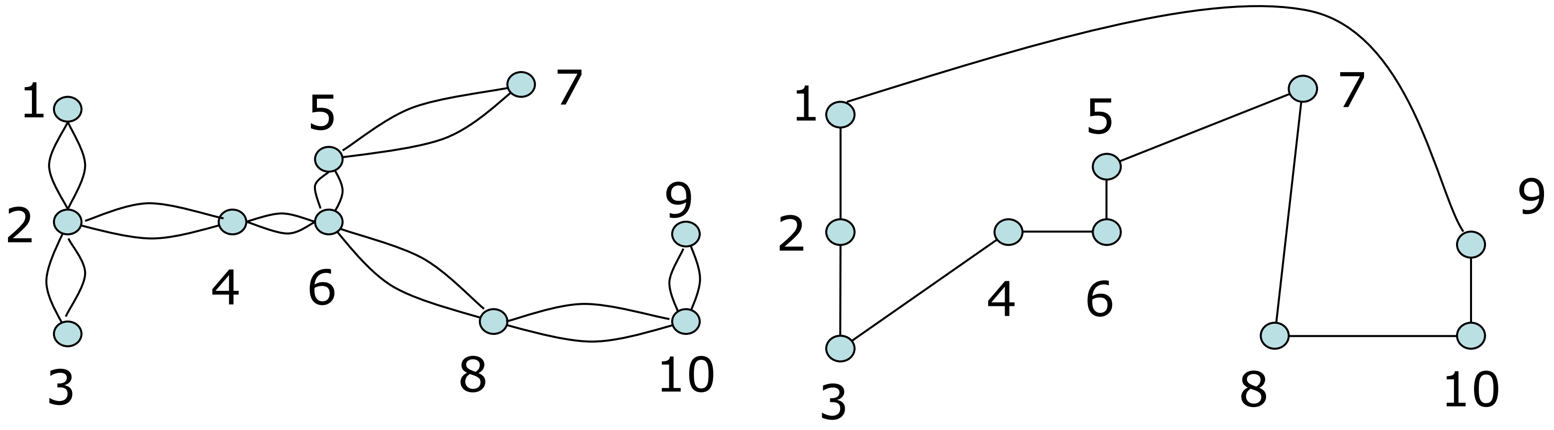
Note: W traverses each edge of T twice: $C(W) = 2\,C(T) \leq 2\,OPT$

Step 4: Find a tour H by "shortcutting" W:

1, 2, 3, ~~2~~, 4, 6, 5, 7, ~~5~~, ~~6~~, 8, 10, 9, ~~10~~, ~~8~~, ~~6~~, ~~4~~, ~~2~~, 1

Final solution H = 1, 2, 3, 4, 6, 5, 7, 8, 10, 9, 1

Δ-TSP: A 2-approximation



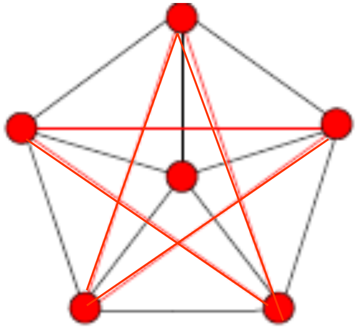
C(H) ≤ C(W) , because of the triangle inequality

Hence: C(H) ≤ C(W) ≤ 2 OPT

QUESTION: What is the complexity of this algorithm ?

