PORTFOLIO MANAGEMENT

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INVESTMENTS

BODIE ◦ KANE ◦ MARCUS

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PART II

CHAPTER SEVEN

7

OPTIMAL RISKY PORTFOLIOS

THE INVESTMENT DECISION can be viewed as a top-down process: (i) *Capital allocation* between the risky portfolio and risk-free assets, (ii) *asset allocation* across broad asset classes (e.g., U.S. stocks, international stocks, and long-term bonds), and (iii) *security selection* of individual assets within each asset class.

Capital allocation, as we saw in Chapter 6, determines the investor's exposure to risk. The optimal capital allocation is determined by risk aversion as well as expectations for the risk–return trade-off of the optimal risky portfolio. In principle, asset allocation and security selection are technically identical; both aim at identifying that optimal risky portfolio, namely, the combination of risky assets that provides the best risk–return trade-off. In practice, however, asset allocation and security selection are typically separated into two steps, in which the broad outlines of the portfolio are established first (asset allocation), while details concerning specific securities are filled in later (security selection). After we show how the optimal risky portfolio may be constructed, we will consider the cost and benefits of pursuing this two-step approach.

We first motivate the discussion by illustrating the potential gains from simple diversification into many assets. We then proceed to examine the process of *efficient* diversification from the ground up, starting with an investment menu of only two risky assets, then adding the risk-free asset, and finally, incorporating the entire universe of available risky securities. We learn how diversification can reduce risk without affecting expected returns. This accomplished, we re-examine the hierarchy of capital allocation, asset allocation, and security selection. Finally, we offer insight into the power of diversification by drawing an analogy between it and the workings of the insurance industry.

The portfolios we discuss in this and the following chapters are of a short-term horizon—even if the overall investment horizon is long, portfolio composition can be rebalanced or updated almost continuously. For these short horizons, the skewness that characterizes long-term compounded returns is absent. Therefore, the assumption of normality is sufficiently accurate to describe holding-period returns, and we will be concerned only with portfolio means and variances.

In Appendix A, we demonstrate how construction of the optimal risky portfolio can easily be accomplished with Excel. Appendix B provides a review of portfolio statistics with emphasis on the intuition behind covariance and correlation measures. Even if you have had a good quantitative methods course, it may well be worth skimming.

7.1 DIVERSIFICATION AND PORTFOLIO RISK

Suppose your portfolio is composed of only one stock, say, Dell Computer Corporation. What would be the sources of risk to this "portfolio"? You might think of two broad sources of uncertainty. First, there is the risk that comes from conditions in the general economy, such as the business cycle, inflation, interest rates, and exchange rates. None of these macroeconomic factors can be predicted with certainty, and all affect the rate of return on Dell stock. In addition to these macroeconomic factors there are firm-specific influences, such as Dell's success in research and development, and personnel changes. These factors affect Dell without noticeably affecting other firms in the economy.

Now consider a naive **diversification** strategy, in which you include additional securities in your portfolio. For example, place half your funds in ExxonMobil and half in Dell. What should happen to portfolio risk? To the extent that the firm-specific influences on the two stocks differ, diversification should reduce portfolio risk. For example, when oil prices fall, hurting ExxonMobil, computer prices might rise, helping Dell. The two effects are offsetting and stabilize portfolio return.

But why end diversification at only two stocks? If we diversify into many more securities, we continue to spread out our exposure to firm-specific factors, and portfolio volatility should continue to fall. Ultimately, however, even with a large number of stocks we cannot avoid risk altogether, because virtually all securities are affected by the common macroeconomic factors. For example, if all stocks are affected by the business cycle, we cannot avoid exposure to business cycle risk no matter how many stocks we hold.

When all risk is firm-specific, as in Figure 7.1, panel A, diversification can reduce risk to arbitrarily low levels. The reason is that with all risk sources independent, the exposure to any particular source of risk is reduced to a negligible level. The reduction of risk to very low levels in the case of independent risk sources is sometimes called the **insurance principle,** because of the notion that an insurance company depends on the risk reduction achieved through diversification when it writes many policies insuring against many independent sources of risk, each policy being a small part of the company's overall portfolio. (See Section 7.5 for a discussion of the insurance principle.)

When common sources of risk affect all firms, however, even extensive diversification cannot eliminate risk. In Figure 7.1, panel B, portfolio standard deviation falls as the number of securities increases, but it cannot be reduced to zero. The risk that remains even after extensive diversification is called **market risk,** risk that is attributable to marketwide risk sources. Such risk is also called **systematic risk,** or **nondiversifiable risk.** In contrast, the risk that *can* be eliminated by diversification is called **unique risk, firm-specific risk, nonsystematic risk,** or **diversifiable risk.**

This analysis is borne out by empirical studies. Figure 7.2 shows the effect of portfolio diversification, using data on NYSE stocks.[1] The figure shows the average standard

[1]Meir Statman, "How Many Stocks Make a Diversified Portfolio?" *Journal of Financial and Quantitative Analysis* 22 (September 1987).

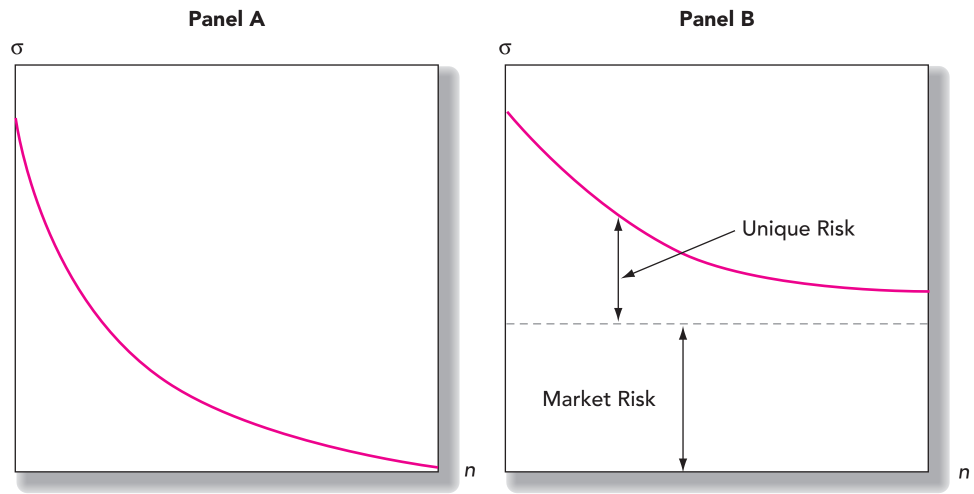


FIGURE 7.1 Portfolio risk as a function of the number of stocks in the portfolio

deviation of equally weighted portfolios constructed by selecting stocks at random as a function of the number of stocks in the portfolio. On average, portfolio risk does fall with diversification, but the power of diversification to reduce risk is limited by systematic or common sources of risk.

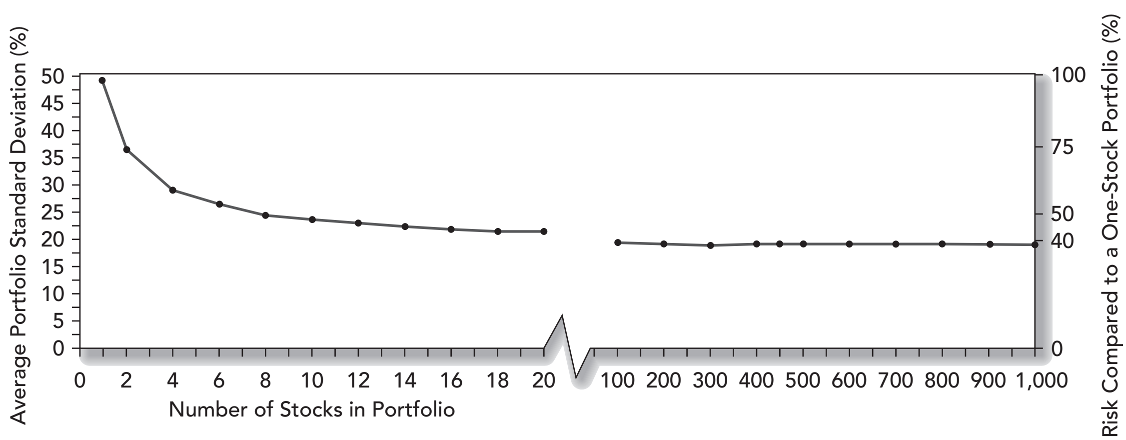


FIGURE 7.2 Portfolio diversification. The average standard deviation of returns of portfolios composed of only one stock was 49.2%. The average portfolio risk fell rapidly as the number of stocks included in the portfolio increased. In the limit, portfolio risk could be reduced to only 19.2%.

Source: From Meir Statman, "How Many Stocks Make a Diversified Portfolio? *Journal of Financial and Quantitative Analysis* 22 (September 1987). Reprinted by permission.

7.2 PORTFOLIOS OF TWO RISKY ASSETS

In the last section we considered naive diversification using equally weighted portfolios of several securities. It is time now to study *efficient* diversification, whereby we construct risky portfolios to provide the lowest possible risk for any given level of expected return. The nearby box provides an introduction to the relationship between diversification and portfolio construction.

Portfolios of two risky assets are relatively easy to analyze, and they illustrate the principles and considerations that apply to portfolios of many assets. It makes sense to think about a two-asset portfolio as an asset allocation decision, and so we consider two mutual funds, a bond portfolio specializing in long-term debt securities, denoted D, and a stock fund that specializes in equity securities, E. Table 7.1 lists the parameters describing the rate-of-return distribution of these funds.

A proportion denoted by w_D is invested in the bond fund, and the remainder, $1 - w_D$, denoted w_E, is invested in the stock fund. The rate of return on this portfolio, r_p, will be[2]

$$r_p = w_D r_D + w_E r_E \tag{7.1}$$

where r_D is the rate of return on the debt fund and r_E is the rate of return on the equity fund.

