$$y_t = \underset{x_{t1}=1}{\beta_1} + \underbrace{x_{t2}\beta_2 + x_{t3}\beta_3 + x_{t4}\beta_4} + \underbrace{\varepsilon_t}$$

$$E(\varepsilon_t) = 0, \quad V(\varepsilon_t) = \sigma^2$$

$$\forall t = 1, 2, \ldots, \underline{\underline{n}}$$

$$
\left.
\begin{aligned}
y_1 &= \beta_1 + x_{12}\beta_2 + x_{13}\beta_3 + x_{14}\beta_4 + \varepsilon_1 \\
y_2 &= \beta_1 + x_{22}\beta_2 + x_{23}\beta_3 + x_{24}\beta_4 + \varepsilon_2 \\
y_3 &= \beta_1 + x_{32}\beta_2 + x_{33}\beta_3 + x_{34}\beta_4 + \varepsilon_3 \\
&\;\;\vdots \\
y_n &= \beta_1 + x_{n2}\beta_2 + x_{n3}\beta_3 + x_{n4}\beta_4 + \varepsilon_n
\end{aligned}
\right\} \Rightarrow
$$

$$
\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{pmatrix}
=
\begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}\beta_1
+
\begin{pmatrix} x_{12} \\ x_{22} \\ x_{32} \\ \vdots \\ x_{n2} \end{pmatrix}\beta_2
+
\begin{pmatrix} x_{13} \\ x_{23} \\ \vdots \\ x_{n3} \end{pmatrix}\beta_3
+
\begin{pmatrix} x_{14} \\ x_{24} \\ x_{34} \\ \vdots \\ x_{n4} \end{pmatrix}\beta_4
+
\begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{pmatrix}
\Rightarrow
$$

$$
\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}
=
\begin{pmatrix}
1 & x_{12} & x_{13} & x_{14} \\
1 & x_{22} & x_{23} & x_{24} \\
1 & x_{32} & x_{33} & x_{34} \\
\vdots & & & \\
1 & x_{n2} & x_{n3} & x_{n4}
\end{pmatrix}
\begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{pmatrix}
+
\begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \vdots \\ \varepsilon_n \end{pmatrix}
$$

$$
\Rightarrow \quad \underset{n \times 1}{Y} = \underset{n \times k}{X} \, \underset{k \times 1}{\beta} + \underset{n \times 1}{\varepsilon}
$$

$$E(\varepsilon) = E\begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{pmatrix} = \begin{pmatrix} E(\varepsilon_1) \\ E(\varepsilon_2) \\ \vdots \\ E(\varepsilon_n) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$V(\varepsilon) = E(\varepsilon\varepsilon') =$$

$$= E\left[\begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{pmatrix} \begin{pmatrix} \varepsilon_1, \varepsilon_2, \ldots \varepsilon_n \end{pmatrix}\right] =$$

$$= E\begin{pmatrix} \varepsilon_1^2 & \varepsilon_1\varepsilon_2 & \varepsilon_1\varepsilon_3 & \cdots & \varepsilon_1\varepsilon_n \\ \varepsilon_2\varepsilon_1 & \varepsilon_2^2 & \varepsilon_2\varepsilon_3 & \cdots & \varepsilon_2\varepsilon_n \\ \varepsilon_3\varepsilon_1 & \varepsilon_3\varepsilon_2 & \varepsilon_3^2 & \cdots & \varepsilon_3\varepsilon_n \\ \vdots & & & & \vdots \\ \varepsilon_n\varepsilon_1 & \varepsilon_n\varepsilon_2 & \cdots & & \varepsilon_n^2 \end{pmatrix} = \begin{pmatrix} \sigma^2 & 0 & 0 & \cdots & 0 \\ 0 & \sigma^2 & 0 & \cdots & 0 \\ 0 & & \ddots & & \vdots \\ \vdots & & & & \\ 0 & \cdots & \cdots & 0 & \sigma^2 \end{pmatrix}$$

$$= \sigma^2 \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & 0 & & \ddots & 1 \end{pmatrix} = \sigma^2 I_n$$

$$V(\varepsilon_t) = \sigma^2 \Rightarrow V(\varepsilon_t) = E(\varepsilon_t - \underbrace{E(\varepsilon_t)}_{0})^2 =$$

$$= E(\varepsilon_t^2) = \sigma^2 \quad \forall t = 1, 2, \ldots, n$$

X ΤΥΧΑΙΑ ΜΕΤΑΒΛΗΤΗ

$$V(X) = E\left(X - E(X)\right)^2$$

Αν $E(X) = 0$

$$\Rightarrow V(X) = E(X^2)$$

$$\underset{n\times 1}{Y} = \underset{n\times k}{X}\,\underset{k\times 1}{b} + \underset{n\times 1}{\varepsilon}, \quad E(\varepsilon)=0, \quad V(\varepsilon)=\sigma^2 I_n$$