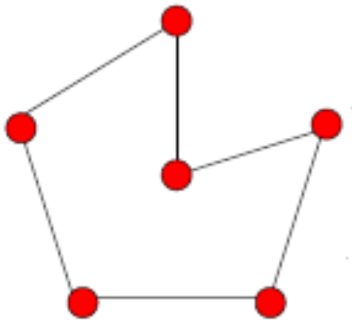
Δ-TSP: Tightness of 2-approximation

Example



Complete graph K_n
Red edges: w = 2
Other edges: w=1 (union of a star + cycle)

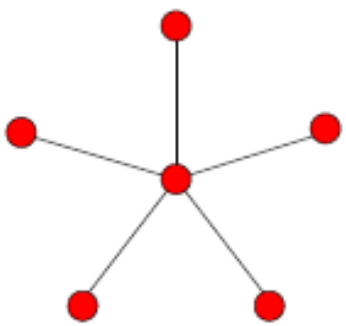
Optimal tour



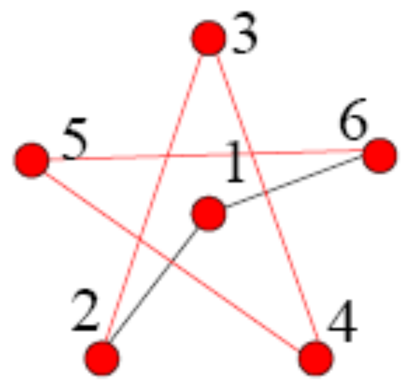
OPT = n

Δ-TSP: Tightness of 2-approximation

Minimum MST



Solution



C(H) = (n-2)*2 + 2*1 = 2n-2

Hence, C(H) / OPT = (2n-2) / n = 2 – (2/n) → 2

Δ-TSP: improvement to ρ = 1.5

Theorem: There is a 1.5-approximation algorithm for Δ-TSP [Chistofides 1976]

Step 1: Start again by finding a minimum spanning tree, T, of cost C(T)

- We cannot now just double the edges, this will not avoid a loss of 2
- But we would still like to create an Eulerian graph starting from T
- What makes T non-Eulerian?
- Problematic vertices: vertices of odd degree
- Claim: The number of odd-degree vertices is even (why?)

Δ-TSP: improvement to ρ = 1.5

Detour on matchings

Consider a graph G = (V, E)

Definition: A matching M is a collection of edges $M \subseteq E$, such that no 2 edges share a common vertex

Given a matching M, a vertex u is called *matched* if there exists an edge $e \in M$ such that e has u as one of its endpoints

Δ-TSP: improvement to $\rho = 1.5$

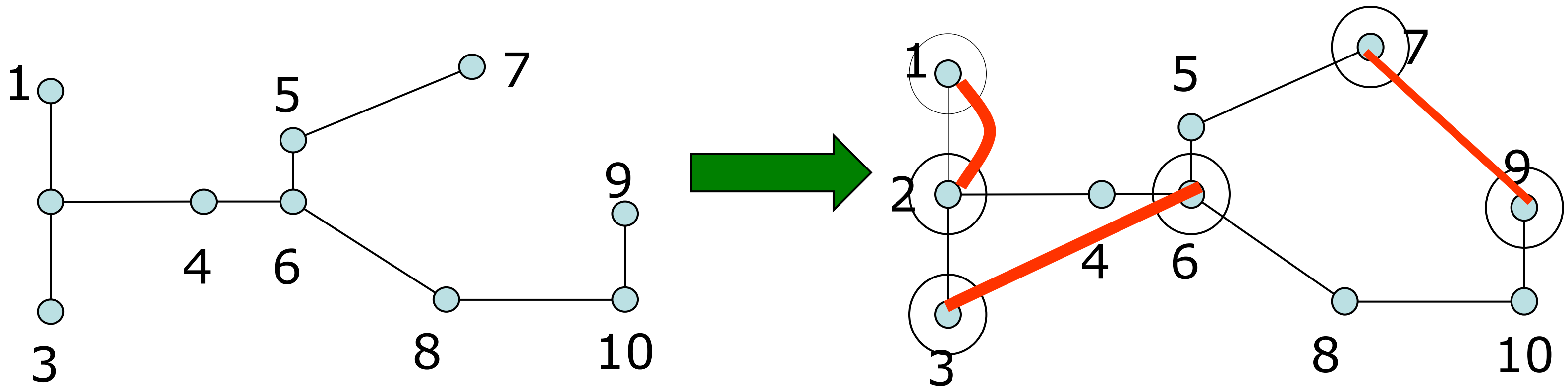
Detour on matchings

Types of matchings we are interested in:

