ΕΣΤΩ ΤΩΡΑ $cov(\varepsilon_i, \varepsilon_j) \neq 0$ ←

ΤΑ ΓΡΑΜΜΙΚΑ ΜΟΝΤΕΛΑ ΠΟΥ ΜΑΣ ΔΙΝΟΥΝ ΑΥΤΟ ΤΟ ΑΠΟΤΕΛ.

ΕΙΝΑΙ ΤΑ

AR AUTOREGRESSIVE

MA MOVING AVERAGE

ARMA AUTOREGR. MOVING AVER.

AR(k)

↑ k is THE ORDER OF AR MODEL

$$\varepsilon_t = \mu + \varphi_1 \varepsilon_{t-1} + \varphi_2 \varepsilon_{t-2} + \dots + \varphi_k \varepsilon_{t-k} + n_t$$

AR(1) $\varepsilon_t = \mu + \varphi \varepsilon_{t-1} + n_t$

$n_t \overset{iid}{\sim} (0, \sigma^2)$

ΑΥΤΟ ΠΟΥ ΘΕΛΟΥΜΕ ΑΠΟ ΜΙΑ ΣΤΟΧΑΣΤΙΚΗ ΔΙΑΔΙΚΑΣΙΑ ΚΑΙ ΕΙΔΙΚΟΤΕΡΑ ΑΠΟ ΜΙΑ ΧΡΟΝΟΛΟΓΙΚΗ ΣΕΙΡΑ, ΕΣΤΩ y_t, ΕΙΝΑΙ ΝΑ ΕΙΝΑΙ ΣΤΑΣΙΜΗ.

ΑΡΑ ΜΙΑ ΧΡΟΝΟΛ. ΣΕΙΡΑ y_t είναι 2ης τάξης ΣΤΑΣΙΜΗ (2nd ORDER STATIONARY) <u>ΑΝΝ</u>

$E(y_t)$, $E(y_t^2)$ ΚΑΙ $Cov(y_t, y_{t-k})$ ΕΙΝΑΙ ΑΝΕΞΑΡΤΗΤΑ ΤΟΥ ΧΡΟΝΟΥ t.

Αυτό που κάνουμε συνήθως είναι να βάζουμε περιορισμούς στις παραμέτρους ώστε η στοχαστική διαδικασία να είναι ΣΤΑΣΙΜΗ 2ης ΤΑΞΗΣ

AR(1) ΕΙΝΑΙ ΣΤΑΣΙΜΟ <u>ΑΝΝ</u> $|\varphi|<1$

$$\varepsilon_t = \mu + \varphi \varepsilon_{t-1} + u_t \qquad u_t \overset{iid}{\sim} (0, \sigma^2)$$

$$E(\varepsilon_t) = \mu + \varphi E(\varepsilon_{t-1}) + \underbrace{E(u_t)}_{=0}$$

$$\Rightarrow E(\varepsilon_t) = \mu + \varphi E(\varepsilon_{t-1})$$

if station.

$$\Rightarrow E(\varepsilon_t)(1-\varphi) = \mu \Rightarrow E(\varepsilon_t) = \frac{\mu}{1-\varphi}$$

$$V(\varepsilon_t) = V(\mu + \varphi\varepsilon_{t-1} + u_t) =$$

$$\Rightarrow V(\varepsilon_t) = V(\varphi\varepsilon_{t-1} + u_t) \Rightarrow$$

$$\Rightarrow V(\varepsilon_t) = \varphi^2 V(\varepsilon_{t-1}) + V(u_t) + 2\varphi\, Cov(\varepsilon_{t-1}, u_t)$$

$$\Rightarrow V(\varepsilon_t) = \varphi^2 V(\varepsilon_{t-1}) + \sigma^2 + 0$$

If STATION. $\Rightarrow$ $V(\varepsilon_t) = \dfrac{\sigma^2}{1-\varphi^2}$

$\varepsilon_t = \mu + \varphi \varepsilon_{t-1} + u_t =$

$= \mu + \varphi\mu + \varphi^2 \varepsilon_{t-2} + u_t + \varphi u_{t-1}$

$= \mu + \varphi\mu + \varphi^2\mu + \varphi^3 \varepsilon_{t-3} + u_t + \varphi u_{t-1} + \varphi^2 \varepsilon_{t-2}$