$\boxed{k+1} \leftarrow$ ΑΓΝΩΣΤΟΥΣ

$$\min_b \sum_{t=1}^{n} \varepsilon_t^2 = \min_b \sum_t \left(y_t - b_1 - x_{t2}b_2 - \dots - x_{t4}b_4\right)^2$$

$$= \min_b (\varepsilon_1\ \varepsilon_2\ \dots\ \varepsilon_n)\begin{pmatrix}\varepsilon_1\\ \varepsilon_2\\ \vdots\\ \varepsilon_n\end{pmatrix} = \min_b \underset{1\times n}{\varepsilon'}\,\underset{n\times 1}{\varepsilon} =$$

$$= \min_b (Y - Xb)'(Y - Xb)$$

$$S = (y - X\beta)'(y - X\beta)$$

$$\underset{1\times 1}{\frac{\partial S}{\partial \beta}}\Big/_{k\times 1} = 0 \Rightarrow \cancel{2}(X'X)\hat{\beta} - \cancel{2}X'y = 0$$

$$\Rightarrow (X'X)\hat{\beta} - X'y = 0 \Rightarrow \underbrace{(X'X)}_{\substack{k\times n \;\; n\times k \\ k\times k}}\underset{k\times 1}{\hat{\beta}} = \underbrace{X'y}_{\substack{k\times n \;\; n\times 1 \\ k\times 1}}$$

NORMAL EQUAT.

$$\Rightarrow \boxed{\hat{\beta} = (X'X)^{-1}X'y}$$

$\underline{\text{IFF}}$ $(X'X)^{-1}$ ΥΠΑΡΧΕΙ $\Leftrightarrow$

$$\Longleftrightarrow \quad |\underbrace{\underset{k\times n}{X'}\,\underset{n\times k}{X}}_{k\times k}| \neq 0$$ δηλαδή $X'X$ ΜΗ-ΙΔΙΑΖΟΥΣΑ

