$$V(\hat{\beta}) = \sigma^2 \underbrace{(X'X)^{-1}}_{\substack{k\times n\; n\times k \\ k\times k}} \Longrightarrow$$

$$V(\hat{\beta}) = E\Big[\underbrace{(\hat{\beta}-\beta)}_{k\times 1}\underbrace{(\hat{\beta}-\beta)'}_{1\times k}\Big]_{k\times k}$$

$$= \begin{pmatrix} V(\hat{\beta}_1) & Cov(\hat{\beta}_1,\hat{\beta}_2) & \cdots & Cov(\hat{\beta}_1,\hat{\beta}_k) \\ Cov(\hat{\beta}_2,\hat{\beta}_1) & V(\hat{\beta}_2) & \cdots & Cov(\hat{\beta}_2,\hat{\beta}_k) \\ \vdots & & & \\ Cov(\hat{\beta}_k,\hat{\beta}_1) & \cdots & \cdots & V(\hat{\beta}_k) \end{pmatrix}$$

$$V(\hat{b}_1) = \left(\sigma^2 (X'X)^{-1}\right)_{(1,1)}$$

$$V(\hat{b}_2) = \left(\sigma^2 (X'X)^{-1}\right)_{(2,2)}$$

$$\vdots$$

$$V(\hat{b}_k) = \left(\sigma^2 (X'X)^{-1}\right)_{(k,k)}$$

$$s^2 = \frac{1}{n-k} \sum_{t=1}^{n} \hat{\varepsilon}_t^2 \, , \quad \hat{\varepsilon}_t = y_t - x_t \hat{b} = y_t - \hat{y}_t$$

$\uparrow$

ESTIMATOR OF σ^2

$$\hat{b} = (X'X)^{-1}X'Y$$

$$\hat{Y} = X\hat{b}, \quad \hat{\varepsilon} = Y - \underbrace{\hat{Y}} \Rightarrow$$

$$\Rightarrow \hat{\varepsilon} = \underbrace{Y} - \underbrace{X\hat{b}}$$

$$s^2 = \frac{1}{n-k}\sum_{t=1}^{n}\hat{\varepsilon}_t^2 = \frac{1}{n-k}\underbrace{\underset{1\times n}{\hat{\varepsilon}'}\ \underset{n\times 1}{\hat{\varepsilon}}}_{1\times 1}$$

$3^*)$ $\quad \underline{\varepsilon} \sim \underline{\underline{N}}(\underline{0}, \sigma^2 I_n)$

$$\hat{\beta} = \underline{\underline{\beta}} + (X'X)^{-1} X' \underline{\underline{\varepsilon}} \Rightarrow$$

$\hat{\beta} \sim N$

$E(\hat{\beta}) = \beta$

$V(\hat{\beta}) = \sigma^2 (X'X)^{-1}$

(ΔΙΟΤΙ ΣΥΝΑΡΤΗΣΗ ΓΡΑΜΜΙΚΗ ΤΟΥ $\varepsilon \sim N$)

$\Rightarrow$

$$\hat{\beta} \sim N\left(\beta, \sigma^2 (X'X)^{-1}\right)$$

$b_3 = 0$, $b_1 + b_2 = 5$, $b_1 - b_2 - b_3 = 0$

$b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}$ ΚΑΙ $b_2 = 0$

$b_3 + b_4 = 3$

$v = 2$

$Rb = q$

ν×κ και ν×1

ν ΓΡΑΜΜΙΚΟΙ ΠΕΡΙΟΡΙΣ.

$$Rb = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$$

$$b_1 - b_2 = 1$$

$$b_3 - 2b_1 = 5$$

$$b_3 + 3b_4 = 0$$

$$\Rightarrow \begin{pmatrix} 1 & -1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ 0 \end{pmatrix}$$

$$b_1 = b_2 \Rightarrow b_1 - b_2 = 0$$

$$3b_1 = 7 - 2b_3 \Rightarrow 3b_1 + 2b_3 = 7 \quad \Rightarrow$$

$$\Rightarrow \begin{pmatrix} 1 & -1 & 0 & 0 \\ 3 & 0 & 2 & 0 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 7 \end{pmatrix}$$

$\underline{\underline{\nu < K}}$ ← ΑΡΙΘΜΟΣ ΕΧΟΓΕΝΩΝ ΜΕΤΑΒ.