- Maximal matching: find a matching where no more edges can be added
- Maximum matching: find a matching with the maximum possible number of edges
- Perfect matching: find a matching where every vertex is matched (if one exists)
- Maximum weight matching: given a weighted graph, find a matching with maximum possible total weight
- Minimum weight perfect matching: given a weighted graph, find a perfect matching with minimum cost

All the above problems can be solved in polynomial time (several algorithms and publications over the last decades)

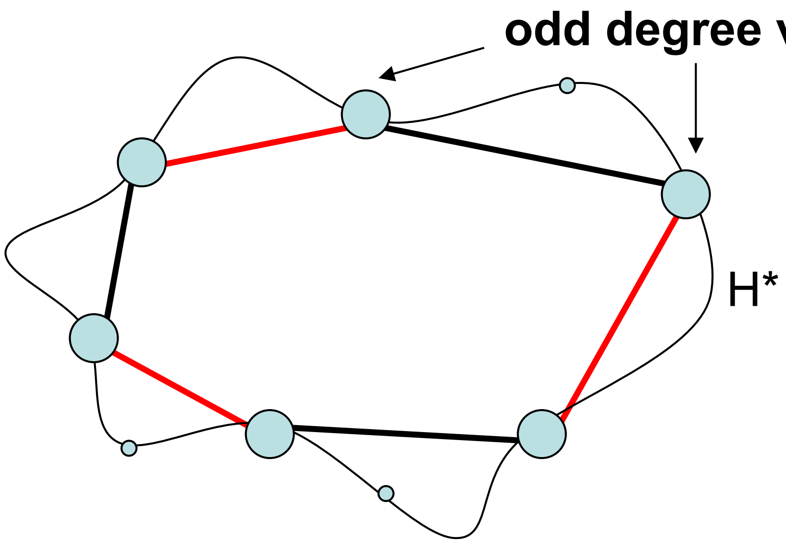
Δ-TSP: improvement to ρ = 1.5



Step 2:

- Find the set of vertices of T of odd degree, say S
- S contains an even number of vertices
- Consider the graph G_S induced by S
- Find a minimum weight perfect matching, M, in G_S

Δ-TSP: improvement to ρ = 1.5



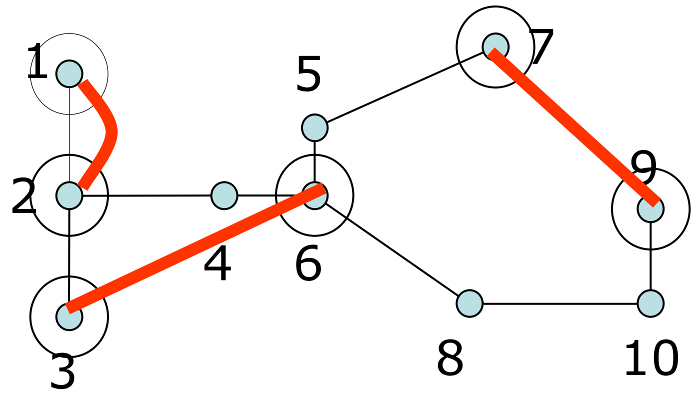
Why is a minimum cost perfect matching useful?