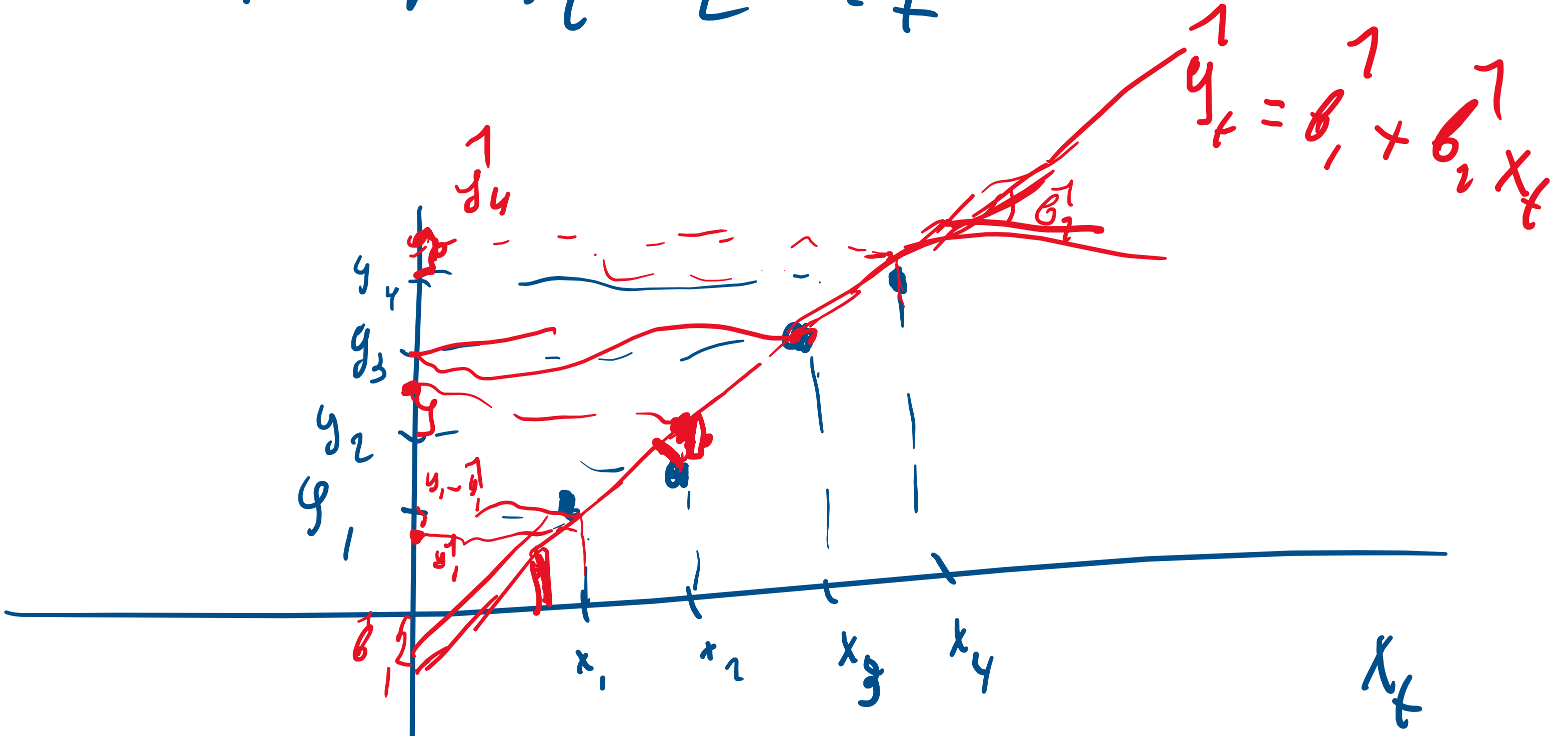
$$\Longleftrightarrow \quad r(\underset{n\times k}{X}) = k \quad \Longleftarrow$$

RANK OF X

ΟΙ k ΕΞΩΓΕΝΕΙΣ ΜΕΤΑΒΛ. ΠΡΕΠΕΙ ΝΑ ΓΡΑΜΜΙΚΑ ΑΝΕΞΑΡΤΗΤΕΣ.

ΔΗΛΑΔΗ ΝΑ ΜΗΝ ΕΧΩ ΠΟΛΥΣΥΓΡΑΜΜΙΚΟΤΗΤΑ (MULTICOLINEARITY)

$$y_t = \beta_1 + x_t \beta_2 + \varepsilon_t$$



$$\hat{\varepsilon}_t = y_t - \hat{y}_t = y_t - \hat{\beta}_1 - \hat{\beta}_2 x_t$$

$$\hat{b} = (X'X)^{-1}X'y$$

$$\hat{y} = X\hat{b}$$

FITTED VALUES

$$\begin{cases} \hat{y}_1 = \hat{b}_1 + x_{12}\hat{b}_2 + x_{13}\hat{b}_3 + x_{14}\hat{b}_4 \\ \hat{y}_2 = \hat{b}_1 + x_{22}\hat{b}_2 + x_{23}\hat{b}_3 + x_{24}\hat{b}_4 \\ \hat{y}_n = \hat{b}_1 + x_{n2}\hat{b}_2 + x_{n3}\hat{b}_3 + x_{n4}\hat{b}_4 \end{cases}$$

$$\hat{\varepsilon} = \underset{=}{Y} - \hat{Y}$$

ΚΑΤΑΛΟΙΠΑ (RESIDUALS)

$$\underset{n \times 1}{\hat{Y}} = X\hat{\beta} = \underbrace{X(X'X)^{-1}X'}_{n \times n} \underset{n \times 1}{Y} = P_x Y$$

$$P_x = X(X'X)^{-1}X'$$ ΣΥΜΜΕΤΡΙΚΗ + ΤΑΥΤΟΔΥΝΑΜΗ

P_x ΣΥΜΜΕΤΡΙΚΗ $\equiv P_x' = P_x$

$$P_x' = \left[X(X'X)^{-1}X'\right]' = \qquad \left((AB\Gamma)' = \Gamma'B'A'\right)$$