$= \mu(1+\varphi+\varphi^2+\dots) + u_t + \varphi u_{t-1} + \varphi^2 u_{t-2} + \varphi^3 u_{t-3} + \dots$

$\Rightarrow E(\varepsilon_t) = \mu(1+\varphi+\varphi^2+\varphi^3+\dots)$

$\Rightarrow E(\varepsilon_t) = \dfrac{\mu}{1-\varphi}$ iff $|\varphi| < 1$ 4

$$V(\varepsilon_t) = V\left(\mu(1+\phi+\phi^2+\dots) + u_t + \phi u_{t-1} + \phi^2 u_{t-2} + \dots\right)$$

$$= V\left(u_t + \phi u_{t-1} + \phi^2 u_{t-2} + \phi^3 u_{t-3} + \dots\right)$$

$$= V(u_t) + \phi^2 V(u_{t-1}) + \phi^4 V(u_{t-2}) + \phi^6 V(u_{t-3}) + \dots$$

$$\Rightarrow V(\varepsilon_t) = \sigma^2\left(1 + \phi^2 + \phi^4 + \phi^6 + \dots\right)$$

$$\Rightarrow V(\varepsilon_t) = \frac{\sigma^2}{1-\phi^2} \quad \underline{\underline{\text{iff}}} \quad \phi^2 < 1 \quad (B)$$

$$(A) + (B) \Rightarrow |\phi| < 1$$

$$\gamma_k = Cov(\varepsilon_t, \varepsilon_{t-k})$$

$$\gamma_1 = Cov(\varepsilon_t, \varepsilon_{t-1})$$

$$Cov(\varepsilon_{t+1}, \varepsilon_t) = \gamma_1$$

$$\varepsilon_t = \mu + \varphi \varepsilon_{t-1} + u_t \rightarrow$$

$$Cov(\varepsilon_t, \varepsilon_{t-k}) = Cov(\mu + \varphi\varepsilon_{t-1} + u_t, \varepsilon_{t-k})$$

$$\Rightarrow \gamma_k = \varphi\, Cov(\varepsilon_{t-1}, \varepsilon_{t-k}) + Cov(u_t, \varepsilon_{t-k})$$

$$\Rightarrow \gamma_k = \varphi \gamma_{k-1} + 0$$

$$\Rightarrow \quad \gamma_k = \phi \gamma_{k-1}$$

$$\gamma_1 = \phi \gamma_0 = \phi \frac{\sigma^2}{1-\phi^2}$$

$$\gamma_2 = \phi \gamma_1 = \phi^2 \gamma_0$$

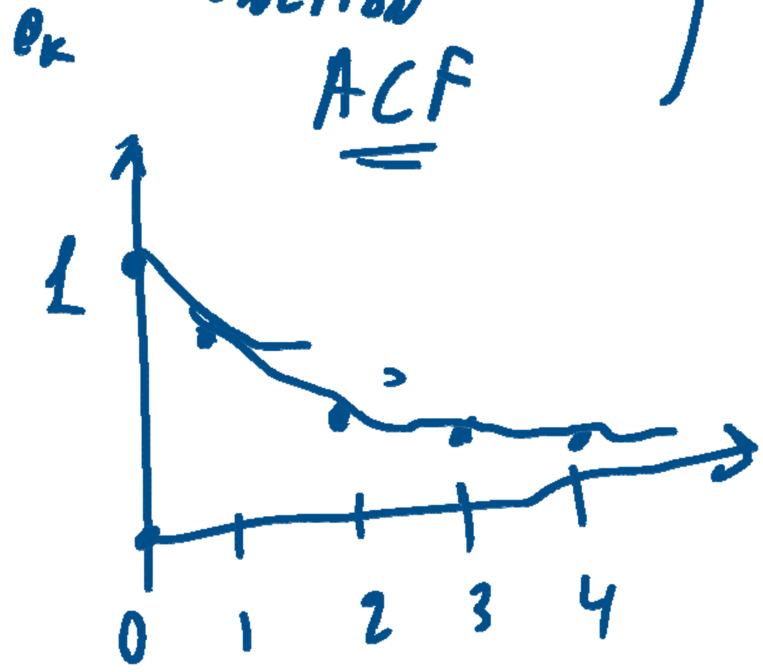
$$\vdots$$

$$\gamma_k = \phi^k \gamma_0$$

$$\rho_k = \frac{\gamma_k}{\gamma_0} = \phi^k$$

(AUTOCORRELATION FUNCTION)