The expected return on the portfolio is a weighted average of expected returns on the component securities with portfolio proportions as weights:

$$E(r_p) = w_D E(r_D) + w_E E(r_E) \tag{7.2}$$

The variance of the two-asset portfolio is

$$\sigma_p^2 = w_D^2\sigma_D^2 + w_E^2\sigma_E^2 + 2w_D w_E \operatorname{Cov}(r_D, r_E) \tag{7.3}$$

Our first observation is that the variance of the portfolio, unlike the expected return, is *not* a weighted average of the individual asset variances. To understand the formula for the portfolio variance more clearly, recall that the covariance of a variable with itself is the variance of that variable; that is

$$\begin{aligned}\operatorname{Cov}(r_D, r_D) &= \sum_{\text{scenarios}} \Pr(\text{scenario})[r_D - E(r_D)][r_D - E(r_D)] \\ &= \sum_{\text{scenarios}} \Pr(\text{scenario})[r_D - E(r_D)]^2 \\ &= \sigma_D^2\end{aligned} \tag{7.4}$$

Therefore, another way to write the variance of the portfolio is

$$\sigma_p^2 = w_D w_D \operatorname{Cov}(r_D, r_D) + w_E w_E \operatorname{Cov}(r_E, r_E) + 2w_D w_E \operatorname{Cov}(r_D, r_E) \tag{7.5}$$

TABLE 7.1

Descriptive statistics for two mutual funds

	Debt	Equity
Expected return, $E(r)$	8%	13%
Standard deviation, σ	12%	20%
Covariance, $\operatorname{Cov}(r_D, r_E)$	72	
Correlation coefficient, ρ_{DE}	.30	

[2]See Appendix B of this chapter for a review of portfolio statistics.

INTRODUCTION TO DIVERSIFICATION

Diversification is a familiar term to most investors. In the most general sense, it can be summed up with this phrase: "Don't put all of your eggs in one basket." While that sentiment certainly captures the essence of the issue, it provides little guidance on the practical implications of the role diversification plays in an investor's portfolio and offers no insight into how a diversified portfolio is actually created.

WHAT IS DIVERSIFICATION?

Taking a closer look at the concept of diversification, the idea is to create a portfolio that includes multiple investments in order to reduce risk. Consider, for example, an investment that consists of only the stock issued by a single company. If that company's stock suffers a serious downturn, your portfolio will sustain the full brunt of the decline. By splitting your investment between the stocks of two different companies, you reduce the potential risk to your portfolio.

Another way to reduce the risk in your portfolio is to include bonds and cash. Because cash is generally used as a short-term reserve, most investors develop an asset allocation strategy for their portfolios based primarily on the use of stocks and bonds. It is never a bad idea to keep a portion of your invested assets in cash, or short-term money-market securities. Cash can be used in case of an emergency, and short-term money-market securities can be liquidated instantly in the event your usual cash requirements spike and you need to sell investments to make payments.

Regardless of whether you are aggressive or conservative, the use of asset allocation to reduce risk through the selection of a balance of stocks and bonds for your portfolio is a more detailed description of how a diversified portfolio is created than the simplistic eggs in one basket concept. The specific balance of stocks and bonds in a given portfolio is designed to create a specific risk-reward ratio that offers the opportunity to achieve a certain rate of return on your investment in exchange for your willingness to accept a certain amount of risk.

WHAT ARE MY OPTIONS?

If you are a person of limited means or you simply prefer uncomplicated investment scenarios, you could choose a single balanced mutual fund and invest all of your assets in the fund. For most investors, this strategy is far too simplistic. Furthermore, while investing in a single mutual fund provides diversification among the basic asset classes of stocks, bonds and cash, the opportunities for diversification go far beyond these basic categories. A host of alternative investments provide the opportunity for further diversification. Real estate investment trusts, hedge funds, art and other investments provide the opportunity to invest in vehicles that do not necessarily move in tandem with the traditional financial markets.

CONCLUSION

Regardless of your means or method, keep in mind that there is no generic diversification model that will meet the needs of every investor. Your personal time horizon, risk tolerance, investment goals, financial means and level of investment experience will play a large role in dictating your investment mix.

Source: Adapted from Jim McWhinney, *Introduction to Diversification*, December 16, 2005, **www.investopedia.com/articles/basics/05/diversification.asp,** retrieved April 25, 2006.

In words, the variance of the portfolio is a weighted sum of covariances, and each weight is the product of the portfolio proportions of the pair of assets in the covariance term.

Table 7.2 shows how portfolio variance can be calculated from a spreadsheet. Panel A of the table shows the *bordered* covariance matrix of the returns of the two mutual funds. The bordered matrix is the covariance matrix with the portfolio weights for each fund placed on the borders, that is, along the first row and column. To find portfolio variance, multiply each element in the covariance matrix by the pair of portfolio weights in its row and column borders. Add up the resultant terms, and you have the formula for portfolio variance given in Equation 7.5.

We perform these calculations in panel B, which is the *border-multiplied* covariance matrix: Each covariance has been multiplied by the weights from the row and the column in the borders. The bottom line of panel B confirms that the sum of all the terms in this matrix (which we obtain by adding up the column sums) is indeed the portfolio variance in Equation 7.5.

This procedure works because the covariance matrix is symmetric around the diagonal, that is, $\text{Cov}(r_D, r_E) = \text{Cov}(r_E, r_D)$. Thus each covariance term appears twice.

TABLE 7.2

Computation of portfolio variance from the covariance matrix

A. Bordered Covariance Matrix		
Portfolio Weights	w_D	w_E
w_D	$\text{Cov}(r_D, r_D)$	$\text{Cov}(r_D, r_E)$
w_E	$\text{Cov}(r_E, r_D)$	$\text{Cov}(r_E, r_E)$
B. Border-multiplied Covariance Matrix		
Portfolio Weights	w_D	w_E
w_D	$w_D w_D \text{Cov}(r_D, r_D)$	$w_D w_E \text{Cov}(r_D, r_E)$
w_E	$w_E w_D \text{Cov}(r_E, r_D)$	$w_E w_E \text{Cov}(r_E, r_E)$
$w_D + w_E = 1$	$w_D w_D \text{Cov}(r_D, r_D) + w_E w_D \text{Cov}(r_E, r_D)$	$w_D w_E \text{Cov}(r_D, r_E) + w_E w_E \text{Cov}(r_E, r_E)$
Portfolio variance	$w_D w_D \text{Cov}(r_D, r_D) + w_E w_D \text{Cov}(r_E, r_D) + w_D w_E \text{Cov}(r_D, r_E) + w_E w_E \text{Cov}(r_E, r_E)$	

This technique for computing the variance from the border-multiplied covariance matrix is general; it applies to any number of assets and is easily implemented on a spreadsheet. Concept Check 1 asks you to try the rule for a three-asset portfolio. Use this problem to verify that you are comfortable with this concept.

CONCEPT CHECK 1

a. First confirm for yourself that our simple rule for computing the variance of a two-asset portfolio from the bordered covariance matrix is consistent with Equation 7.3.

b. Now consider a portfolio of three funds, *X, Y, Z,* with weights w_X, w_Y, and w_Z. Show that the portfolio variance is

$$w_X^2\sigma_X^2 + w_Y^2\sigma_Y^2 + w_Z^2\sigma_Z^2 + 2w_Xw_Y\text{Cov}(r_X, r_Y)$$
$$+2w_Xw_Z\text{Cov}(r_X, r_Z) + 2w_Yw_Z\text{Cov}(r_Y, r_Z)$$

Equation 7.3 reveals that variance is reduced if the covariance term is negative. It is important to recognize that even if the covariance term is positive, the portfolio standard deviation *still* is less than the weighted average of the individual security standard deviations, unless the two securities are perfectly positively correlated.

To see this, notice that the covariance can be computed from the correlation coefficient, ρ_{DE}, as

$$\text{Cov}(r_D, r_E) = \rho_{DE}\sigma_D\sigma_E \quad \textbf{(7.6)}$$

Therefore,

$$\sigma_p^2 = w_D^2\sigma_D^2 + w_E^2\sigma_E^2 + 2w_Dw_E\sigma_D\sigma_E\rho_{DE} \quad \textbf{(7.7)}$$

Other things equal, portfolio variance is higher when ρ_{DE} is higher. In the case of perfect positive correlation, $\rho_{DE} = 1$, the right-hand side of Equation 7.7 is a perfect square and simplifies to

$$\sigma_p^2 = (w_D\sigma_D + w_E\sigma_E)^2 \quad \textbf{(7.8)}$$

or

$$\sigma_p = w_D\sigma_D + w_E\sigma_E \quad \textbf{(7.9)}$$

Therefore, the standard deviation of the portfolio with perfect positive correlation is just the weighted average of the component standard deviations. In all other cases, the correlation coefficient is less than 1, making the portfolio standard deviation *less* than the weighted average of the component standard deviations.

A hedge asset has *negative* correlation with the other assets in the portfolio. Equation 7.7 shows that such assets will be particularly effective in reducing total risk. Moreover, Equation 7.2 shows that expected return is unaffected by correlation between returns. Therefore, other things equal, we will always prefer to add to our portfolios assets with low or, even better, negative correlation with our existing position.

Because the portfolio's expected return is the weighted average of its component expected returns, whereas its standard deviation is less than the weighted average of the component standard deviations, *portfolios of less than perfectly correlated assets always offer better risk–return opportunities than the individual component securities on their own.* The lower the correlation between the assets, the greater the gain in efficiency.

How low can portfolio standard deviation be? The lowest possible value of the correlation coefficient is −1, representing perfect negative correlation. In this case, Equation 7.7 simplifies to

$$\sigma_p^2 = (w_D\sigma_D - w_E\sigma_E)^2 \qquad \textbf{(7.10)}$$

and the portfolio standard deviation is

$$\sigma_p = \text{Absolute value } (w_D\sigma_D - w_E\sigma_E) \qquad \textbf{(7.11)}$$

When $\rho = -1$, a perfectly hedged position can be obtained by choosing the portfolio proportions to solve

$$w_D\sigma_D - w_E\sigma_E = 0$$

The solution to this equation is

$$\begin{aligned} w_D &= \frac{\sigma_E}{\sigma_D + \sigma_E} \\ w_E &= \frac{\sigma_D}{\sigma_D + \sigma_E} = 1 - w_D \end{aligned} \qquad \textbf{(7.12)}$$

These weights drive the standard deviation of the portfolio to zero.