$$= (X')'\left((X'X)^{-1}\right)'X'$$

$$= X\left((X'X)'\right)^{-1}X' =$$

$$= X\left(X'(X')'\right)^{-1}X' = X(X'X)^{-1}X' = P_x$$

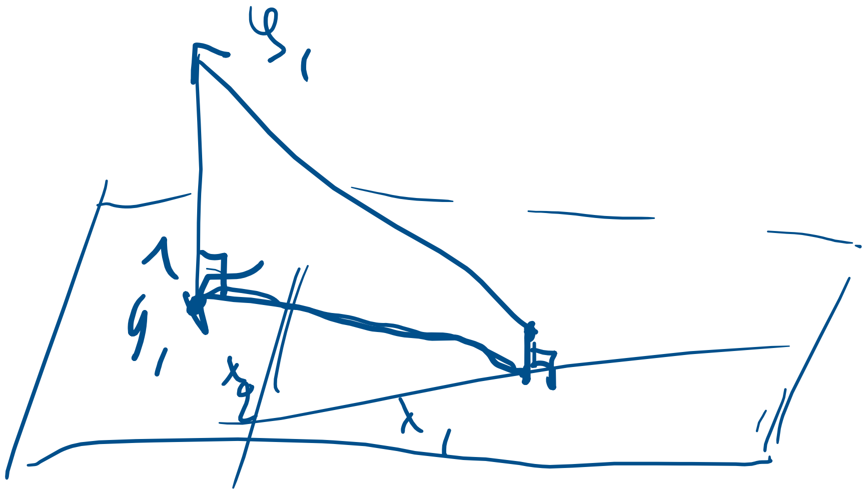
ΟΝΤΩΣ P_x ΣΥΜΜΕΤΡΙΚΗ

P_X ΤΑΥΤΟΔΥΝΑΜΗ $\iff P_X^2 = P_X \cdot P_X = P_X$

$$P_X^2 = P_X \cdot P_X = \left(X(X'X)^{-1}X'\right)\left(X(X'X)^{-1}X'\right)$$

$$= X(X'X)^{-1}\underbrace{X'X(X'X)^{-1}}_{=I_k}X' =$$

$$= X(X'X)^{-1} I_k X' = X(X'X)^{-1}X' = P_X$$



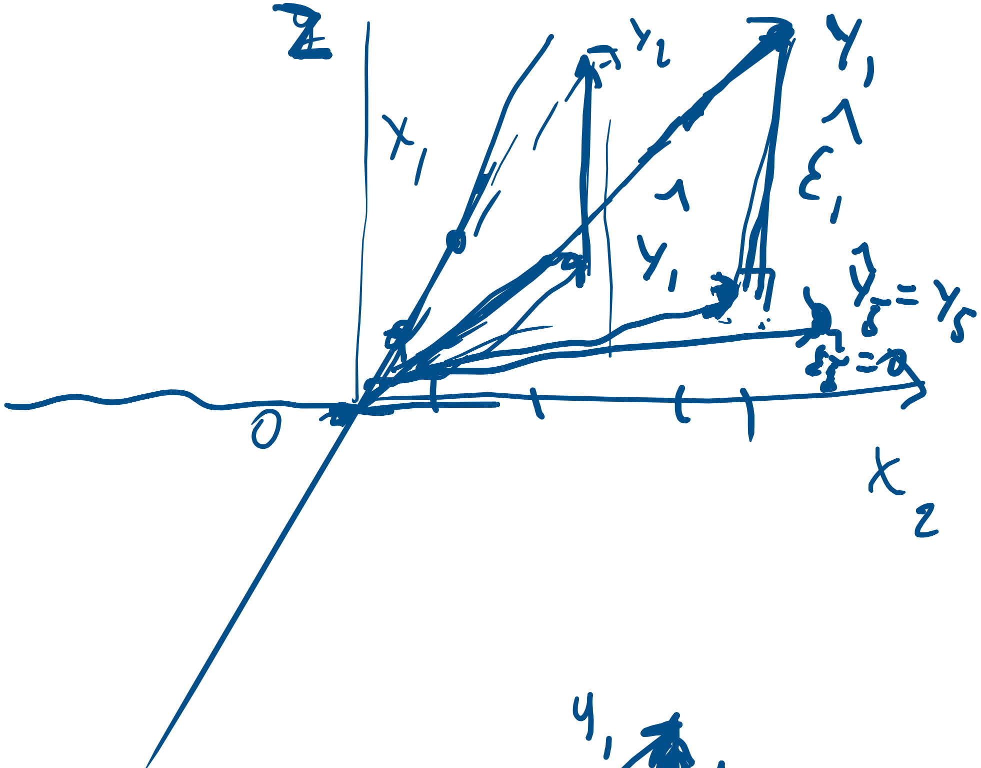
$$\hat{y} = P_X \underline{\underline{y}}$$