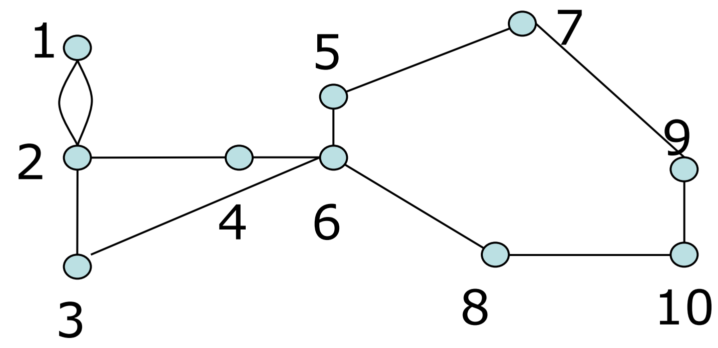
- Let H* be an optimal TSP tour
- Shortcut the tour to vertices of S
- This leads to a tour over S
- By triangle inequality, cost of S-tour ≤ C(H*) = OPT(I)
- S-tour can be decomposed into 2 perfect matchings of S (the red (M_1), and the black (M_2))

Then $C(H^*) \geq C(M_1) + C(M_2) \geq C(M) + C(M)$,
since M is a minimum weight perfect matching

Hence, $C(M) \leq C(H^*) / 2 = OPT(I)/2$

Δ-TSP: improvement to ρ = 1.5





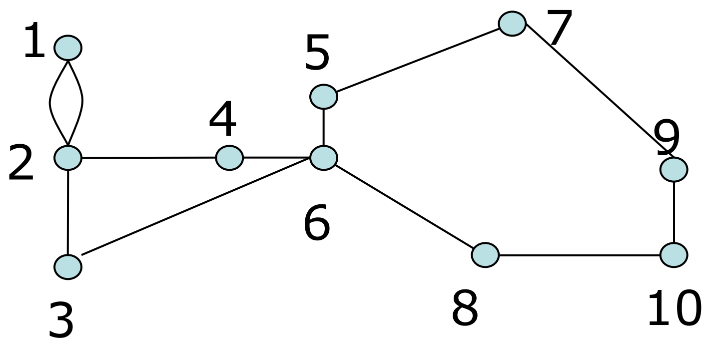
Step 3:

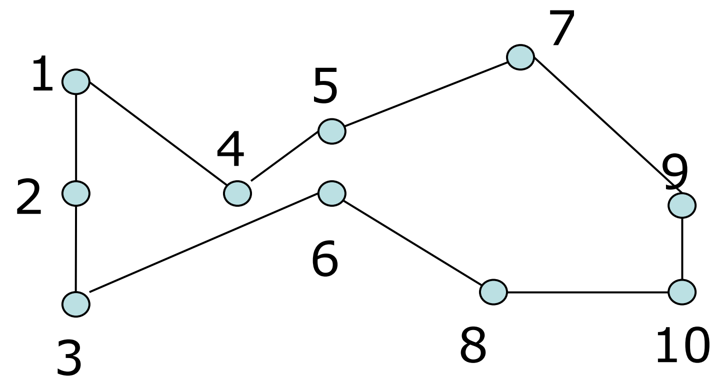
- Add the edges of M to T and let T' be the obtained (multi)graph
- All vertices of T' are of even degree now, hence T' is Eulerian
- Find an Euler cycle, W, in T'

Euler cycle W: 1, 2, 3, 6, 8, 10, 9, 7, 5, 6, 4, 2, 1

$C(W) = C(T) + C(M) \leq C(H^*) + C(H^*) / 2 = 1.5\ C(H^*)$

Δ-TSP: improvement to ρ = 1.5





Step 4:

Find a tour H by shortcutting the Euler tour W:

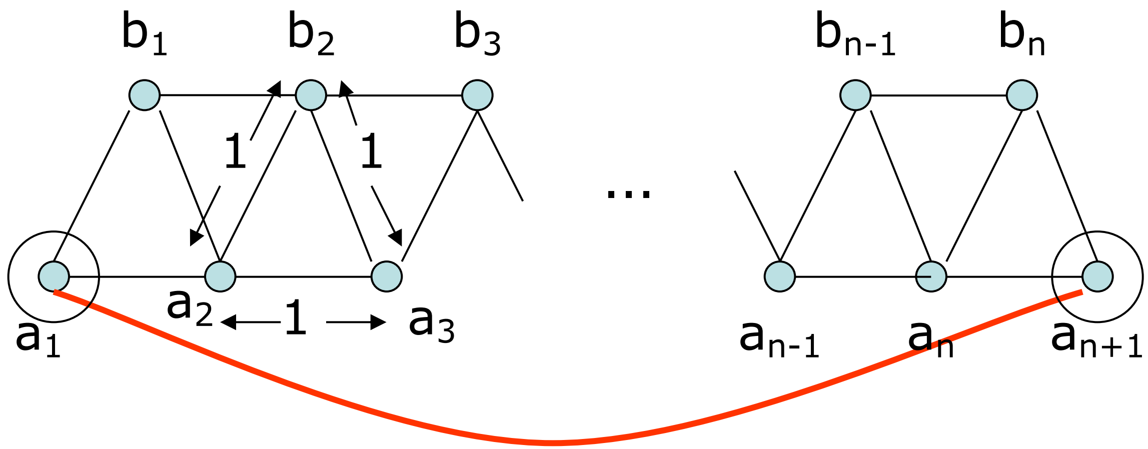
H: 1, 2, 3, 6, 8, 10, 9, 7, 5, ~~6~~, 4, ~~2~~, 1

$C(H) \leq C(W)$, by use of the triangle inequality

Hence, overall: $SOL(I) = C(H) \leq C(W) \leq 1.5\ C(H^*) = 1.5\ OPT(I)$

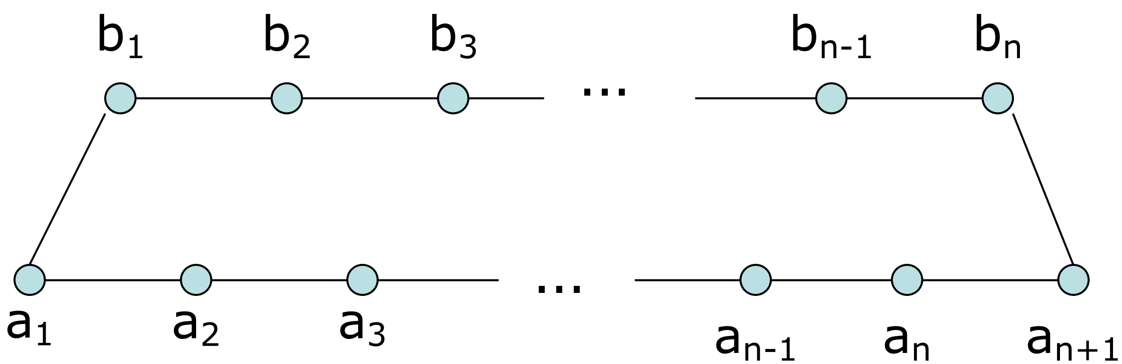
QUESTION: What is the complexity of this algorithm ?

Δ-TSP: Tightness of 1.5-approximation



- All edges with cost 1, apart from the red edge of cost n
- Shortcutting may pick the red edge and the zig-zag MST

C(H)=n+n+n=3n



For the optimal tour H*

C(H*) = n + (n-1) + 2 = 2n+1

C(H) / C(H*) → 3/2

Asymmetric Δ-TSP

- So far we assumed the graph is undirected
- For directed graphs the problem is more difficult (non-symmetric)
- [Frieze, Galbiati, Maffioli 1982]: O(logn)-approximation
  - Relatively simple algorithm
- [Asadpour, Goemans, Madry, Oveis Gharan, Saberi, 2011]: O(logn/loglogn)- approximation
  - Way more involved algorithm, based on Linear Programming and LP-rounding techniques
  - Randomized algorithm
  - It produces a solution with cost at most O(logn/loglogn) OPT(I) with high probability (approaching 1)
- More Recent, [Svensson, Tarnawski, Végh 2017]: constant approximation algorithm.