EXAMPLE 7.1 Portfolio Risk and Return

Let us apply this analysis to the data of the bond and stock funds as presented in Table 7.1. Using these data, the formulas for the expected return, variance, and standard deviation of the portfolio as a function of the portfolio weights are

$$\begin{aligned} E(r_p) &= 8w_D + 13w_E \\ \sigma_p^2 &= 12^2 w_D^2 + 20^2 w_E^2 + 2 \times 12 \times 20 \times .3 \times w_D w_E \\ &= 144w_D^2 + 400w_E^2 + 144w_D w_E \\ \sigma_p &= \sqrt{\sigma_p^2} \end{aligned}$$

We can experiment with different portfolio proportions to observe the effect on portfolio expected return and variance. Suppose we change the proportion invested in bonds. The effect on expected return is tabulated in Table 7.3 and plotted in Figure 7.3. When the proportion invested in debt varies from zero to 1 (so that the proportion in equity varies from

TABLE 7.3

Expected return and standard deviation with various correlation coefficients

w_D	w_E	$E(r_P)$	Portfolio Standard Deviation for Given Correlation: $\rho = -1$	$\rho = 0$	$\rho = .30$	$\rho = 1$
0.00	1.00	13.00	20.00	20.00	20.00	20.00
0.10	0.90	12.50	16.80	18.04	18.40	19.20
0.20	0.80	12.00	13.60	16.18	16.88	18.40
0.30	0.70	11.50	10.40	14.46	15.47	17.60
0.40	0.60	11.00	7.20	12.92	14.20	16.80
0.50	0.50	10.50	4.00	11.66	13.11	16.00
0.60	0.40	10.00	0.80	10.76	12.26	15.20
0.70	0.30	9.50	2.40	10.32	11.70	14.40
0.80	0.20	9.00	5.60	10.40	11.45	13.60
0.90	0.10	8.50	8.80	10.98	11.56	12.80
1.00	0.00	8.00	12.00	12.00	12.00	12.00
			Minimum Variance Portfolio			
		w_D	0.6250	0.7353	0.8200	—
		w_E	0.3750	0.2647	0.1800	—
		$E(r_P)$	9.8750	9.3235	8.9000	—
		σ_P	0.0000	10.2899	11.4473	—

1 to zero), the portfolio expected return goes from 13% (the stock fund's expected return) to 8% (the expected return on bonds).

What happens when $w_D > 1$ and $w_E < 0$? In this case portfolio strategy would be to sell the equity fund short and invest the proceeds of the short sale in the debt fund. This will decrease

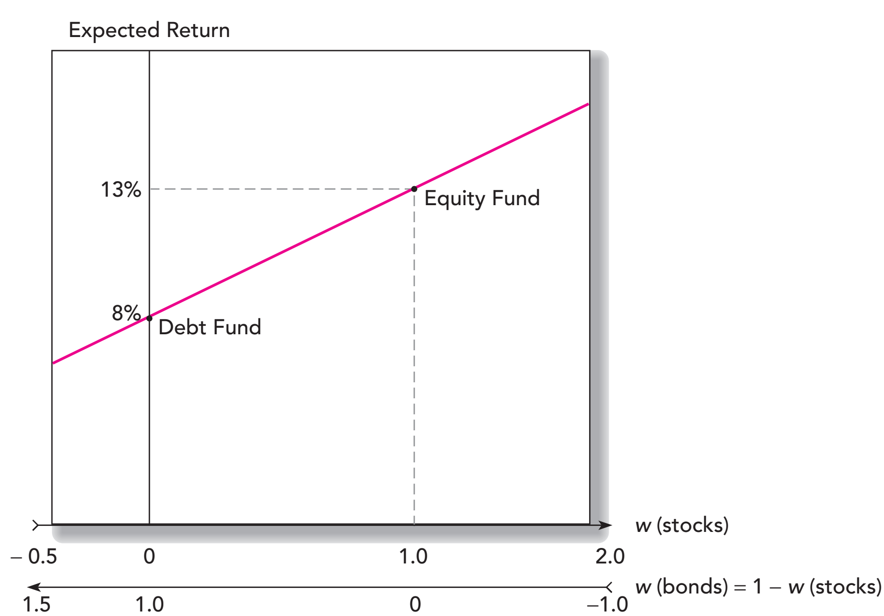


FIGURE 7.3 Portfolio expected return as a function of investment proportions

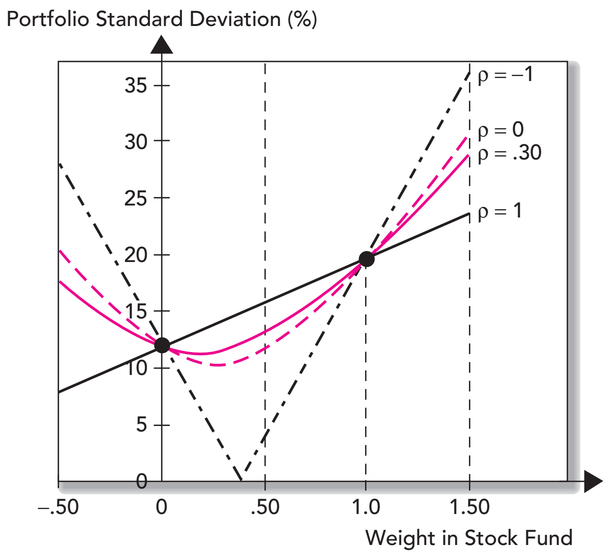


FIGURE 7.4 Portfolio standard deviation as a function of investment proportions

the expected return of the portfolio. For example, when $w_D = 2$ and $w_E = -1$, expected portfolio return falls to $2 \times 8 + (-1) \times 13 = 3\%$. At this point the value of the bond fund in the portfolio is twice the net worth of the account. This extreme position is financed in part by short-selling stocks equal in value to the portfolio's net worth.

The reverse happens when $w_D < 0$ and $w_E > 1$. This strategy calls for selling the bond fund short and using the proceeds to finance additional purchases of the equity fund.

Of course, varying investment proportions also has an effect on portfolio standard deviation. Table 7.3 presents portfolio standard deviations for different portfolio weights calculated from Equation 7.7 using the assumed value of the correlation coefficient, .30, as well as other values of ρ. Figure 7.4 shows the relationship between standard deviation and portfolio weights. Look first at the solid curve for $\rho_{DE} = .30$. The graph shows that as the portfolio weight in the equity fund increases from zero to 1, portfolio standard deviation first falls with the initial diversification from bonds into stocks, but then rises again as the portfolio becomes heavily concentrated in stocks, and again is undiversified. This pattern will generally hold as long as the correlation coefficient between the funds is not too high.[3] For a pair of assets with a large positive correlation of returns, the portfolio standard deviation will increase monotonically from the low-risk asset to the high-risk asset. Even in this case, however, there is a positive (if small) value from diversification.

What is the minimum level to which portfolio standard deviation can be held? For the parameter values stipulated in Table 7.1, the portfolio weights that solve this minimization problem turn out to be[4]

$$w_{\text{Min}}(D) = .82$$

$$w_{\text{Min}}(E) = 1 - .82 = .18$$

This minimum-variance portfolio has a standard deviation of

$$\sigma_{\text{Min}} = [(.82^2 \times 12^2) + (.18^2 \times 20^2) + (2 \times .82 \times .18 \times 72)]^{1/2} = 11.45\%$$

as indicated in the last line of Table 7.3 for the column $\rho = .30$.

The solid colored line in Figure 7.4 plots the portfolio standard deviation when $\rho = .30$ as a function of the investment proportions. It passes through the two undiversified portfolios

[3]As long as $\rho < \sigma_D/\sigma_E$, volatility will initially fall when we start with all bonds and begin to move into stocks.

[4]This solution uses the minimization techniques of calculus. Write out the expression for portfolio variance from Equation 7.3, substitute $1 - w_D$ for w_E, differentiate the result with respect to w_D, set the derivative equal to zero, and solve for w_D to obtain

$$w_{\text{Min}}(D) = \frac{\sigma_E^2 - \text{Cov}(r_D, r_E)}{\sigma_D^2 + \sigma_E^2 - 2\text{Cov}(r_D, r_E)}$$

Alternatively, with a spreadsheet program such as Excel, you can obtain an accurate solution by using the Solver to minimize the variance. See Appendix A for an example of a portfolio optimization spreadsheet.

of $w_D = 1$ and $w_E = 1$. Note that the **minimum-variance portfolio** has a standard deviation *smaller than that of either of the individual component assets.* This illustrates the effect of diversification.

The other three lines in Figure 7.4 show how portfolio risk varies for other values of the correlation coefficient, holding the variances of each asset constant. These lines plot the values in the other three columns of Table 7.3.

The solid dark line connecting the undiversified portfolios of all bonds or all stocks, $w_D = 1$ or $w_E = 1$, shows portfolio standard deviation with perfect positive correlation, $\rho = 1$. In this case there is no advantage from diversification, and the portfolio standard deviation is the simple weighted average of the component asset standard deviations.

The dashed colored curve depicts portfolio risk for the case of uncorrelated assets, $\rho = 0$. With lower correlation between the two assets, diversification is more effective and portfolio risk is lower (at least when both assets are held in positive amounts). The minimum portfolio standard deviation when $\rho = 0$ is 10.29% (see Table 7.3), *again lower than the standard deviation of either asset.*

Finally, the triangular broken line illustrates the perfect hedge potential when the two assets are perfectly negatively correlated ($\rho = -1$). In this case the solution for the minimum-variance portfolio is, by Equation 7.12,

$$w_{\text{Min}}(D; \rho = -1) = \frac{\sigma_E}{\sigma_D + \sigma_E} = \frac{20}{12 + 20} = .625$$

$$w_{\text{Min}}(E; \rho = -1) = 1 - .625 = .375$$

and the portfolio variance (and standard deviation) is zero.