ACF



$\rho_k^{\bullet}$ = PARTIAL AUTOCOR. OF ORDER k

= AUTOCOR. ΤΟΥ ε_t και ε_{t-k} ΑΦΟΥ λάβω υπ' όψιν μου ΤΑ AUTOCOR. ΤΟΥ $(\varepsilon_t, \varepsilon_{t-1}), (\varepsilon_t, \varepsilon_{t-2}), \dots, (\varepsilon_t, \varepsilon_{t-k+1})$

$$\varepsilon_t = c^* + \rho_k \varepsilon_{t-k} + v_t$$

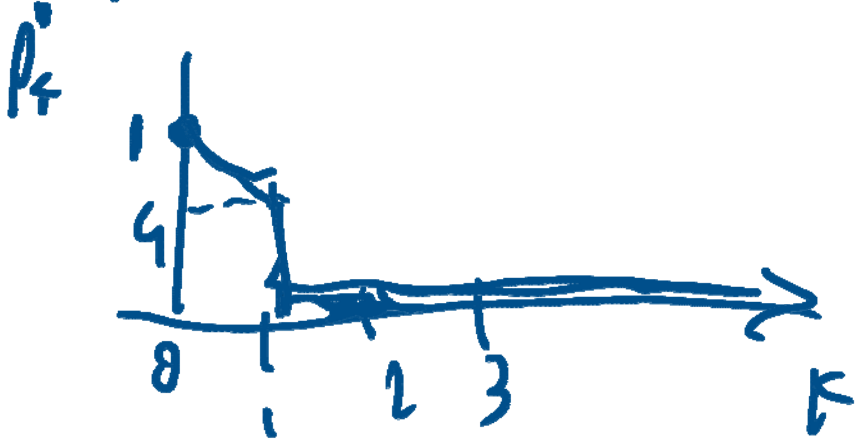
$$\varepsilon_t = c^{**} + \delta_1 \varepsilon_{t-1} + \delta_2 \varepsilon_{t-2} + \dots + \delta_{k-1} \varepsilon_{t-k+1} + \rho_k^{\bullet} \varepsilon_{t-k} + \hat{v}_t$$

$$\rho_1^* = \rho_1 = \varphi$$

$$\varepsilon_t = \mu + \varphi\varepsilon_{t-1} + \rho_2^*\varepsilon_{t-2} + u_t$$

ε_t όσον $EX①$ $\varepsilon_t = \mu + \varphi\varepsilon_{t-1} + u_t$ $\Rightarrow$

$$\Rightarrow \rho_2^* = 0 = \rho_3^* = \rho_4^* = \ldots$$



ΕΣΤΩ ΤΩΡΑ ΟΤΙ ΤΑ ΛΑΘΗ ε_t ΑΚΟΛΟΥΘΟΥΝ AR(1)

$$\begin{cases} \underset{1\times 1}{y_t} = \underset{1\times k}{x_t}\,\underset{k\times 1}{\beta} + \varepsilon_t \\ \varepsilon_t = \varphi\,\varepsilon_{t-1} + u_t, \quad u_t \overset{iid}{\sim} (0, \sigma^2) \end{cases}$$

$$\underset{n\times 1}{Y} = \underset{n\times k}{X}\,\underset{k\times 1}{\beta} + \underset{n\times 1}{\varepsilon} \qquad E(\varepsilon) = 0$$