ΑΡΙΘΜΟΣ ΓΡΑΜ. ΠΕΡΙΟΡ.

$$\hat{b} \sim N\left(b, \sigma^2 (X'X)^{-1}\right)$$

$$E(R\hat{b}) = R\,E(\hat{b}) = Rb$$

$$V(R\hat{b}) = R\,V(\hat{b})\,R' = \sigma^2 R (X'X)^{-1} R' \quad \Rightarrow$$

$$R\hat{b} \sim N$$

$$R\hat{b} \sim N\left[Rb,\ \sigma^2 R(X'X)^{-1}R'\right]$$

$$X \sim N(\mu, A) \Rightarrow (X-\mu)'A^{-1}(X-\mu) \sim \chi^2_{r(A)}$$

$$\Rightarrow (R\hat{b} - Rb)'\left[\sigma^2 R(X'X)^{-1}R'\right]^{-1}(R\hat{b} - Rb)$$

$$\sim \chi^2_{r(R(X'X)^{-1}R')} = \chi^2_v$$

$H_0: \underset{v \times k}{R} \underset{k \times 1}{\beta} = \underset{v \times 1}{q}$

$$(R\hat{\beta} - q)' \left[\sigma^2 R (X'X)^{-1} R'\right]^{-1} (R\hat{\beta} - q) \overset{H_0}{\sim} \chi^2_v$$

$$\cancel{\hat{\beta} = (X'X)^{-1}X'y}$$

$$\sum_{t=1}^{n} \hat{\varepsilon}_t^2 = \hat{\varepsilon}'\hat{\varepsilon}$$

$$\hat{\varepsilon} = Y - X\hat{b} = Y - X(X'X)^{-1}X'Y =$$

$$= \underbrace{\left[I_n - X(X'X)^{-1}X' \right]}_{M} Y = MY$$

$$M' = \left(I_n - X(X'X)^{-1}X'\right)' = I_n' - \left(X(X'X)^{-1}X'\right)' =$$

$$= I_n - (X')'\left((X'X)^{-1}\right)'X' = I_n - X(X'X)^{-1}X' = M$$

M ΣΥΜΜΕΤΡΙΚΗ

$$M \cdot M = [I_n - X(X'X)^{-1}X'][I_n - X(X'X)^{-1}X']$$

$$= I_n - X(X'X)^{-1}X' - X(X'X)^{-1}X'$$

$$+ X(X'X)^{-1}\underbrace{X'X(X'X)^{-1}}_{I}X' =$$

$$= I_n - X(X'X)^{-1}X' - \cancel{X(X'X)^{-1}X' + X(X'X)X'}$$

$$= I_n - X(X'X)^{-1}X' = M \Rightarrow$$

M ΤΑΥΤΟΔΥΝΑΜΗ.

$$M'X = MX = \left(I_n - X(X'X)^{-1}X'\right)X =$$

$$= X - X\underbrace{(X'X)^{-1}X'X}_{I_k} = X - X = 0$$

M ΟΡΘΟΓΩΝΙΑ ΤΗΣ X.