We can combine Figures 7.3 and 7.4 to demonstrate the relationship between portfolio risk (standard deviation) and expected return—given the parameters of the available assets. This is done in Figure 7.5. For any pair of investment proportions, w_D, w_E, we read the expected return from Figure 7.3 and the standard deviation from Figure 7.4. The resulting pairs of expected return and standard deviation are tabulated in Table 7.3 and plotted in Figure 7.5.

The solid colored curve in Figure 7.5 shows the **portfolio opportunity set** for $\rho = .30$. We call it the portfolio opportunity set because it shows all combinations of portfolio expected return and standard deviation that can be constructed from the two available assets. The other lines show the portfolio opportunity set for other values of the correlation coefficient. The solid black line connecting the two funds shows that there is no benefit from diversification when the correlation between the two is perfectly positive ($\rho = 1$). The opportunity set is not "pushed" to the northwest. The dashed colored line demonstrates the greater benefit from diversification when the correlation coefficient is lower than .30.

Finally, for $\rho = -1$, the portfolio opportunity set is linear, but now it offers a perfect hedging opportunity and the maximum advantage from diversification.

To summarize, although the expected return of any portfolio is simply the weighted average of the

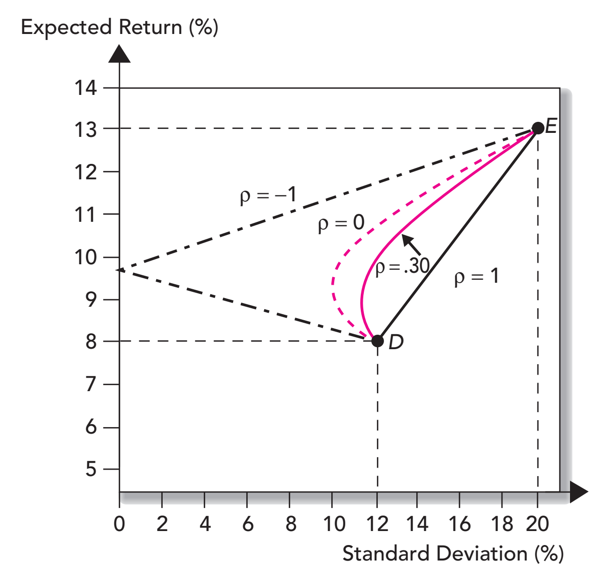


FIGURE 7.5 Portfolio expected return as a function of standard deviation

asset expected returns, this is not true of the standard deviation. Potential benefits from diversification arise when correlation is less than perfectly positive. The lower the correlation, the greater the potential benefit from diversification. In the extreme case of perfect negative correlation, we have a perfect hedging opportunity and can construct a zero-variance portfolio.

Suppose now an investor wishes to select the optimal portfolio from the opportunity set. The best portfolio will depend on risk aversion. Portfolios to the northeast in Figure 7.5 provide higher rates of return but impose greater risk. The best trade-off among these choices is a matter of personal preference. Investors with greater risk aversion will prefer portfolios to the southwest, with lower expected return but lower risk.[5]

CONCEPT CHECK 2

Compute and draw the portfolio opportunity set for the debt and equity funds when the correlation coefficient between them is $\rho = .25$.

7.3 ASSET ALLOCATION WITH STOCKS, BONDS, AND BILLS

In the previous chapter we examined the capital allocation decision, the choice of how much of the portfolio to leave in risk-free money market securities versus in a risky portfolio. Now we have taken a further step, specifying that the risky portfolio comprises a stock and a bond fund. We still need to show how investors can decide on the proportion of their risky portfolios to allocate to the stock versus the bond market. This is an asset allocation decision. As the nearby box emphasizes, most investment professionals recognize that "the really critical decision is how to divvy up your money among stocks, bonds and supersafe investments such as Treasury bills."

In the last section, we derived the properties of portfolios formed by mixing two risky assets. Given this background, we now reintroduce the choice of the third, risk-free, portfolio. This will allow us to complete the basic problem of asset allocation across the three key asset classes: stocks, bonds, and risk-free money market securities. Once you understand this case, it will be easy to see how portfolios of many risky securities might best be constructed.

The Optimal Risky Portfolio with Two Risky Assets and a Risk-Free Asset

What if our risky assets are still confined to the bond and stock funds, but now we can also invest in risk-free T-bills yielding 5%? We start with a graphical solution. Figure 7.6 shows the opportunity set based on the properties of the bond and stock funds, using the data from Table 7.1.

[5]Given a level of risk aversion, one can determine the portfolio that provides the highest level of utility. Recall from Chapter 6 that we were able to describe the utility provided by a portfolio as a function of its expected return, $E(r_p)$, and its variance, σ_p^2, according to the relationship $U = E(r_p) - 0.5A\sigma_p^2$. The portfolio mean and variance are determined by the portfolio weights in the two funds, w_E and w_D, according to Equations 7.2 and 7.3. Using those equations and some calculus, we find the optimal investment proportions in the two funds. A warning: to use the following equation (or any equation involving the risk aversion parameter, A), you must express returns in decimal form.

$$w_D = \frac{E(r_D) - E(r_E) + A(\sigma_E^2 - \sigma_D\sigma_E\rho_{DE})}{A(\sigma_D^2 + \sigma_E^2 - 2\sigma_D\sigma_E\rho_{DE})}$$

$$w_E = 1 - w_D$$

Here, too, Excel's Solver or similar software can be used to maximize utility subject to the constraints of Equations 7.2 and 7.3, plus the portfolio constraint that $w_D + w_E = 1$ (i.e., that portfolio weights sum to 1).

RECIPE FOR SUCCESSFUL INVESTING: FIRST, MIX ASSETS WELL

First things first.

If you want dazzling investment results, don't start your day foraging for hot stocks and stellar mutual funds. Instead, say investment advisers, the really critical decision is how to divvy up your money among stocks, bonds, and supersafe investments such as Treasury bills.

In Wall Street lingo, this mix of investments is called your asset allocation. "The asset-allocation choice is the first and most important decision," says William Droms, a finance professor at Georgetown University. "How much you have in [the stock market] really drives your results."

"You cannot get [stock market] returns from a bond portfolio, no matter how good your security selection is or how good the bond managers you use," says William John Mikus, a managing director of Financial Design, a Los Angeles investment adviser.

For proof, Mr. Mikus cites studies such as the 1991 analysis done by Gary Brinson, Brian Singer and Gilbert Beebower. That study, which looked at the 10-year results for 82 large pension plans, found that a plan's asset-allocation policy explained 91.5% of the return earned.

DESIGNING A PORTFOLIO

Because your asset mix is so important, some mutual fund companies now offer free services to help investors design their portfolios.

Gerald Perritt, editor of the *Mutual Fund Letter,* a Chicago newsletter, says you should vary your mix of assets depending on how long you plan to invest. The further away your investment horizon, the more you should have in stocks. The closer you get, the more you should lean toward bonds and money-market instruments, such as Treasury bills. Bonds and money-market instruments may generate lower returns than stocks. But for those who need money in the near future, conservative investments make more sense, because there's less chance of suffering a devastating short-term loss.

SUMMARIZING YOUR ASSETS

"One of the most important things people can do is summarize all their assets on one piece of paper and figure out their asset allocation," says Mr. Pond.

Once you've settled on a mix of stocks and bonds, you should seek to maintain the target percentages, says Mr. Pond. To do that, he advises figuring out your asset allocation once every six months. Because of a stock-market plunge, you could find that stocks are now a far smaller part of your portfolio than you envisaged. At such a time, you should put more into stocks and lighten up on bonds.

When devising portfolios, some investment advisers consider gold and real estate in addition to the usual trio of stocks, bonds and money-market instruments. Gold and real estate give "you a hedge against hyper-inflation," says Mr. Droms. "But real estate is better than gold, because you'll get better long-run returns."

Source: Jonathan Clements, "Recipe for Successful Investing: First, Mix Assets Well," *The Wall Street Journal,* October 6, 1993. Reprinted by permission of *The Wall Street Journal,*

Two possible capital allocation lines (CALs) are drawn from the risk-free rate ($r_f = 5\%$) to two feasible portfolios. The first possible CAL is drawn through the minimum-variance portfolio *A*, which is invested 82% in bonds and 18% in stocks (Table 7.3, bottom panel, last column). Portfolio *A*'s expected return is 8.90%, and its standard deviation is 11.45%. With a T-bill rate of 5%, the **reward-to-volatility (Sharpe) ratio,** which is the slope of the CAL combining T-bills and the minimum-variance portfolio, is

$$S_A = \frac{E(r_A) - r_f}{\sigma_A} = \frac{8.9 - 5}{11.45} = .34$$

Now consider the CAL that uses portfolio *B* instead of *A*. Portfolio *B* invests 70% in bonds and 30% in stocks. Its expected return is 9.5% (a risk premium of 4.5%), and its standard deviation is 11.70%. Thus the reward-to-volatility ratio on the CAL that is supported by portfolio *B* is

$$S_B = \frac{9.5 - 5}{11.7} = .38$$

which is higher than the reward-to-volatility ratio of the CAL that we obtained using the minimum-variance portfolio and T-bills. Hence, portfolio *B* dominates *A*.