$$V(\varepsilon) = E(\varepsilon\varepsilon')$$

$$= E\begin{pmatrix} \varepsilon_1^2 & \varepsilon_1\varepsilon_2 & \varepsilon_1\varepsilon_3 & \cdots & \varepsilon_1\varepsilon_n \\ \varepsilon_2\varepsilon_1 & \varepsilon_2^2 & \varepsilon_2\varepsilon_3 & \cdots & \varepsilon_2\varepsilon_n \\ \vdots & & & & \\ \varepsilon_n\varepsilon_1 & \varepsilon_n\varepsilon_2 & \cdots & \cdots & \varepsilon_n^2 \end{pmatrix} = \begin{pmatrix} \gamma_0 & \gamma_1 & \gamma_2 & \cdots & \gamma_{n-1} \\ \gamma_1 & \gamma_0 & \gamma_1 & \cdots & \gamma_{n-2} \\ \vdots & & & & \\ \gamma_{n-1} & \cdots & & \cdots & \gamma_0 \end{pmatrix} =$$

$$= \gamma_0 \begin{pmatrix} 1 & \rho_1 & \rho_2 & \cdots & \rho_{n-1} \\ \rho_1 & 1 & \rho_1 & & \rho_{n-2} \\ \vdots & & & & \\ \rho_{n-1} & \rho_{n-2} & \cdots & \cdots & 1 \end{pmatrix} =$$

$$= \frac{\sigma^2}{1-\phi^2} \begin{pmatrix} 1 & \phi & \phi^2 & \cdots & \phi^{n-1} \\ \phi & 1 & \phi & \cdots & \phi^{n-2} \\ \vdots & & & & \\ \phi^{n-1} & \phi^{n-2} & \cdots & \cdots & 1 \end{pmatrix} = \sigma^{*2} \Omega$$

$V(\varepsilon) \neq \sigma^2 I_n \Rightarrow \hat{\beta} = (X'X)^{-1}X'Y$ is UNBIASED BUT INEFFICIENT.

Η ΛΥΣΗ ΕΙΝΑΙ ΕΚΤΙΜΗΣΗ ΜΕ MAXIMUM LIKELIHOOD

ΕΣΤΩ

(1) $y_t = x_t' b + \varepsilon_t$

(2) $\varepsilon_t = \varphi \varepsilon_{t-1} + u_t$ ΚΑΙ $u_t \overset{iid}{\sim} N(0, \sigma^2)$

ΑΝΤΙΚΑΘΙΣΤΩΝΤΑΣ ΤΟ ε_t ΣΤΗΝ (1) ΕΧΟΥΜΕ

$y_t = x_t b + \varphi \varepsilon_{t-1} + u_t$ ΚΑΙ ΑΠΟ (1) $\varepsilon_{t-1} = y_{t-1} - x_{t-1}' b$

$\Rightarrow y_t = x_t' b + \varphi y_{t-1} - \varphi x_t' b + u_t$

$\Rightarrow$ (3) $y_t - \varphi y_{t-1} = (x_t' - \varphi x_{t-1}') b + u_t$

$$y_t = x_t' b + \varphi y_{t-1} - \varphi x_{t-1}' b + u_t$$

$$u_t \overset{iid}{\sim} N(0, \sigma^2)$$

$$I_t = \{x_{t+1}, y_t, x_t, y_{t-1}, x_{t-1}, y_{t-2}, x_{t-2}, \dots\}$$

$$E(y_t \mid I_{t-1}) = x_t' b + \varphi y_{t-1} - \varphi x_{t-1}' b$$

$$V(y_t \mid I_{t-1}) = V(u_t) = \sigma^2$$

$$\ell(y_1, y_2, \dots, y_n, b, \sigma^2, \varphi) =$$

$$= \sum_{t=2}^{n} \ell(y_t \mid I_{t-1}) = \sum_{t=2}^{n} -\frac{1}{2}\ln 2\pi - \frac{1}{2}\ln \sigma^2 - \frac{1}{2\sigma^2}(y_t - x_t' b - \varphi y_{t-1} + \varphi x_{t-1}' b)^2$$

ALTERNATIVELY WE COULD ASSUME an MA Model.

$$\varepsilon_t = \mu + u_t - \vartheta u_{t-1} \qquad MA(1)$$

$$E(\varepsilon_t) = \mu + 0 - \vartheta \cdot 0 = \mu \qquad \forall \vartheta$$

$$V(\varepsilon_t) = V(\mu + u_t - \vartheta u_{t-1}) = V(u_t) + \vartheta^2 V(u_{t-1})$$
$$- 2\vartheta Cov(u_t, u_{t-1}) = \sigma^2 + \sigma^2\vartheta^2 - 0 \Rightarrow$$