$$\hat{\varepsilon} = My \Rightarrow \hat{\varepsilon} = M(X\beta + \varepsilon) \Rightarrow$$

$$\Rightarrow \hat{\varepsilon} = \underbrace{MX}_{=0}\beta + M\varepsilon \Rightarrow \boxed{\hat{\varepsilon} = M\varepsilon}$$

$$M = I_n - X(X'X)^{-1}X'$$

$$r(M) \underset{\text{ΤΑΥΤ.}}{\overset{\text{sym.}}{=}} tr(M) = tr\left(I_n - X(X'X)^{-1}X'\right)$$

$$= tr(I_n) - tr\left(X(X'X)^{-1}X'\right) =$$

$$= n - tr\left((X'X)^{-1}X'X\right) = n - tr(I_k) =$$

$$= n - k < n \Rightarrow |M| = 0 \Rightarrow M^{-1} \text{ ΔΕΝ ΥΠΑΡΧΕΙ}$$

$$\varepsilon \sim N(0, \sigma^2 I_n)$$

$$\hat{\varepsilon} = M\varepsilon \Rightarrow \hat{\varepsilon} \sim N(0, \sigma^2 M)$$

$$E(\hat{\varepsilon}) = E(M\varepsilon) = M E(\varepsilon) = M \cdot 0 = 0$$

$$V(\hat{\varepsilon}) = V(M\varepsilon) = M V(\varepsilon) M' =$$

$$= M \sigma^2 I_n M' = \sigma^2 M M = \sigma^2 M$$

$$\hat{\varepsilon}' \hat{\varepsilon} = (M\varepsilon)'(M\varepsilon) = \varepsilon' M' M \varepsilon =$$

$$= \varepsilon' M M \varepsilon = \varepsilon' M \varepsilon$$

$$\varepsilon \sim N(0, \sigma^2 I_n) \Leftrightarrow$$

$$\frac{1}{\sigma}\varepsilon \sim N(0, I_n) \Rightarrow$$

$$\left(\frac{1}{\sigma}\varepsilon\right)' M \frac{1}{\sigma}\varepsilon \sim \chi^2_{r(M)} \Rightarrow$$

$$\frac{1}{\sigma^2}\varepsilon' M \varepsilon \sim \chi^2_{n-k} \quad \Big| \Rightarrow \frac{1}{\sigma^2}\hat{\varepsilon}'\hat{\varepsilon} \sim \chi^2_{n-k}$$

$$\frac{1}{\sigma^2}\hat{\varepsilon}'\hat{\varepsilon} = \frac{1}{\sigma^2}\varepsilon' M \varepsilon$$

$$(R\hat{\beta}-q)'\left[R(X'X)^{-1}R'\right]^{-1}(R\hat{\beta}-q)\frac{1}{\sigma^2} \sim \chi^2_v$$

$$\frac{1}{\sigma^2}\hat{\varepsilon}'\hat{\varepsilon} \sim \chi^2_{n-K}$$

$\Rightarrow$

$$\frac{(R\hat{\beta}-q)'\left[R(X'X)^{-1}R'\right]'(R\hat{\beta}-q)\frac{1}{\cancel{\sigma}^2 v}}{\frac{1}{\cancel{\sigma}^2}\frac{\hat{\varepsilon}'\hat{\varepsilon}}{n-K}} \overset{H_0}{\sim} F(v, n-K)$$