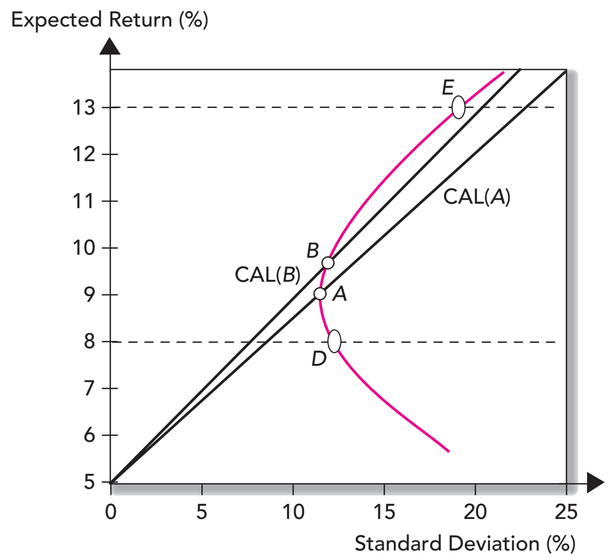


FIGURE 7.6 The opportunity set of the debt and equity funds and two feasible CALs

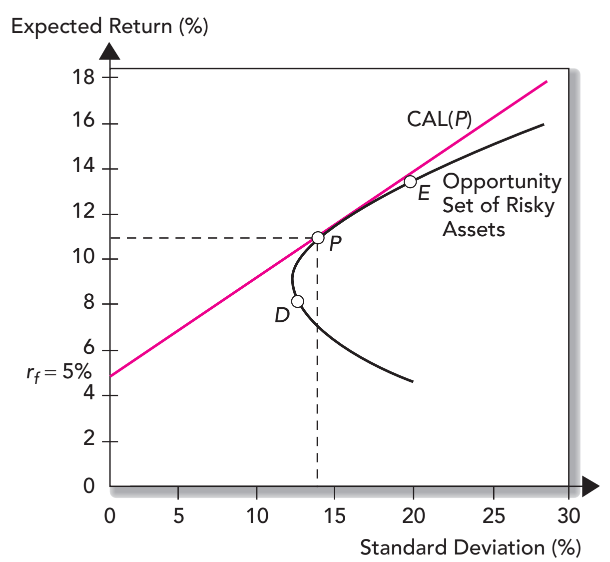


FIGURE 7.7 The opportunity set of the debt and equity funds with the optimal CAL and the optimal risky portfolio

But why stop at portfolio B? We can continue to ratchet the CAL upward until it ultimately reaches the point of tangency with the investment opportunity set. This must yield the CAL with the highest feasible reward-to-volatility ratio. Therefore, the tangency portfolio, labeled P in Figure 7.7, is the optimal risky portfolio to mix with T-bills. We can read the expected return and standard deviation of portfolio P from the graph in Figure 7.7:

$$E(r_P) = 11\%$$
$$\sigma_P = 14.2\%$$

In practice, when we try to construct optimal risky portfolios from more than two risky assets, we need to rely on a spreadsheet or another computer program. The spreadsheet we present in Appendix A can be used to construct efficient portfolios of many assets. To start, however, we will demonstrate the solution of the portfolio construction problem with only two risky assets (in our example, long-term debt and equity) and a risk-free asset. In this simpler two-asset case, we can derive an explicit formula for the weights of each asset in the optimal portfolio. This will make it easy to illustrate some of the general issues pertaining to portfolio optimization.

The objective is to find the weights w_D and w_E that result in the highest slope of the CAL (i.e., the weights that result in the risky portfolio with the highest reward-to-volatility ratio). Therefore, the objective is to maximize the slope of the CAL for any possible portfolio, p. Thus our *objective function* is the slope (equivalently, the Sharpe ratio) S_p:

$$S_p = \frac{E(r_p) - r_f}{\sigma_p}$$

For the portfolio with two risky assets, the expected return and standard deviation of portfolio p are

$$\begin{aligned} E(r_p) &= w_D E(r_D) + w_E E(r_E) \\ &= 8w_D + 13w_E \\ \sigma_p &= [w_D^2\sigma_D^2 + w_E^2\sigma_E^2 + 2w_D w_E \mathrm{Cov}(r_D, r_E)]^{1/2} \\ &= [144w_D^2 + 400w_E^2 + (2 \times 72 w_D w_E)]^{1/2} \end{aligned}$$

When we maximize the objective function, S_p, we have to satisfy the constraint that the portfolio weights sum to 1.0 (100%), that is, $w_D + w_E = 1$. Therefore, we solve an optimization problem formally written as

$$\underset{w_i}{\text{Max}}\ S_p = \frac{E(r_p) - r_f}{\sigma_p}$$

subject to $\Sigma w_i = 1$. This is a nonlinear problem that can be solved using standard tools of calculus.

In the case of two risky assets, the solution for the weights of the **optimal risky portfolio,** P, is given by Equation 7.13. Notice that the solution employs *excess* rates of return (denoted R) rather than total returns (denoted r).[6]

$$w_D = \frac{E(R_D)\sigma_E^2 - E(R_E)\text{Cov}(R_D, R_E)}{E(R_D)\sigma_E^2 + E(R_E)\sigma_D^2 - [E(R_D) + E(R_E)]\text{Cov}(R_D, R_E)} \qquad \textbf{(7.13)}$$
$$w_E = 1 - w_D$$

EXAMPLE 7.2 Optimal Risky Portfolio

Using our data, the solution for the optimal risky portfolio is

$$w_D = \frac{(8-5)400 - (13-5)72}{(8-5)400 + (13-5)144 - (8-5+13-5)72} = .40$$
$$w_E = 1 - .40 = .60$$

The expected return and standard deviation of this optimal risky portfolio are

$$E(r_P) = (.4 \times 8) + (.6 \times 13) = 11\%$$
$$\sigma_P = [(.4^2 \times 144) + (.6^2 \times 400) + (2 \times .4 \times .6 \times 72)]^{1/2} = 14.2\%$$

The CAL of this optimal portfolio has a slope of

$$S_P = \frac{11-5}{14.2} = .42$$

which is the reward-to-volatility (Sharpe) ratio of portfolio P. Notice that this slope exceeds the slope of any of the other feasible portfolios that we have considered, as it must if it is to be the slope of the best feasible CAL.

In Chapter 6 we found the optimal *complete* portfolio given an optimal *risky* portfolio and the CAL generated by a combination of this portfolio and T-bills. Now that we have constructed the optimal risky portfolio, P, we can use the individual investor's degree of risk aversion, A, to calculate the optimal proportion of the complete portfolio to invest in the risky component.

[6]The solution procedure for two risky assets is as follows. Substitute for $E(r_P)$ from Equation 7.2 and for σ_P from Equation 7.7. Substitute $1 - w_D$ for w_E. Differentiate the resulting expression for S_p with respect to w_D, set the derivative equal to zero, and solve for w_D.

EXAMPLE 7.3 Optimal Complete Portfolio

An investor with a coefficient of risk aversion $A = 4$ would take a position in portfolio P of[7]

$$y = \frac{E(r_P) - r_f}{A\sigma_P^2} = \frac{.11 - .05}{4 \times .142^2} = .7439 \qquad (7.14)$$

Thus the investor will invest 74.39% of his or her wealth in portfolio P and 25.61% in T-bills. Portfolio P consists of 40% in bonds, so the fraction of wealth in bonds will be $yw_D = .4 \times .7439 = .2976$, or 29.76%. Similarly, the investment in stocks will be $yw_E = .6 \times .7439 = .4463$, or 44.63%. The graphical solution of this asset allocation problem is presented in Figures 7.8 and 7.9.

Once we have reached this point, generalizing to the case of many risky assets is straightforward. Before we move on, let us briefly summarize the steps we followed to arrive at the complete portfolio.

1. Specify the return characteristics of all securities (expected returns, variances, covariances).
2. Establish the risky portfolio:
   a. Calculate the optimal risky portfolio, P (Equation 7.13).
   b. Calculate the properties of portfolio P using the weights determined in step (*a*) and Equations 7.2 and 7.3.

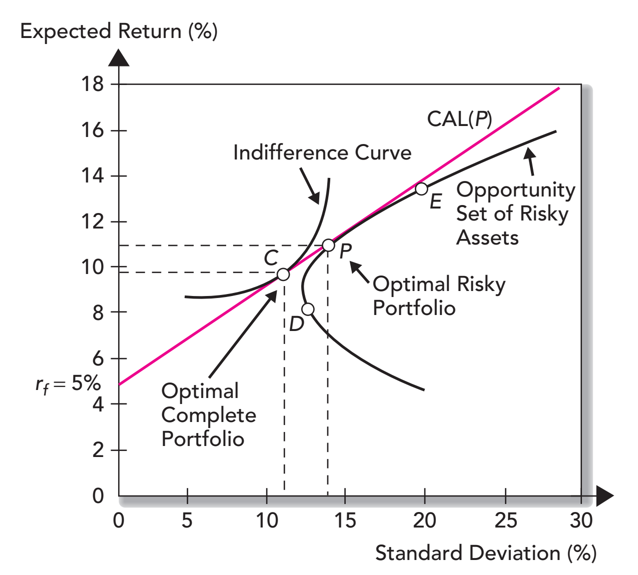


FIGURE 7.8 Determination of the optimal complete portfolio

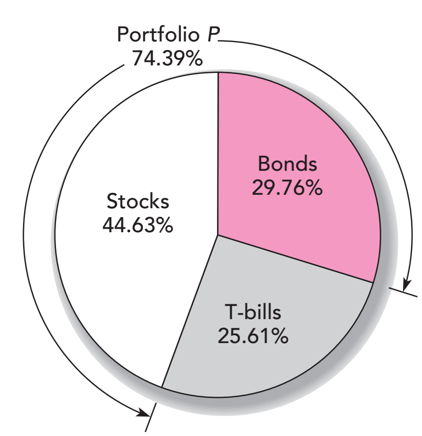


FIGURE 7.9 The proportions of the optimal complete portfolio

[7]Notice that we express returns as decimals in Equation 7.14. This is necessary when using the risk aversion parameter, A, to solve for capital allocation.

3. Allocate funds between the risky portfolio and the risk-free asset:
   a. Calculate the fraction of the complete portfolio allocated to portfolio *P* (the risky portfolio) and to T-bills (the risk-free asset) (Equation 7.14).
   b. Calculate the share of the complete portfolio invested in each asset and in T-bills.

Recall that our two risky assets, the bond and stock mutual funds, are already diversified portfolios. The diversification *within* each of these portfolios must be credited for a good deal of the risk reduction compared to undiversified single securities. For example, the standard deviation of the rate of return on an average stock is about 50% (see Figure 7.2). In contrast, the standard deviation of our stock-index fund is only 20%, about equal to the historical standard deviation of the S&P 500 portfolio. This is evidence of the importance of diversification within the asset class. Optimizing the asset allocation between bonds and stocks contributed incrementally to the improvement in the reward-to-volatility ratio of the complete portfolio. The CAL with stocks, bonds, and bills (Figure 7.7) shows that the standard deviation of the complete portfolio can be further reduced to 18% while maintaining the same expected return of 13% as the stock portfolio.