P_X ΣΥΜ + ΤΑΥ

$$r(P_X) = r\left(X(X'X)^{-1}X'\right) =$$

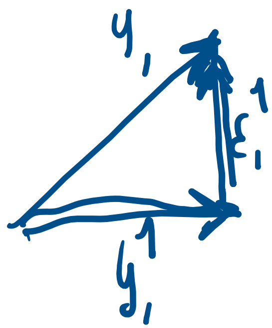
$$= tr\left(X(X'X)^{-1}X'\right) = tr\left((X'X)^{-1}X'X\right) =$$

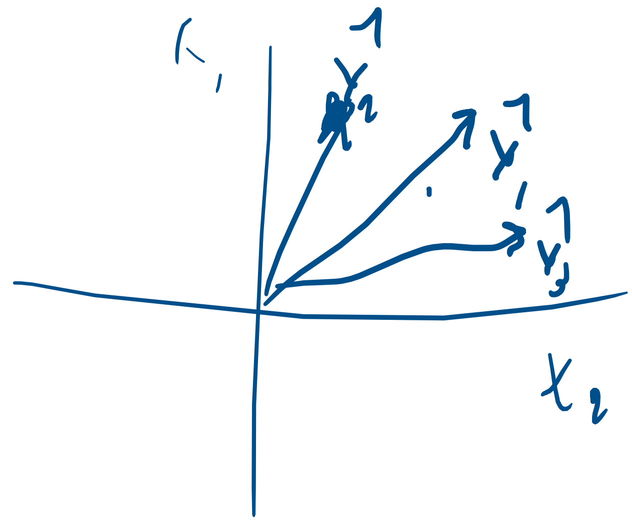
$$= tr(I_k) = k \quad (4)$$



$$\hat{\varepsilon}_1 = y_1 - \hat{y}_1$$

$$\hat{\varepsilon}_1 + \hat{y}_1 = y_1$$





$r(\underset{n \times k}{A})$ = RANK(A) ΒΑΘΜΟΣ ΤΗΣ Α

= ΑΡΙΘΜΟΣ ΤΩΝ ΓΡΑΜΜΙΚΑ ΑΝΕΞΑΡΤΗΤΩΝ ΓΡΑΜΜΩΝ Ή ΣΤΗΛΩΝ

$r(A) \leq \min(n, k)$

Π.χ. $\underset{10 \times 5}{A}$ $r(A)$ ΤΟ ΠΟΛΥ ΘΑ ΕΙΝΑΙ 5

$A_{n \times n}$ (ΤΕΤΡΑΓΩΝΙΚΗ)

ΤΟΤΕ $tr(A) = trace(A) =$

$= \text{ΙΧΝΟΣ}(A) = \sum_{i=1}^{n} \alpha_{ii} = \alpha_{11} + \alpha_{22} + \ldots + \alpha_{nn}$

= ΑΘΡΟΙΣΜΑ ΤΩΝ ΣΤΟΙΧΕΙΩΝ ΤΗΣ ΚΥΡΙΑΣ ΔΙΑΓΩΝΙΟΥ.

$\underset{n\times n}{A}$ ΣΥΜΜΕΤΡΙΚΗ $\underline{\underline{\text{ΑΝΝ}}}$ $A' = A$

$$A = \begin{pmatrix} 2 & 3 \\ 5 & 7 \end{pmatrix} \Rightarrow A' = \begin{pmatrix} 2 & 5 \\ 3 & 7 \end{pmatrix}$$

$$tr(AB\Gamma) = tr(B\Gamma A) = tr(\Gamma AB)$$

$$r(AA') = r(A'A) = r(A)$$

$$\underset{n\times 1}{Y} = \underset{n\times k}{X}\underset{k\times 1}{\beta} + \underset{n\times 1}{\varepsilon} \qquad \underline{\underline{E(\varepsilon) = 0}} \qquad V(\varepsilon) = \sigma^2 I_n$$

$$\hat{\beta} = (X'X)^{-1}X'Y = (X'X)^{-1}X'X\beta + (X'X)^{-1}X'\varepsilon$$

$$= \beta + (X'X)^{-1}X'\varepsilon$$

$$E(\hat{\beta}) = E\left[\beta + (X'X)^{-1}X'\varepsilon\right] =$$

$$= E(\beta) + E\left[(X'X)^{-1}X'\varepsilon\right] =$$

$$= \beta + (X'X)^{-1}X'\underset{\substack{\shortparallel\\ 0}}{E(\varepsilon)} = \beta$$