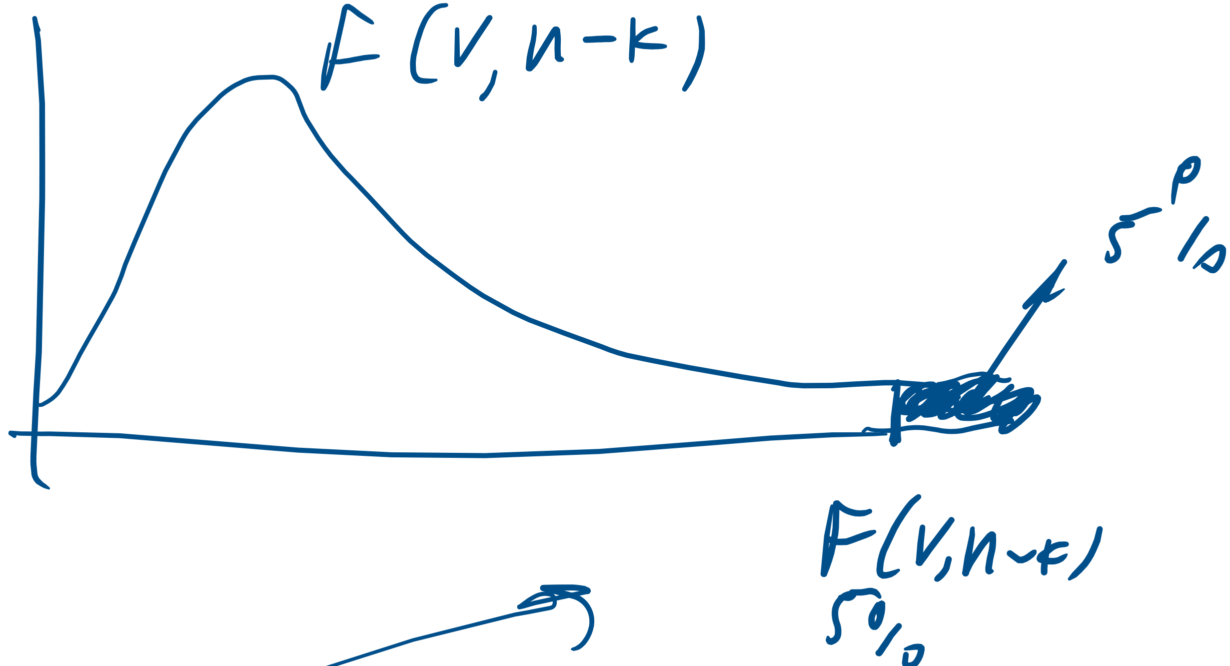
$$\Rightarrow F = \frac{\dfrac{[R\hat{\beta} - q]'\left[R(X'X)^{-1}R'\right]^{-1}(R\hat{\beta} - q)}{v}}{\dfrac{\hat{\varepsilon}'\hat{\varepsilon}}{n-k}} \sim F(v, n-k)$$

$$\frac{\hat{\varepsilon}'\hat{\varepsilon}}{n-k} = s^2$$

Reject H_0 iff

$$F > F_{5\%}(v, n-k)$$

(ΔΕΞΙΑ ΟΥΡΑ)



$F >$ Reject H_0

$$b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} \qquad H_0: b_3 = 0 \qquad q = (0)$$

$$R = (0, 0, 1, 0)$$

$$R\vec{b} = (0, 0, 1, 0) \begin{pmatrix} \vec{b}_1 \\ \vec{b}_2 \\ \vec{b}_3 \\ \vec{b}_4 \end{pmatrix} = \vec{b}_3$$

$$R\hat{\beta} - q = \hat{\beta}_3 - 0 = \hat{\beta}_3$$

$$R\,(X'X)^{-1}R' =$$

$$(0,0,1,0)\,(X'X)^{-1}\begin{pmatrix}0\\0\\1\\0\end{pmatrix} =$$

$$= \left((X'X)^{-1}\right)_{(3,3)}$$

$$\frac{\hat{b}_3 \left(X'X\right)^{-1}_{(3,3)} \hat{b}_3 / 1}{s^2} \sim F(1, n-k)$$

$$\left(\frac{\hat{b}_3}{V(\hat{b}_3)}\right)^2 \sim F(1, n-k) \Rightarrow$$

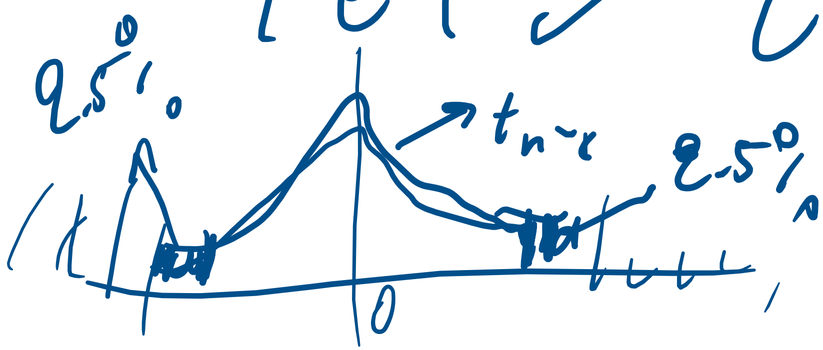
$$\frac{\hat{b}_3}{\sqrt{V(\hat{b}_3)}} \sim t_{n-k}$$