CONCEPT CHECK 3

The universe of available securities includes two risky stock funds, *A* and *B*, and T-bills. The data for the universe are as follows:

	Expected Return	Standard Deviation
A	10%	20%
B	30	60
T-bills	5	0

The correlation coefficient between funds *A* and *B* is −.2.

a. Draw the opportunity set of funds *A* and *B*.
b. Find the optimal risky portfolio, *P*, and its expected return and standard deviation.
c. Find the slope of the CAL supported by T-bills and portfolio *P*.
d. How much will an investor with $A = 5$ invest in funds *A* and *B* and in T-bills?

7.4 THE MARKOWITZ PORTFOLIO SELECTION MODEL

Security Selection

We can generalize the portfolio construction problem to the case of many risky securities and a risk-free asset. As in the two risky assets example, the problem has three parts. First, we identify the risk–return combinations available from the set of risky assets. Next, we identify the optimal portfolio of risky assets by finding the portfolio weights that result in the steepest CAL. Finally, we choose an appropriate complete portfolio by mixing the risk-free asset with the optimal risky portfolio. Before describing the process in detail, let us first present an overview.

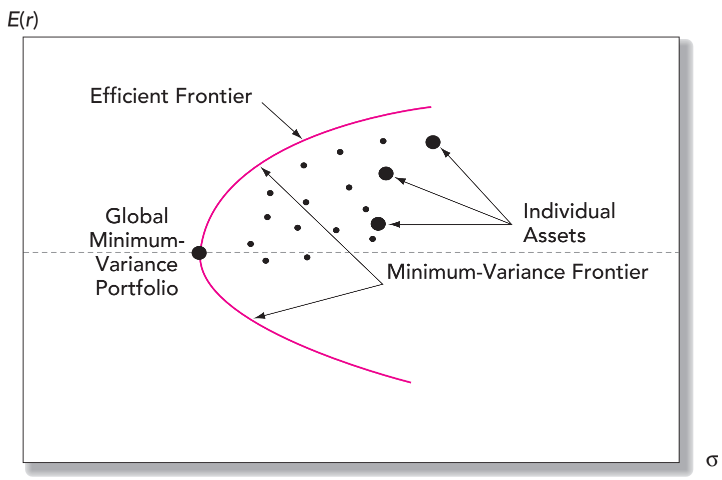


FIGURE 7.10 The minimum-variance frontier of risky assets

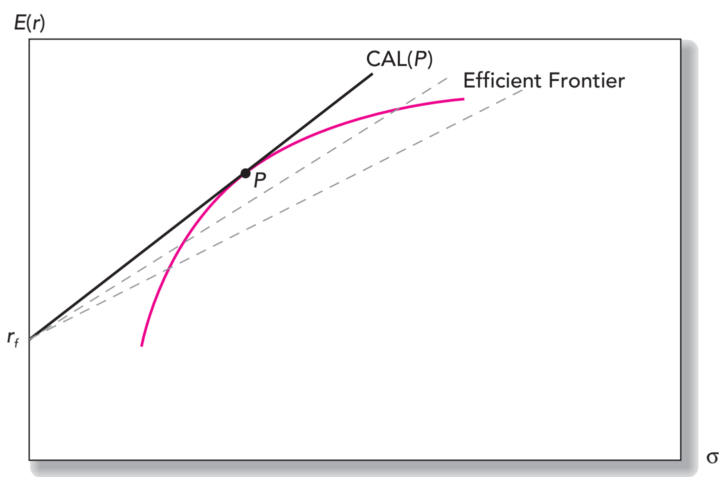


FIGURE 7.11 The efficient frontier of risky assets with the optimal CAL

The first step is to determine the risk–return opportunities available to the investor. These are summarized by the **minimum-variance frontier** of risky assets. This frontier is a graph of the lowest possible variance that can be attained for a given portfolio expected return. Given the input data for expected returns, variances, and covariances, we can calculate the minimum-variance portfolio for any targeted expected return. The plot of these expected return–standard deviation pairs is presented in Figure 7.10.

Notice that all the individual assets lie to the right inside the frontier, at least when we allow short sales in the construction of risky portfolios.[8] This tells us that risky portfolios comprising only a single asset are inefficient. Diversifying investments leads to portfolios with higher expected returns and lower standard deviations.

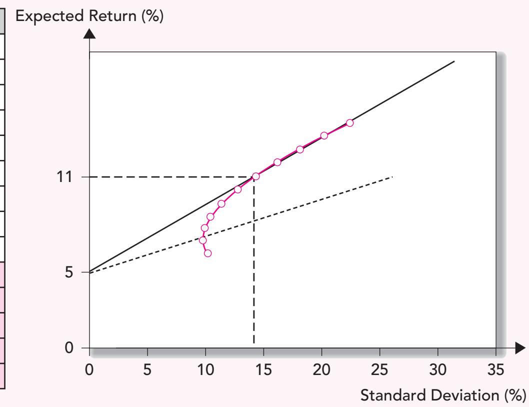
All the portfolios that lie on the minimum-variance frontier from the global minimum-variance portfolio and upward provide the best risk–return combinations and thus are candidates for the optimal portfolio. The part of the frontier that lies above the global minimum-variance portfolio, therefore, is called the **efficient frontier of risky assets.** For any portfolio on the lower portion of the minimum-variance frontier, there is a portfolio with the same standard deviation and a greater expected return positioned directly above it. Hence the bottom part of the minimum-variance frontier is inefficient.

The second part of the optimization plan involves the risk-free asset. As before, we search for the capital allocation line with the highest reward-to-volatility ratio (that is, the steepest slope) as shown in Figure 7.11.

[8]When short sales are prohibited, single securities may lie on the frontier. For example, the security with the highest expected return must lie on the frontier, as that security represents the *only* way that one can obtain a return that high, and so it must also be the minimum-variance way to obtain that return. When short sales are feasible, however, portfolios can be constructed that offer the same expected return and lower variance. These portfolios typically will have short positions in low-expected-return securities.

The accompanying spreadsheet can be used to measure the return and risk of a portfolio of two risky assets. The model calculates the return and risk for varying weights of each security along with the optimal risky and minimum-variance portfolio. Graphs are automatically generated for various model inputs. The model allows you to specify a target rate of return and solves for optimal combinations using the risk-free asset and the optimal risky portfolio. The spreadsheet is constructed with the two-security return data from Table 7.1. This spreadsheet is available at **www.mhhe.com/bkm.**

	A	B	C	D	E	F
1	**Asset Allocation Analysis: Risk and Return**					
2		**Expected**	**Standard**	**Correlation**		
3		**Return**	**Deviation**	**Coefficient**	**Covariance**	
4	Security 1	0.08	0.12	0.3	0.0072	
5	Security 2	0.13	0.2			
6	T-Bill	0.05	0			
7						
8	**Weight**	**Weight**		**Expected**	**Standard**	**Reward to**
9	**Security 1**	**Security 2**		**Return**	**Deviation**	**Volatility**
10	1	0		0.08000	0.12000	0.25000
11	0.9	0.1		0.08500	0.11559	0.30281
12	0.8	0.2		0.09000	0.11454	0.34922
13	0.7	0.3		0.09500	0.11696	0.38474
14	0.6	0.4		0.10000	0.12264	0.40771



The CAL that is supported by the optimal portfolio, *P*, is tangent to the efficient frontier. This CAL dominates all alternative feasible lines (the broken lines that are drawn through the frontier). Portfolio *P*, therefore, is the optimal risky portfolio.

Finally, in the last part of the problem the individual investor chooses the appropriate mix between the optimal risky portfolio *P* and T-bills, exactly as in Figure 7.8.

Now let us consider each part of the portfolio construction problem in more detail. In the first part of the problem, risk–return analysis, the portfolio manager needs as inputs a set of estimates for the expected returns of each security and a set of estimates for the covariance matrix. (In Part Five on security analysis we will examine the security valuation techniques and methods of financial analysis that analysts use. For now, we will assume that analysts already have spent the time and resources to prepare the inputs.)

The portfolio manager is now armed with the n estimates of $E(r_i)$ and the $n \times n$ estimates of the covariance matrix in which the n diagonal elements are estimates of the variances, σ_i^2, and the $n^2 - n = n(n-1)$ off-diagonal elements are the estimates of the covariances between each pair of asset returns. (You can verify this from Table 7.2 for the case $n = 2$.) We know that each covariance appears twice in this table, so actually we have $n(n-1)/2$ different covariance estimates. If our portfolio management unit covers 50 securities, our security analysts need to deliver 50 estimates of expected returns, 50 estimates of variances, and $50 \times 49/2 = 1{,}225$ different estimates of covariances. This is a daunting task! (We show later how the number of required estimates can be reduced substantially.)

Once these estimates are compiled, the expected return and variance of any risky portfolio with weights in each security, w_i, can be calculated from the bordered covariance matrix or, equivalently, from the following formulas:

$$E(r_p) = \sum_{i=1}^{n} w_i E(r_i) \qquad \textbf{(7.15)}$$

$$\sigma_p^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \text{Cov}(r_i, r_j) \quad (7.16)$$

An extended worked example showing you how to do this using a spreadsheet is presented in Appendix A of this chapter.

We mentioned earlier that the idea of diversification is age-old. The phrase "don't put all your eggs in one basket" existed long before modern finance theory. It was not until 1952, however, that Harry Markowitz published a formal model of portfolio selection embodying diversification principles, thereby paving the way for his 1990 Nobel Prize in Economics.[9] His model is precisely step one of portfolio management: the identification of the efficient set of portfolios, or the *efficient frontier of risky assets.*

The principal idea behind the frontier set of risky portfolios is that, for any risk level, we are interested only in that portfolio with the highest expected return. Alternatively, the frontier is the set of portfolios that minimizes the variance for any target expected return.

Indeed, the two methods of computing the efficient set of risky portfolios are equivalent. To see this, consider the graphical representation of these procedures. Figure 7.12 shows the minimum-variance frontier.