Back to symmetric Δ-TSP

- Inspired by the ideas for the progress on asymmetric TSP
- An interesting special case: graphic TSP: given a weighted graph G = (V, E), for edges that are not present, the weight is given by the shortest path
    - Also referred to as shortest path metrics
- [Asadpour, Goemans, Madry, Oveis Gharan, Saberi, 2011]: A randomized approximation of $3/2 - \varepsilon$, where $\varepsilon \approx 10^{-12}$
- [Momke, Svensson, 2011]: ≈ 1.461-approximation
- [Mucha, 2012]: 13/9 ≈ 1.444-approximation
- Conjecture: **4/3**

Coping with NP-complete problems

1. Small instances
2. Special cases
3. **Exponential algorithms (Dynamic Programming, Branch and Bound,...)**
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Branch-and-Bound

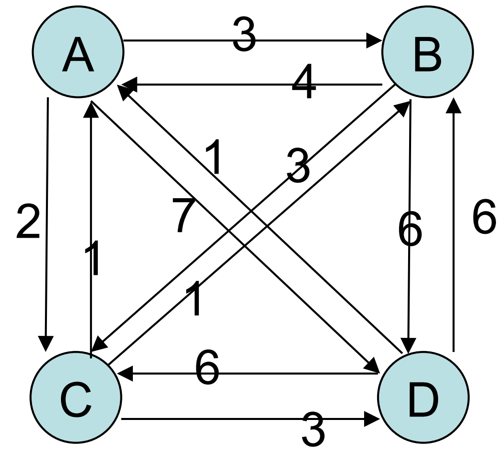
A different lower bound on the optimal solution:

$$\frac{1}{2}\sum_{i=1}^{n}(\min_{j\neq i}\{w_{i,j}\}+\min_{j\neq i}\{w_{j,i}\})$$

- the half of the sum of minimum elements of each row and each column
- For every node one edge of the tour has to come towards i and one has to leave from i

Σ_0

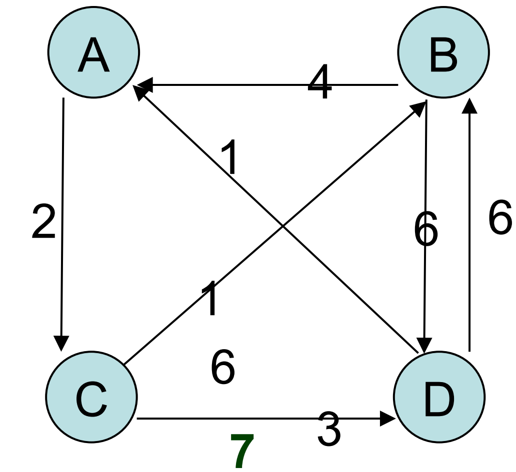
	A	B	C	D	
A	x	3	2	7	2
B	4	x	3	6	3
C	1	1	x	3	1
D	1	6	6	x	1
	1	1	2	3	LB = 14/2 = 7



Branch-and-Bound

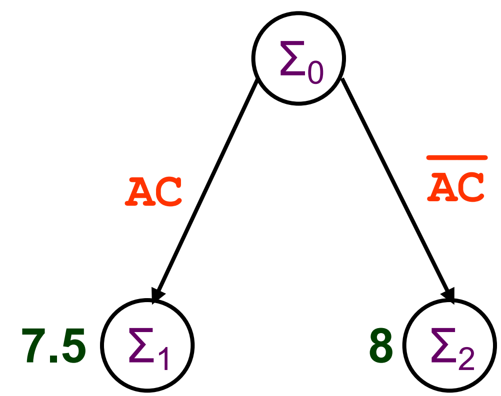
Σ_1 **Branch 1: edge AC in the tour ➔ CA, AB, AD, BC, DC not in tour (why ?)**