H_0: $\hat{b}_3 = 0$ $\quad$ H_1: $b_3 \neq 0$

$$t = \frac{\hat{b}_3}{s.e.(\hat{b}_3)} \sim t_{n-k}$$

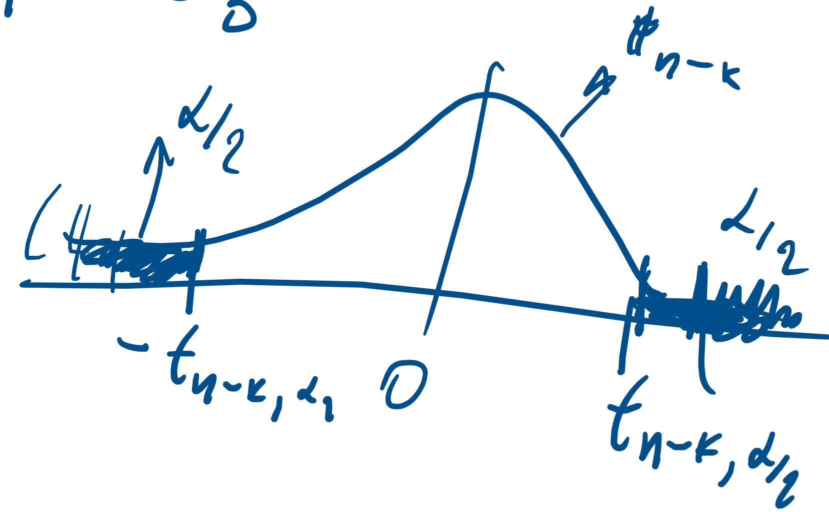
Reject H_0 iff

$$|t| > t_{n-k,\, 2.5\%} \quad \left(5\% \underset{\text{test}}{\text{size}}\right)$$