$\hat{\beta}$ UNBIASED ESTIMATOR OF β.

$$V(\hat{\beta}) = E\left\{\left[\hat{\beta} - E(\hat{\beta})\right]\left[\hat{\beta} - E(\hat{\beta})\right]'\right\}$$

$$\left.\begin{array}{l} = E\left[(\hat{\beta}-\beta)(\hat{\beta}-\beta)'\right] \\ \hat{\beta}-\beta = (X'X)^{-1}X'\varepsilon \end{array}\right\} \Rightarrow$$

$$V(\hat{\beta}) = E\left[(X'X)^{-1}X'\varepsilon\left[(X'X)^{-1}X'\varepsilon\right]'\right]$$

$$= E\left[(X'X)^{-1}X'\varepsilon\varepsilon'X(X'X)^{-1}\right] =$$

$$= (X'X)^{-1}X'E(\varepsilon\varepsilon')X(X'X)^{-1} =$$

$$= (X'X)^{-1} X' \sigma^2 I_n X (X'X)^{-1} =$$

$$= \sigma^2 (X'X)^{-1} \cancel{X'X (X'X)^{-1}} =$$

$$= \sigma^2 (X'X)^{-1}$$

$$E(\hat{\beta}) = \beta, \quad V(\hat{\beta}) = \sigma^2 (X'X)^{-1}$$

$$\underset{n\times 1}{Y} = \underset{n\times k}{X}\,\underset{k\times 1}{\beta} + \underset{n\times 1}{\varepsilon}$$

1) Η ΣΧΕΣΗ ΣΩΣΤΗ

2) $r(X)=k$ (NO MULTICOL.)

3) $E(\varepsilon)=0$, $V(\varepsilon)=\sigma^2 I_n$

$$V(\varepsilon) = E(\varepsilon\varepsilon') = E\left[\begin{pmatrix}\varepsilon_1\\ \varepsilon_2\\ \vdots\\ \varepsilon_n\end{pmatrix}(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)\right]$$

$$= E\begin{pmatrix}
\varepsilon_1^2 & \varepsilon_1\varepsilon_2 & \varepsilon_1\varepsilon_3 & \varepsilon_1\varepsilon_4 & \cdots & \varepsilon_1\varepsilon_n \\
\varepsilon_2\varepsilon_1 & \varepsilon_2^2 & \varepsilon_2\varepsilon_3 & \varepsilon_2\varepsilon_4 & \cdots & \varepsilon_2\varepsilon_n \\
\varepsilon_3\varepsilon_1 & \varepsilon_3\varepsilon_2 & \varepsilon_3^2 & \varepsilon_3\varepsilon_4 & \cdots & \varepsilon_3\varepsilon_n \\
\vdots & & & & & \\
\varepsilon_n\varepsilon_1 & \varepsilon_n\varepsilon_2 & \cdots & & \cdots & \varepsilon_n^2
\end{pmatrix} =$$

$$= \begin{pmatrix} E\xi_1^2 & E\xi_1\xi_2 & \cdots & E\xi_1\xi_n \\ E\xi_2\xi_1 & E\xi_2^2 & \cdots & E\xi_2\xi_n \\ \vdots & & \ddots & \vdots \\ E\xi_n\xi_1 & \cdots & \cdots & E(\xi_n^2) \end{pmatrix} =$$

$$= \begin{pmatrix} V(\xi_1) & Cov(\xi_1,\xi_2) & \cdots & Cov(\xi_1,\xi_n) \\ Cov(\xi_2,\xi_1) & V(\xi_2) & \cdots & \vdots \\ \vdots & & \ddots & \\ Cov(\xi_n,\xi_1) & \cdots & \cdots & V(\xi_n) \end{pmatrix} =$$

$$= \begin{pmatrix} \sigma^2 & 0 & 0 & \cdots & 0 \\ 0 & \sigma^2 & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & \cdots & \cdots & 0 & \sigma^2 \end{pmatrix} = \sigma^2 I_n$$

Theorem GAUSS-MARKOV ✓

$\hat{\beta}$ BEST LINEAR UNBIASED ESTIMATOR of β.