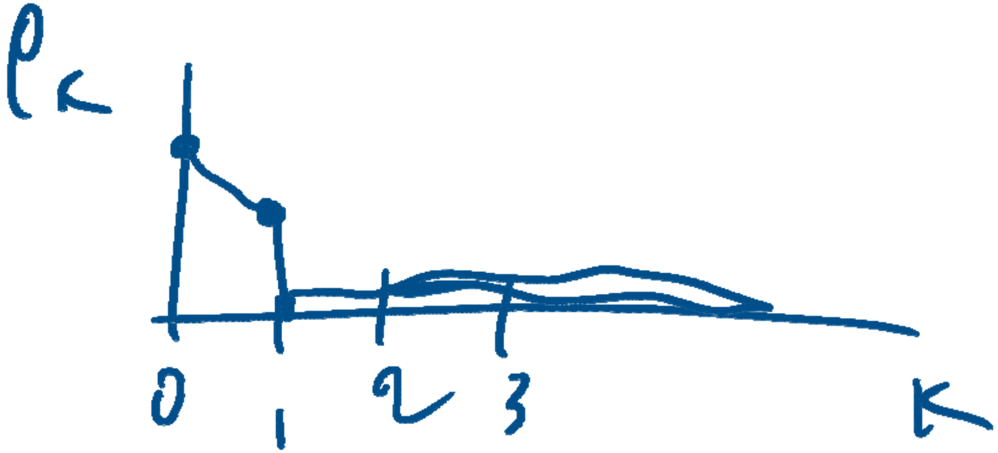
$$\Rightarrow V(\varepsilon_t) = \sigma^2(1+\vartheta^2) \qquad \forall \vartheta,$$

$\Rightarrow$ MA(1) ΠΑΝΤΑ ΣΤΑΣΙΜΟ ($\forall \vartheta$ is station 2^{nd} order)

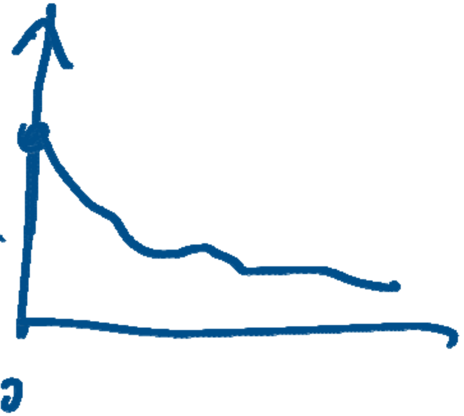
$$Cov(\varepsilon_t, \varepsilon_{t-k}) = \gamma_k = \begin{cases} \dfrac{-\vartheta}{(1+\vartheta)^2 \sigma^2} & k=1 \\ 0 & \forall k>1 \end{cases}$$

p

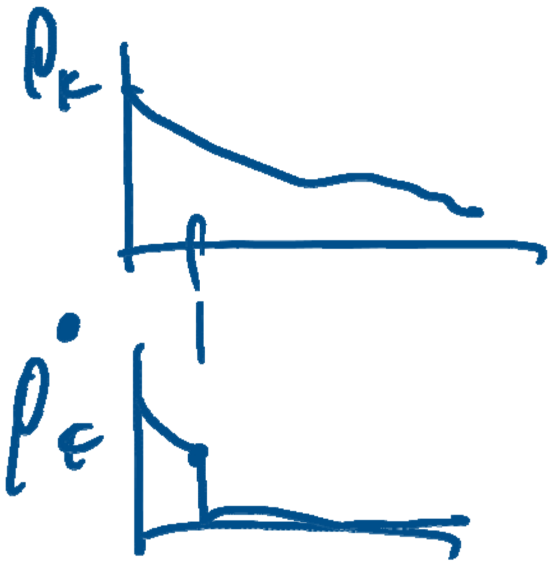
$$\rho_k = \begin{cases} -\dfrac{\theta}{1+\theta^2} & k=1 \\ 0 & k \geq 2 \end{cases}$$