The points marked by squares are the result of a variance-minimization program. We first draw the constraints, that is, horizontal lines at the level of required expected returns. We then look for the portfolio with the lowest standard deviation that plots on each horizontal line—we look for the portfolio that will plot farthest to the left (smallest standard deviation) on that line. When we repeat this for many levels of required expected returns, the shape of the minimum-variance frontier emerges. We then discard the bottom (dashed) half of the frontier, because it is inefficient.

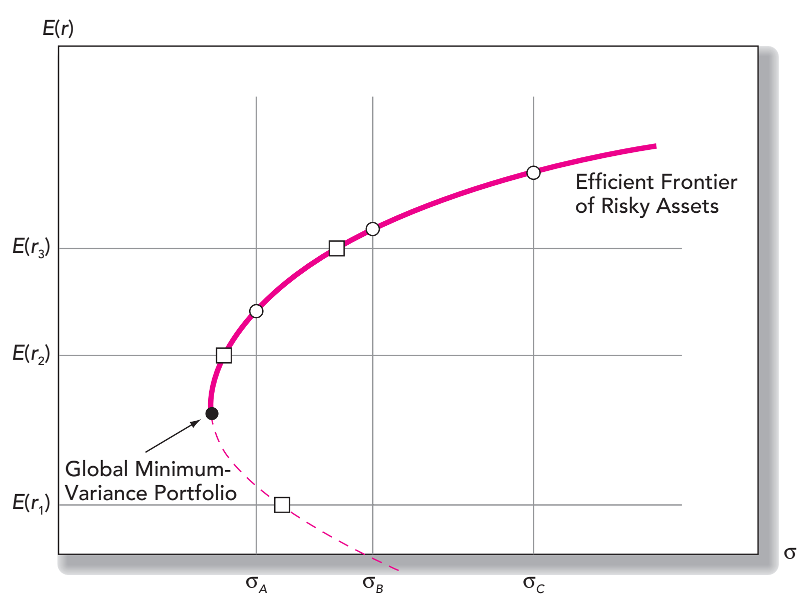


FIGURE 7.12 The efficient portfolio set

[9]Harry Markowitz, "Portfolio Selection," *Journal of Finance*, March 1952.

In the alternative approach, we draw a vertical line that represents the standard deviation constraint. We then consider all portfolios that plot on this line (have the same standard deviation) and choose the one with the highest expected return, that is, the portfolio that plots highest on this vertical line. Repeating this procedure for many vertical lines (levels of standard deviation) gives us the points marked by circles that trace the upper portion of the minimum-variance frontier, the efficient frontier.

When this step is completed, we have a list of efficient portfolios, because the solution to the optimization program includes the portfolio proportions, w_i, the expected return, $E(r_p)$, and the standard deviation, σ_p.

Let us restate what our portfolio manager has done so far. The estimates generated by the security analysts were transformed into a set of expected rates of return and a covariance matrix. This group of estimates we shall call the **input list.** This input list is then fed into the optimization program.

Before we proceed to the second step of choosing the optimal risky portfolio from the frontier set, let us consider a practical point. Some clients may be subject to additional constraints. For example, many institutions are prohibited from taking short positions in any asset. For these clients the portfolio manager will add to the optimization program constraints that rule out negative (short) positions in the search for efficient portfolios. In this special case it is possible that single assets may be, in and of themselves, efficient risky portfolios. For example, the asset with the highest expected return will be a frontier portfolio because, without the opportunity of short sales, the only way to obtain that rate of return is to hold the asset as one's entire risky portfolio.

Short-sale restrictions are by no means the only such constraints. For example, some clients may want to ensure a minimal level of expected dividend yield from the optimal portfolio. In this case the input list will be expanded to include a set of expected dividend yields $d_1, \ldots, d_n$ and the optimization program will include an additional constraint that ensures that the expected dividend yield of the portfolio will equal or exceed the desired level, d.

Portfolio managers can tailor the efficient set to conform to any desire of the client. Of course, any constraint carries a price tag in the sense that an efficient frontier constructed subject to extra constraints will offer a reward-to-volatility ratio inferior to that of a less constrained one. The client should be made aware of this cost and should carefully consider constraints that are not mandated by law.

Another type of constraint is aimed at ruling out investments in industries or countries considered ethically or politically undesirable. This is referred to as *socially responsible investing,* which entails a cost in the form of a lower reward-to-volatility on the resultant constrained, optimal portfolio. This cost can be justifiably viewed as a contribution to the underlying cause.

Capital Allocation and the Separation Property

Now that we have the efficient frontier, we proceed to step two and introduce the risk-free asset. Figure 7.13 shows the efficient frontier plus three CALs representing various portfolios from the efficient set. As before, we ratchet up the CAL by selecting different portfolios until we reach portfolio *P*, which is the tangency point of a line from *F* to the efficient frontier. Portfolio *P* maximizes the reward-to-volatility ratio, the slope of the line from *F* to portfolios on the efficient frontier. At this point our portfolio manager is done. Portfolio *P* is the optimal risky portfolio for the manager's clients. This is a good time to ponder our results and their implementation.

eXcel APPLICATIONS: optimal portfolios

A spreadsheet model featuring optimal risky portfolios is available on the Online Learning Center at **www.mhhe.com/bkm.** It contains a template that is similar to the template developed in this section. The model can be used to find optimal mixes of securities for targeted levels of returns for both restricted and unrestricted portfolios. Graphs of the efficient frontier are generated for each set of inputs. The example available at our Web site applies the model to portfolios constructed from equity indexes (called WEBS securities) of several countries.

	A	B	C	D	E	F
1	Efficient Frontier for World Equity Benchmark Securities (WEBS)					
2						
3		Mean	Standard			
4	WEBS	Return	Deviation	Country		
5	EWD	15.5393	26.4868	Sweden		
6	EWH	6.3852	41.1475	Hong Kong		
7	EWI	26.5999	26.0514	Italy		
8	EWJ	1.4133	26.0709	Japan		
9	EWL	18.0745	21.6916	Switzerland		
10	EWP	18.6347	25.0779	Spain		
11	EWW	16.2243	38.7686	Mexico		
12	S&P 500	17.2306	17.1944			

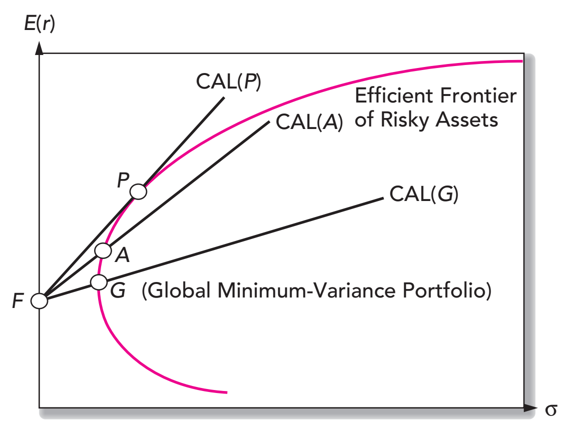


FIGURE 7.13 Capital allocation lines with various portfolios from the efficient set

The most striking conclusion is that a portfolio manager will offer the same risky portfolio, *P*, to all clients regardless of their degree of risk aversion.[10] The degree of risk aversion of the client comes into play only in the selection of the desired point along the CAL. Thus the only difference between clients' choices is that the more risk-averse client will invest more in the risk-free asset and less in the optimal risky portfolio than will a less risk-averse client. However, both will use portfolio *P* as their optimal risky investment vehicle.

This result is called a **separation property;** it tells us that the portfolio choice problem may be separated into two independent tasks.[11] The first task, determination of the optimal risky portfolio, is purely technical. Given the manager's input list, the best risky portfolio is the same for all clients, regardless of risk aversion. The second task, however, allocation of the complete portfolio to T-bills versus the risky portfolio, depends on personal preference. Here the client is the decision maker.

The crucial point is that the optimal portfolio *P* that the manager offers is the same for all clients. Put another way, investors with varying degrees of risk aversion would be satisfied with a universe of only two mutual funds: a money market fund for risk-free investments and a mutual fund that hold the optimal risky portfolio, *P*, on the tangency point of the CAL and the efficient frontier. This result makes professional management more

[10]Clients who impose special restrictions (constraints) on the manager, such as dividend yield, will obtain another optimal portfolio. Any constraint that is added to an optimization problem leads, in general, to a different and inferior optimum compared to an unconstrained program.

[11]The separation property was first noted by Nobel laureate James Tobin, "Liquidity Preference as Behavior toward Risk," *Review of Economic Statistics* 25 (February 1958), pp. 65–86.

efficient and hence less costly. One management firm can serve any number of clients with relatively small incremental administrative costs.

In practice, however, different managers will estimate different input lists, thus deriving different efficient frontiers, and offer different "optimal" portfolios to their clients. The source of the disparity lies in the security analysis. It is worth mentioning here that the universal rule of GIGO (garbage in–garbage out) also applies to security analysis. If the quality of the security analysis is poor, a passive portfolio such as a market index fund will result in a better CAL than an active portfolio that uses low-quality security analysis to tilt portfolio weights toward seemingly favorable (mispriced) securities.

One particular input list that would lead to a worthless estimate of the efficient frontier is based on recent security average returns. If sample average returns over recent years are used as proxies for the true expected return on the security, the noise in those estimates will make the resultant efficient frontier virtually useless for portfolio construction.

Consider a stock with an annual standard deviation of 50%. Even if one were to use a 10-year average to estimate its expected return (and 10 years is almost ancient history in the life of a corporation), the standard deviation of that estimate would still be $50/\sqrt{10} = 15.8\%$. The chances that this average represents expected returns for the coming year are negligible.[12] In Chapter 25, we see an example demonstrating that efficient frontiers constructed from past data may be wildly optimistic in terms of the *apparent* opportunities they offer to improve Sharpe ratios.

As we have seen, optimal risky portfolios for different clients also may vary because of portfolio constraints such as dividend-yield requirements, tax considerations, or other client preferences. Nevertheless, this analysis suggests that a limited number of portfolios may be sufficient to serve the demands of a wide range of investors. This is the theoretical basis of the mutual fund industry.