$H_0: \beta_3 = 0$ v.s. $H_1: \beta_3 \neq 0$ size test α

$$t = \frac{\hat{\beta}_3 - 0}{s.e.(\hat{\beta}_3)}$$



~~H_1~~ Reject H_0 iff $|t| > t_{n-k,\alpha/2}$

Reject H_0 iff $P.V.(t) < \alpha$

$$P.V.(t) = P(t_{n-k} < -|t|) + P(t_{n-k} > |t|)$$

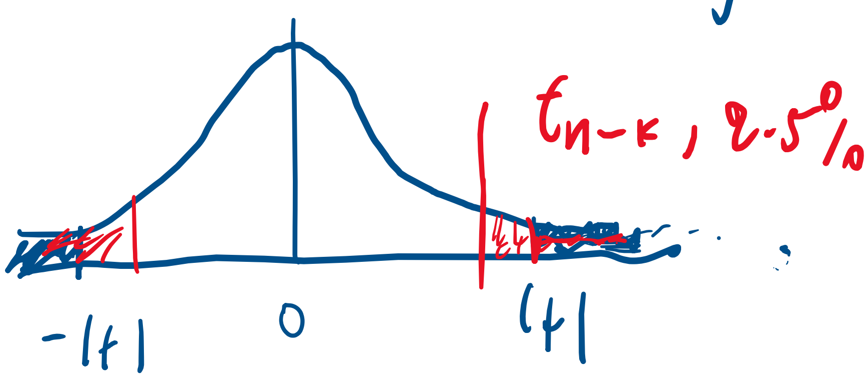
$$= 2P(t_{n-k} > |t|)$$

$H_0: b_2 = 0$ v.s. $H_1: b_2 \neq 0$

Reject H_0 <u>iff</u> $P.V.(t) < 5\%$

$$t = \frac{\hat{b}_2}{s.e.(\hat{b}_2)} \overset{H_0}{\sim} t_{n-k}$$

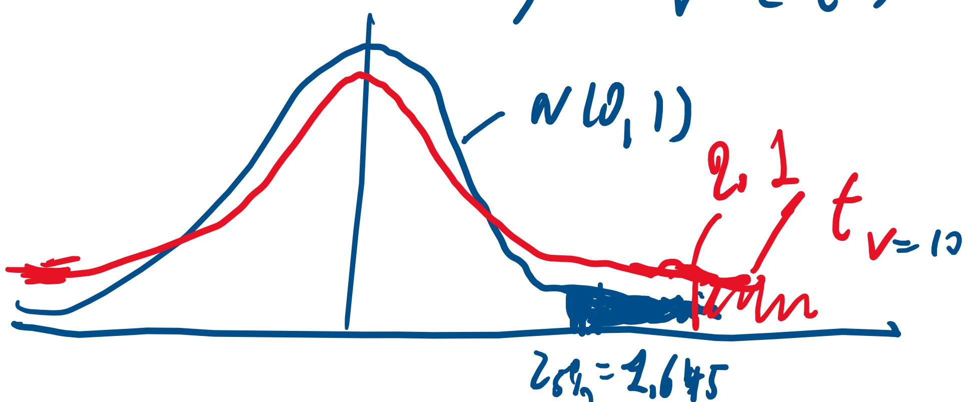
$$P.V.(t) = P(t_{n-k} < -|t|) + P(t_{n-k} > |t|) = 2P(t_{n-k} > |t|)$$



t_v συμμετρική ως προς 0

$$t \sim t_v$$

$$E(t) = 0, \quad V(t) = \frac{v}{v-2} \quad v>2$$



Για $v > 50$ (ΜΕΓΑΛΟ) $\Rightarrow$

$$\boxed{t_v \overset{A}{\sim} N(0,1)}$$

Dependent Variable: R1-RF
Method: Least Squares
Date: 10/24/19 Time: 10:23
Sample: 1960M01 2003M12
Included observations: 528

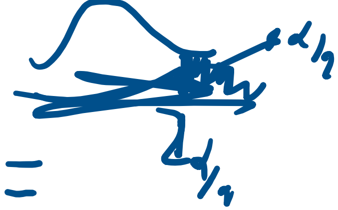
Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	-0.012192	0.082434	-0.147905	0.8825
RMRF	0.896798	0.019628	45.69048	0.0000
HML	0.272386	0.030257	9.002418	0.0000
SMB	1.252069	0.025830	48.47303	0.0000
UMD	0.004131	0.019786	0.208788	0.8347

R-squared	0.922954	Mean dependent var	0.791042
Adjusted R-squared	0.922365	S.D. dependent var	6.384388
S.E. of regression	1.778889	Akaike info criterion	3.999280
Sum squared resid	1655.006	Schwarz criterion	4.039707
Log likelihood	-1050.810	Hannan-Quinn criter.	4.015107
F-statistic	1566.287	Durbin-Watson stat	2.088172
Prob(F-statistic)	0.000000		