	A	B	C	D	
A	x	x	2	x	2
B	4	x	x	6	4
C	x	1	x	3	1
D	1	6	x	x	1
	1	1	2	3	LB = 15/2 = 7.5



Σ_2 **Branch 2: AC not in tour**

	A	B	C	D	
A	x	3	x	7	3
B	4	x	3	6	3
C	1	1	x	3	1
D	1	6	6	x	1
	1	1	3	3	LB = 16/2 = 8



Branch-and-Bound

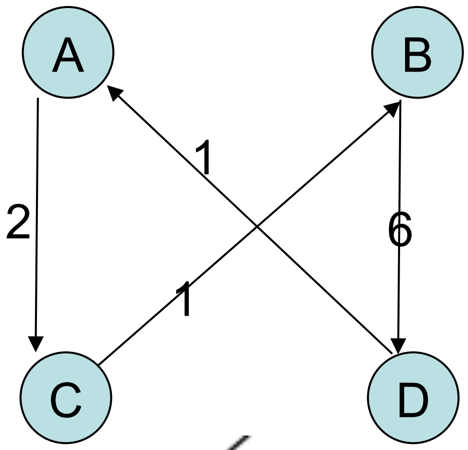
Σ_3

AC in tour ➔ CA, AB, AD, BC, DC not in tour

CB in tour ➔ CD, DB, BA not in tour

	A	B	C	D	
A	x	x	2	x	2
B	x	x	x	6	6
C	x	1	x	x	1
D	1	x	x	x	1
	1	1	2	6	LB = 20/2 = 10

A feasible Solution

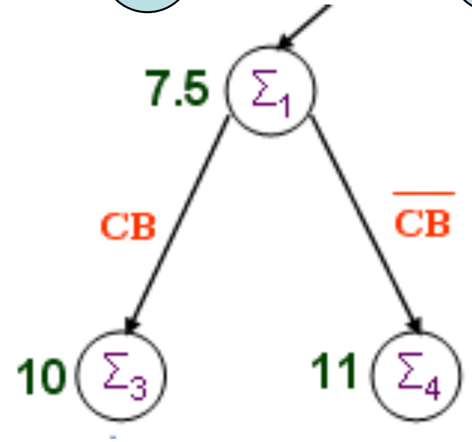


Σ_4

AC in tour ➔ CA, AB, AD, BC, DC not in tour

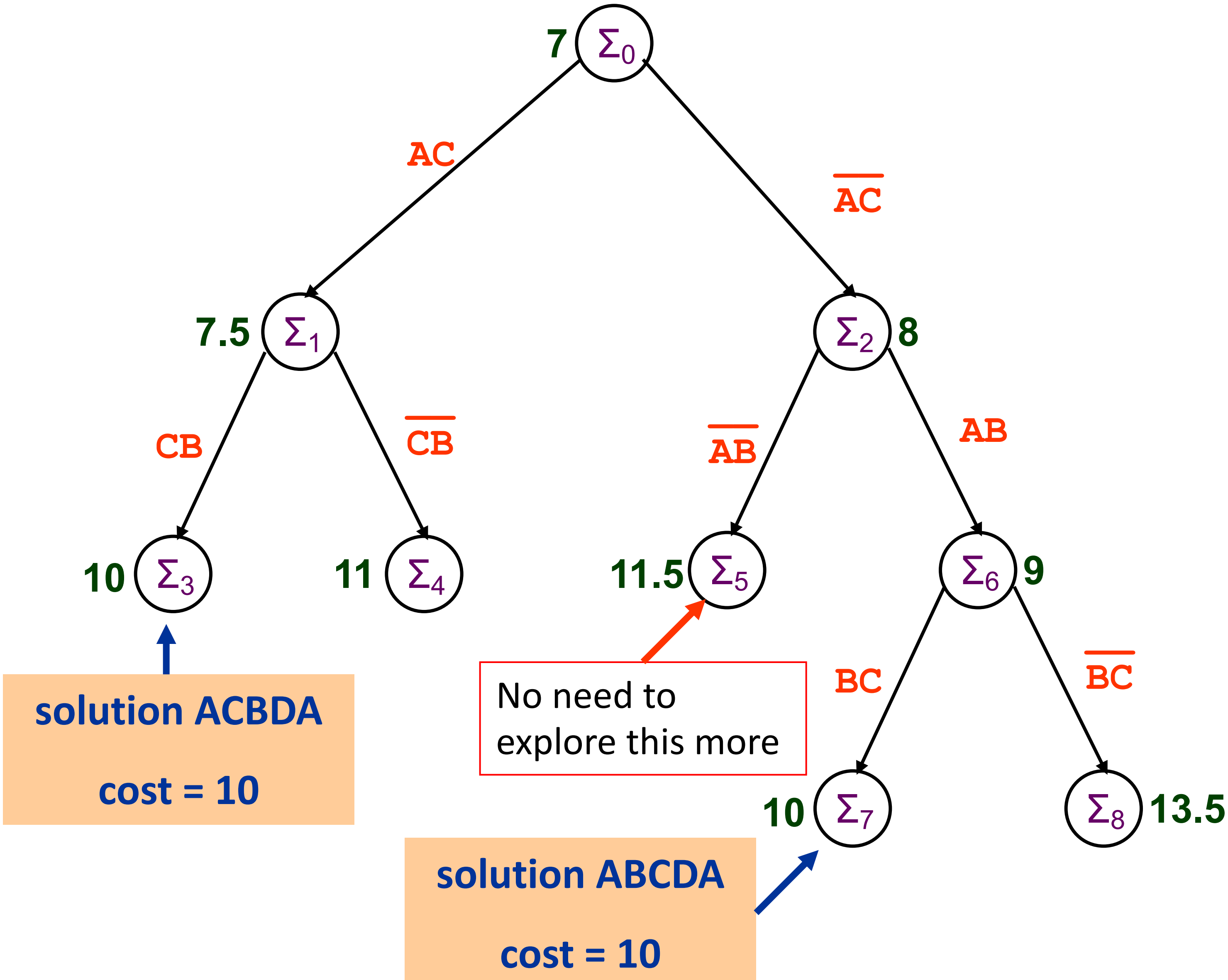
CB not in tour

	A	B	C	D	
A	x	x	2	x	2
B	4	x	x	6	4
C	x	x	x	3	3
D	1	6	x	x	1
	1	6	2	3	LB = 22/2 = 11



and so on …

Branch-and-Bound



Branch-and-Bound

Parameters

- Maintain a set S of active states
- Initially S = {Σ_0} (nothing has been expanded yet)
- In each step extract state Σ from S (Σ is the state to be expanded)
- UB is a global upper bound of the optimum solution
  - For minimization problems we initially set UB = **+**∞
- LB(Σ) is a lower bound on all solutions represented by state Σ (i.e. from all solutions that can arise after expanding Σ)
- Whenever we reach a terminal node with LB(Σ) ≤ UB, then we can update our current UB
- During the process, we do not need to examine any further the nodes where their LB is higher than UB!

Branch-and-Bound

```
Algorithm Branch and Bound
{  S = {Σ0};
   UB = +∞
   while S ≠ ∅ do
   {   get a node Σ from S;
       //which node ? FIFO/LIFO/Best LB
       S:= S - {Σ};
       for all possible "1-step" extensions Σj of Σ do
       {      create Σj and find LB(Σj);
              if LB(Σj) ≤ UB then
                     if Σj is terminal then
                        {  UB:= LB(Σj);
                           optimum:= Σj   }
                     else add Σj to S     }      }
```

Branch-and-Bound

See Chapter 9 (Section 9.1.2) in DPV book, for a different branch and bound algorithm for TSP.