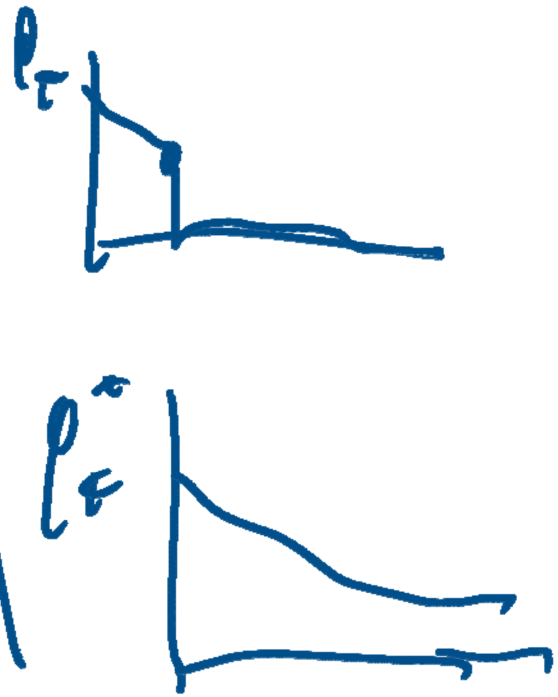
$$\rho_k^* = \theta^k \cdot ct.$$



AR(1)



MA(1)



ARMA(1,1) IS THE MIXED MODEL OF AR AND MA, i.e.

$$\varepsilon_t = \mu + \varphi \varepsilon_{t-1} + \eta_t - \theta \varepsilon_{t-1}$$

IN THIS CASE ρ_k 

and ρ^*_k 

$AR(k)$, $MA(k)$, $ARMA(p,q)$

ESTIMATION VIA MAX. LIKELIHOOD...

TESTING AUTOCOR.

$H_0: \rho_k = 0$ v.s $H_1: \rho_k \neq 0$

$$\hat{\rho}_k \underset{H_0}{\overset{A}{\sim}} N\left(0, \frac{1}{n}\right) \Longleftrightarrow \sqrt{n}\,\hat{\rho}_k \underset{H_0}{\overset{A}{\sim}} N(0,1)$$

Reject H_0 <u>Iff</u>

$$|\sqrt{n}\,\hat{\rho}_k| > 1.96 \quad (5\% \text{ size } 2\text{-sided})$$

$\Rightarrow \quad n\hat{\rho}_k^2 \underset{H_0}{\overset{A}{\sim}} \chi^2_1 \Rightarrow$ Reject H_0 Iff $n\hat{\rho}_k^2 > \chi^2_{1,5\%}$

As now $\hat{\rho}_i$ INDEPENDENT of $\hat{\rho}_j$ for $i \neq j$ we get:

$$Q_k = n(\hat{\rho}_1^2 + \hat{\rho}_2^2 + \dots + \hat{\rho}_k^2) \overset{H_0}{\underset{A}{\sim}} \chi^2_k$$

$H_0: \rho_1 = \rho_2 = \dots = \rho_k = 0$ V.S. H_1:

At least one $\neq 0$. Q_k is the BOX-Ljung test OR PORTMANTEAU test

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$$y_t = x_t' \hat{b} + \hat{\varepsilon}_t$$

$\boxed{\text{LM-TEST}}$

$$\hat{\varepsilon}_t = x_t' \delta + \alpha_1 \hat{\varepsilon}_{t-1} + \alpha_2 \hat{\varepsilon}_{t-2} + \dots + \alpha_k \hat{\varepsilon}_{t-k}$$