DO NOT Reject

0 x 1

Reject H_0



$t_{n-k} =$

$t_{528-5} = t_{523} \sim N(0,1)$

$\alpha = 5\%$

$z_{\alpha/2} = z_{2.5\%} = 1.96$

$H_0: \beta_i = 0$ v.s $H_1: \beta_i \neq 0$

$i = 1, 2, 3, 4, 5$

Reject iff $|t| > z_{\alpha/2}$

$= \left| \frac{\hat{\beta}_i}{s.e.(\hat{\beta}_i)} \right| > 1.96$

$H_0: b_i = 0$ v.s. $H_1: b_i \neq 0$ $\quad i = 1, 2, \dots, k$

$$t = \frac{\hat{b}_i}{s.e.(\hat{b}_i)}$$

Reject H_0 <u>iff</u> $|t| > t_{n-k, \alpha/2}$ or $\equiv$

Reject H_0 <u>iff</u> $P_{H_0}(t) < \alpha$

If H_0 is <u>NOT</u> Rejected $\Rightarrow$ η μεταβλητή στατιστικά ΜΗ-ΣΥΜΑΝΤΙΚΗ ΔΗΛΑΔΗ ο b_i συντελεστής είναι 0.

If H_0 is Rejected $\Rightarrow$

Η μεταβλητή είναι στατιστικά σημαντική δηλαδή $\beta_i \neq 0$.

R1−RF → RISK FREE RATE

↑ ΑΠΟΔΟΣΗ ΧΑΡΤΟΦΥΛΑΚΙΟΥ 1

RMRF = R_m − RF = ΑΠΟΔΟΣΗ ΔΕΙΚΤΗ − RISK FREE RATE

→ ΥΠΕΡΒΑΛΛΟΥΣΑ ΑΠΟΔΟΣΗ ΤΟΥ ΧΑΡΤ. 1

→ " " " ΤΗΣ ΑΓΟΡΑΣ

HML = ΥΠΕΡΒΑΛΛΟΥΣΑ ΑΠΟΔΟΣΗ ΕΝΟΣ ΧΑΡΤΟΦΥΛ. ΕΜΠΕΡΙΕΧΕΙ ΑΠΟΔΟΣΗ ΜΕΤΟΧΩΝ ΥΨΗΛΗΣ ΑΠΟΔΟΣ.

− ΑΠΟΔ. ΜΕΤΟΧΩΝ ΧΑΜΗΛΗΣ

= ΠΑΡΑΓΟΝΤΑΣ (FACTOR)

UMD, SMB FACTORS

ΥΠΕΡΒΑΛ. ΑΠΟΔΟΣΗ ΧΑΡΤΟΦΥΛ. 1

$= b_1 + b_2$ ΥΠΕΡΒ. ΑΓΟΡΑΣ $+ b_3$ FACTOR1

σταθερά

$+ b_4$ FACTOR2 $+ b_5$ FACTOR3 $+ \varepsilon$

MULTI FACTOR CAPM

ΥΠΕΡΒΑΛΛΟΥΣΑ ΜΕΤΟΧΗΣ Ή ΧΑΡΤΟΦΥΛ.

$$r_t = \underbrace{\boxed{b_1} + b_2 \, r_{Mt}}_{\text{classic}} + b_3 r_{3t} + b_4 r_{4t} + b_5 r_{5t}$$

ΥΠΕΡΒΑΛ. ΑΓΟΡΑΣ

$$r_t = R_t - RF_{t-1}$$

RISK FREE RATE $(t-1)$

ΑΠΟΔΟΣ. ΜΕΤΟΧΗΣ $= (\ln P_t - \ln P_{t-1}) \cdot 100$

ΥΠΕΡΒΑΛΛΟΥΣΑ ΑΠΟΔΟΣΗ (EXCESS Return over and above the Risk Free Rate)

$$H_0: \underset{v\times k}{R}\beta = q \quad \text{v.s.} \quad H_1: R\beta \neq q$$

k = NO. EXPLAN. VAR.

v = NO. RESTRICTIONS

$$F = \frac{(R\hat{\beta} - q)' \left[R(X'X)^{-1}R' \right]^{-1} (R\hat{\beta} - q) / v}{\hat{\varepsilon}'\hat{\varepsilon} / (n-k)} \overset{H_0}{\sim} F(v, n-k)$$