$\hat{\beta} = (X'X)^{-1} X' Y$ (LINEAR IN Y)

BEST $\Leftrightarrow$ $V(\hat{\beta}) \leq V(\tilde{\beta})$

όπου $\tilde{\beta}$ οποιοσδήποτε γραμμικός αμερόληπτος εκτιμητής του β.

Έστω $\tilde{\beta}$ άλλος linear + unbiased estimator $\Rightarrow$

$$V(\hat{\beta}) \leq V(\tilde{\beta}) \Rightarrow$$

$V(\hat{\beta}) - V(\tilde{\beta})$ negative definite matrix.

$$V(\hat{\beta}) = \sigma^2 \underbrace{(X'X)^{-1}}_{\substack{k\times n \; n\times k \\ k\times k}}$$

$$V(\hat{\beta}) = E\left[\underbrace{(\hat{\beta}-\beta)}_{k\times 1}\underbrace{(\hat{\beta}-\beta)'}_{1\times k}\right]$$

$k \times k$

$$= \begin{pmatrix} V(\hat{\beta}_1) & Cov(\hat{\beta}_1, \hat{\beta}_2) & \cdots & Cov(\hat{\beta}_1, \hat{\beta}_k) \\ Cov(\hat{\beta}_2, \hat{\beta}_1) & V(\hat{\beta}_2) & \cdots & Cov(\hat{\beta}_2, \hat{\beta}_k) \\ \vdots & & & \\ Cov(\hat{\beta}_k, \hat{\beta}_1) & \cdots & \cdots & V(\hat{\beta}_k) \end{pmatrix}$$

$$\sum_{t=1}^{n} \hat{\varepsilon}_t^2 = \hat{\varepsilon}'\hat{\varepsilon}$$

$$\hat{\varepsilon} = Y - X\hat{b} = Y - X(X'X)^{-1}X'Y =$$

$$= \underbrace{\left[I_n - X(X'X)^{-1}X' \right]}_{M} Y = MY$$

$$M' = \left(I_n - X(X'X)^{-1}X'\right)' = I_n' - \left(X(X'X)^{-1}X'\right)' =$$

$$= I_n - (X')'\left((X'X)^{-1}\right)'X' = I_n - X(X'X)^{-1}X' = M$$

M ΣΥΜΜΕΤΡΙΚΗ

$$M \cdot M = [I_n - X(X'X)^{-1}X'][I_n - X(X'X)^{-1}X']$$

$$= I_n - X(X'X)^{-1}X' - X(X'X)^{-1}X' + X(X'X)^{-1}\underbrace{X'X(X'X)^{-1}}_{I}X' =$$

$$= I_n - X(X'X)^{-1}X' - \cancel{X(X'X)^{-1}X'} + \cancel{X(X'X)X'}$$

$$= I_n - X(X'X)^{-1}X' = M \Rightarrow$$

M ΤΑΥΤΟΔΥΝΑΜΗ.

$$M'X = MX = \left(I_n - X(X'X)^{-1}X'\right)X =$$

$$= X - X\underbrace{(X'X)^{-1}X'X}_{I_k} = X - X = 0$$

M ΟΡΘΟΓΩΝΙΑ ΤΗΣ X.

$$\hat{\varepsilon} = My \Rightarrow \hat{\varepsilon} = M(X\beta + \varepsilon) \Rightarrow$$

$$\Rightarrow \hat{\varepsilon} = \underbrace{MX}_{=0}\beta + M\varepsilon \Rightarrow \boxed{\hat{\varepsilon} = M\varepsilon}$$

$$M = I_n - X(X'X)^{-1}X'$$

$$r(M) \underset{\text{ΤΑΥΤ.}}{\overset{\text{SYM.}}{=\!=}} tr(M) = tr\left(I_n - X(X'X)^{-1}X'\right)$$

$$= tr(I_n) - tr\left(X(X'X)^{-1}X'\right) =$$

$$= n - tr\left((X'X)^{-1}X'X\right) = n - tr(I_k) =$$

$$= n - k < n \Rightarrow |M| = 0 \Rightarrow M^{-1} \text{ ΔΕΝ ΥΠΑΡΧΕΙ}$$