$\downarrow R^2$

$$nR^2 \overset{H_0}{\underset{A}{\sim}} \chi^2_k$$

$H_0: \alpha_1 = \alpha_2 = \dots = \alpha_k = 0$

CONDITIONAL HETEROSKEDASTICITY

$$y_t = x_t' b + \varepsilon_t, \quad \varepsilon_t | I_{t-1} \sim N(0, \sigma_t^2)$$

$$\sigma_t^2 = \omega + \alpha \varepsilon_{t-1}^2$$

Autoregressive CONDITIONALLY HETEROSKEDASTIC of order 1

ARCH(1)

$$I_{t-1} = \{ y_{t-1}, x_{t-1}, \ldots \}$$

$$\sim \{ \varepsilon_{t-1}, \varepsilon_{t-2}, \ldots \}$$

$$\sigma_t^2 = \omega + \alpha \varepsilon_{t-1}^2 \Rightarrow \varepsilon_t^2 = \omega + \alpha \varepsilon_{t-1}^2 + \underbrace{\varepsilon_t^2 - \sigma_t^2}_{v_t}$$

$$\Rightarrow \varepsilon_t^2 = \omega + \alpha \varepsilon_{t-1}^2 + v_t$$

AR(1) 6Tq ε_t^2 <u>ANN</u>

$$E(v_t) = 0$$

$$E(v_t v_{t-k}) = 0$$

$$E(\varepsilon_{t-1}^2 v_t) = 0$$

$$E(v_t) = E\left[E(v_t \mid I_{t-1})\right] =$$

$$= E\left[E(\varepsilon_t^2 - \sigma_t^2) \mid I_{t-1}\right] =$$

$$= E\left[E(\varepsilon_t^2 \mid I_{t-1}) - \sigma_t^2\right] = E(\sigma_t^2 - \sigma_t^2) = 0$$

For $|\alpha| < 1 \Rightarrow E(\varepsilon_t^2) = \omega + \alpha E(\varepsilon_{t-1}^2) + E(v_t)$

$\Rightarrow E(\varepsilon_t^2)(1-\alpha) = \omega \Rightarrow E(\varepsilon_t^2) = \frac{\omega}{1-\alpha}$

$\sigma_t^2 = \omega + \alpha \varepsilon_{t-1}^2$

POSITIVITY CONSTRAINTS.

CONSTRAINTS ON THE Parameters of the COND. VAR EQUATION such that

$P(\sigma_t^2 > 0) = 1 \Leftrightarrow P(\sigma_t^2 < 0) = 0$

FOR ARCH(1) $\omega > 0$, $\alpha \geq 0$

For σ_t^2 with posit. constr + stationa

$\Rightarrow 0 \leq \alpha < 1$

$$\sigma_t^2 = w + \alpha \varepsilon_{t-1}^2$$

$$\sigma_t^2 = w + \alpha_1 \varepsilon_{t-1}^2 + \alpha_2 \varepsilon_{t-2}^2 + \dots + \alpha_q \varepsilon_{t-q}^2$$

ARCH (q)

$$\sigma_t^2 = w + \alpha \varepsilon_{t-1}^2 + \beta \sigma_{t-1}^2 \quad \text{GARCH}(1,1)$$

$$\Leftrightarrow \varepsilon_t^2 = w + (\alpha + \beta) \varepsilon_{t-1}^2 + v_t - \beta v_{t-1}$$

ARMA (1, 1)

ε_t^2 stationary iff $(\alpha+\beta) < 1$

$$\mathrm{Corr}\left(\varepsilon_t^2, \varepsilon_{t-k}^2\right) = (\alpha+\beta)^{k-1} \rho_1$$

$$\rho_1 = \mathrm{Corr}\left(\varepsilon_t^2, \varepsilon_{t-1}^2\right)$$

$\alpha + \beta =$ PERSISTANCE

POSITIVITY CONSTRAINTS

$\omega > 0, \quad \alpha \geq 0, \quad \beta \geq 0$

GARCH (2,1)

$$\sigma_t^2 = \omega + \beta_1 \sigma_{t-1}^2 + \beta_2 \sigma_{t-2}^2 + \alpha \varepsilon_{t-1}^2$$

GARCH (1,2)

$$\sigma_t^2 = \omega + \beta \sigma_{t-1}^2 + \alpha_1 \varepsilon_{t-1}^2 + \alpha_2 \varepsilon_{t-2}^2$$

GARCH (1,1)

POSIT. CONST + station $\Rightarrow$

$\omega > 0$

$\alpha \geq 0 \quad 0 \leq \alpha + \beta < 1$

$\beta \geq 0$