An $\varepsilon \sim N(0, \sigma^2 I_n)$

$$Y = X\beta + \varepsilon \Rightarrow \hat{\beta} = (X'X)^{-1}X'Y \Rightarrow \hat{\varepsilon} = Y - X\hat{\beta}$$

ΕΣΤΩ $\tilde{\beta}$ Η ΕΚΤΙΜΗΤΡΙΑ ΤΟΥ β ΑΦΟΥ ΕΧΩ ΕΝΣΩΜΑΤΩΣΕΙ ΤΟΥΣ ΠΕΡΙΟΡΙΣΜΟΥΣ $\Rightarrow \tilde{\varepsilon} = Y - X\tilde{\beta}$

$$F = \frac{(\tilde{\varepsilon}'\tilde{\varepsilon} - \hat{\varepsilon}'\hat{\varepsilon})/v}{\hat{\varepsilon}'\hat{\varepsilon}/n-k} \overset{H_0}{\sim} F(v, n-k)$$

$$\frac{(RRSS - URSS)/v}{URSS/n-k} \overset{H_0}{\sim} F(v, n-k)$$

$RRSS = \tilde{\varepsilon}'\tilde{\varepsilon}$ = RESTRICTED RESID. SUM. SQUA.

$URSS = \hat{\varepsilon}'\hat{\varepsilon}$ = UNRESTRICTED " " "

$$y_i = \beta_1 + \beta_2 x_{i2} + \beta_3 x_{i3} + \cdots + \beta_k x_{ik} + \varepsilon_i$$

$i = 1, 2, \ldots, n$

$H_0: \beta_2 = \beta_3 = \cdots = \beta_k = 0$ V.S. H_1: AT LEAST ONE $\neq 0$

RESTRICTED MODEL

$$y_i = \beta_1 + \varepsilon_i \Rightarrow \hat{\beta}_1 = \bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$$

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \beta_1 + \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{pmatrix}$$

$$Y = X\beta_1 + \varepsilon$$

$$\hat{\beta}_1 = (X'X)^{-1} X'Y = \left((1\ 1\ 1 \dots 1) \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \right)^{-1} (1\ 1 \dots 1) \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

$$= n^{-1} \sum_{i=1}^{n} y_i = \frac{1}{n} \sum y_i = \bar{y}$$

$$F = \frac{\left(\tilde{\varepsilon}'\tilde{\varepsilon} - \hat{\varepsilon}'\hat{\varepsilon}\right)/k-1}{\dfrac{\hat{\varepsilon}'\hat{\varepsilon}}{n-k}} \sim F(k-1, n-k)$$

ΤΥΠΩΝΕΤΑΙ ΑΠΟ ΟΛΑ ΤΑ ΟΙΚΟΝΟΜ. ΠΑΚΕΤΑ

ΕΙΝΑΙ F STATISTIC ΓΙΑ ΤΗΝ ΥΠΟΘΕΣΗ ΟΤΙ ΟΛΕΣ ΟΙ ΜΕΤΑΒΛΗΤΕΣ ΠΛΗΝ ΤΗΣ ΣΤΑΘΕΡΑΣ ΕΙΝΑΙ ΜΗΔΕΝ. Δηλαδή ΕΛΕΓΧΩ ΤΗΝ ΣΤΑΤΙΣΤΙΚΗ ΣΥΜΑΝΤΙΚΟΤΗΤΑ ΟΛΩΝ ΤΩΝ ΜΕΤΑΒΛΗΤΩΝ (ΠΛΗΝ ΣΤΑΘΕΡΑΣ) ΑΠΟ ΚΟΙΝΟΥ.

$H_0: \beta_1 = \beta_5 = 0$ vs. H_1: At least one $\neq 0$

$$F = \frac{(RRSS - URSS)/2}{\frac{URSS}{n-k}} =$$

$$= \frac{(1655.775 - 1655.006)/2}{\frac{1655.006}{528-5}} \overset{H_0}{\sim} F(2, 523)$$

Reject H_0 iff $F > F_{5\%}(2, 523)$