The (computerized) optimization technique is the easiest part of the portfolio construction problem. The real arena of competition among portfolio managers is in sophisticated security analysis. This analysis, as well as its proper interpretation, is part of the art of portfolio construction.[13]

CONCEPT CHECK 4

Suppose that two portfolio managers who work for competing investment management houses each employ a group of security analysts to prepare the input list for the Markowitz algorithm. When all is completed, it turns out that the efficient frontier obtained by portfolio manager *A* seems to dominate that of manager *B*. By dominate, we mean that *A*'s optimal risky portfolio lies northwest of *B*'s. Hence, given a choice, investors will all prefer the risky portfolio that lies on the CAL of *A*.

a. What should be made of this outcome?

b. Should it be attributed to better security analysis by *A*'s analysts?

c. Could it be that *A*'s computer program is superior?

d. If you were advising clients (and had an advance glimpse at the efficient frontiers of various managers), would you tell them to periodically switch their money to the manager with the most northwesterly portfolio?

[12]Moreover, you cannot avoid this problem by observing the rate of return on the stock more frequently. In Chapter 5 we showed that the accuracy of the sample average as an estimate of expected return depends on the length of the sample period, and is not improved by sampling more frequently within a given sample period.

[13]You can find a nice discussion of some practical issues in implementing efficient diversification in a white paper prepared by Wealthcare Capital Management at this address: **www.financeware.com/ruminations/WP_EfficiencyDeficiency.pdf.** A copy of the report is also available at the Online Learning Center for this text, **www.mhhe.com/bkm.**

The Power of Diversification

Section 7.1 introduced the concept of diversification and the limits to the benefits of diversification resulting from systematic risk. Given the tools we have developed, we can reconsider this intuition more rigorously and at the same time sharpen our insight regarding the power of diversification.

Recall from Equation 7.16, restated here, that the general formula for the variance of a portfolio is

$$\sigma_p^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \text{Cov}(r_i, r_j) \tag{7.16}$$

Consider now the naive diversification strategy in which an *equally weighted* portfolio is constructed, meaning that $w_i = 1/n$ for each security. In this case Equation 7.16 may be rewritten as follows, where we break out the terms for which $i = j$ into a separate sum, noting that $\text{Cov}(r_i, r_i) = \sigma_i^2$:

$$\sigma_p^2 = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{n} \sigma_i^2 + \sum_{\substack{j=1 \\ j \neq i}}^{n} \sum_{i=1}^{n} \frac{1}{n^2} \text{Cov}(r_i, r_j) \tag{7.17}$$

Note that there are n variance terms and $n(n - 1)$ covariance terms in Equation 7.17.

If we define the average variance and average covariance of the securities as

$$\bar{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} \sigma_i^2 \tag{7.18}$$

$$\overline{\text{Cov}} = \frac{1}{n(n-1)} \sum_{\substack{j=1 \\ j \neq i}}^{n} \sum_{i=1}^{n} \text{Cov}(r_i, r_j) \tag{7.19}$$

we can express portfolio variance as

$$\sigma_p^2 = \frac{1}{n} \bar{\sigma}^2 + \frac{n-1}{n} \overline{\text{Cov}} \tag{7.20}$$

Now examine the effect of diversification. When the average covariance among security returns is zero, as it is when all risk is firm-specific, portfolio variance can be driven to zero. We see this from Equation 7.20. The second term on the right-hand side will be zero in this scenario, while the first term approaches zero as n becomes larger. Hence when security returns are uncorrelated, the power of diversification to reduce portfolio risk is unlimited.

However, the more important case is the one in which economy-wide risk factors impart positive correlation among stock returns. In this case, as the portfolio becomes more highly diversified (n increases) portfolio variance remains positive. Although firm-specific risk, represented by the first term in Equation 7.20, is still diversified away, the second term simply approaches $\overline{\text{Cov}}$ as n becomes greater. [Note that $(n - 1)/n = 1 - 1/n$, which approaches 1 for large n.] Thus the irreducible risk of a diversified portfolio depends on the covariance of the returns of the component securities, which in turn is a function of the importance of systematic factors in the economy.

To see further the fundamental relationship between systematic risk and security correlations, suppose for simplicity that all securities have a common standard deviation, σ, and all security pairs have a common correlation coefficient, ρ. Then the covariance between all pairs of securities is $\rho\sigma^2$, and Equation 7.20 becomes

$$\sigma_p^2 = \frac{1}{n}\sigma^2 + \frac{n-1}{n}\rho\sigma^2 \tag{7.21}$$

The effect of correlation is now explicit. When $\rho = 0$, we again obtain the insurance principle, where portfolio variance approaches zero as n becomes greater. For $\rho > 0$, however, portfolio variance remains positive. In fact, for $\rho = 1$, portfolio variance equals σ^2 regardless of n, demonstrating that diversification is of no benefit: In the case of perfect correlation, all risk is systematic. More generally, as n becomes greater, Equation 7.21 shows that systematic risk becomes $\rho\sigma^2$.

Table 7.4 presents portfolio standard deviation as we include ever-greater numbers of securities in the portfolio for two cases, $\rho = 0$ and $\rho = .40$. The table takes σ to be 50%. As one would expect, portfolio risk is greater when $\rho = .40$. More surprising, perhaps, is that portfolio risk diminishes far less rapidly as n increases in the positive correlation case. The correlation among security returns limits the power of diversification.

Note that for a 100-security portfolio, the standard deviation is 5% in the uncorrelated case—still significant compared to the potential of zero standard deviation. For $\rho = .40$, the standard deviation is high, 31.86%, yet it is very close to undiversifiable systematic risk in the infinite-sized security universe, $\sqrt{\rho\sigma^2} = \sqrt{.4 \times 50^2} = 31.62\%$. At this point, further diversification is of little value.

Perhaps the most important insight from the exercise is this: When we hold diversified portfolios, the contribution to portfolio risk of a particular security will depend on the *covariance* of that security's return with those of other securities, and *not* on the security's variance. As we shall see in Chapter 9, this implies that fair risk premiums also should depend on covariances rather than total variability of returns.

CONCEPT CHECK 5

Suppose that the universe of available risky securities consists of a large number of stocks, identically distributed with $E(r) = 15\%$, $\sigma = 60\%$, and a common correlation coefficient of $\rho = .5$.

a. What are the expected return and standard deviation of an equally weighted risky portfolio of 25 stocks?

b. What is the smallest number of stocks necessary to generate an efficient portfolio with a standard deviation equal to or smaller than 43%?

c. What is the systematic risk in this security universe?

d. If T-bills are available and yield 10%, what is the slope of the CAL?

Asset Allocation and Security Selection

As we have seen, the theories of security selection and asset allocation are identical. Both activities call for the construction of an efficient frontier, and the choice of a particular portfolio from along that frontier. The determination of the optimal combination of securities proceeds in the same manner as the analysis of the optimal combination of asset classes. Why, then, do we (and the investment community) distinguish between asset allocation and security selection?

Three factors are at work. First, as a result of greater need and ability to save (for college educations, recreation, longer life in retirement, health care needs, etc.), the demand

TABLE 7.4

Risk reduction of equally weighted portfolios in correlated and uncorrelated universes

		$\rho = 0$		$\rho = .4$	
Universe Size *n*	Portfolio Weights $w = 1/n$ (%)	Standard Deviation (%)	Reduction in σ	Standard Deviation (%)	Reduction in σ
1	100	50.00	14.64	50.00	8.17
2	50	35.36		41.83	
5	20	22.36	1.95	36.06	0.70
6	16.67	20.41		35.36	
10	10	15.81	0.73	33.91	0.20
11	9.09	15.08		33.71	
20	5	11.18	0.27	32.79	0.06
21	4.76	10.91		32.73	
100	1	5.00	0.02	31.86	0.00
101	0.99	4.98		31.86	

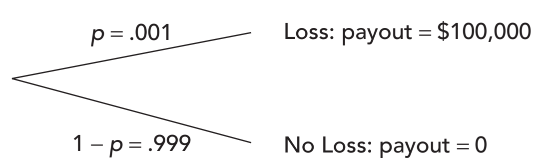
for sophisticated investment management has increased enormously. Second, the widening spectrum of financial markets and financial instruments has put sophisticated investment beyond the capacity of many amateur investors. Finally, there are strong economies of scale in investment analysis. The end result is that the size of a competitive investment company has grown with the industry, and efficiency in organization has become an important issue.

A large investment company is likely to invest both in domestic and international markets and in a broad set of asset classes, each of which requires specialized expertise. Hence the management of each asset-class portfolio needs to be decentralized, and it becomes impossible to simultaneously optimize the entire organization's risky portfolio in one stage, although this would be prescribed as optimal on *theoretical* grounds.

The practice is therefore to optimize the security selection of each asset-class portfolio independently. At the same time, top management continually updates the asset allocation of the organization, adjusting the investment budget allotted to each asset-class portfolio.

7.5 RISK POOLING, RISK SHARING, AND RISK IN THE LONG RUN

Consider an insurance company that offers a 1-year policy on a residential property valued at $100,000. Suppose the following event tree gives the probability distribution of year-end payouts on the policy:



Assume for simplicity that the insurance company sets aside $100,000 to cover its potential payout on the policy. The funds may be invested in T-bills for the coverage year, earning the risk-free rate of 5%. Of course, the *expected* payout on the policy is far smaller; it equals $p \times$ potential payout = .001 × 100,000 = $100. The insurer may charge an up-front premium of $120. The $120 yields (with 5% interest) $126 by year-end. Therefore, the insurer's expected profit on the policy is $126 − $100 = $26, which makes for a risk premium of 2.6 basis points (.026%) on the $100,000 set aside to cover potential losses. Relative to what appears a paltry expected profit of $26, the standard deviation is enormous, $3,160.70 (try checking this); this implies a standard deviation of return of $\sigma = 3.16\%$ of the $100,000 investment, compared to a risk premium of only 0.26%.

By now you may be thinking about diversification and the insurance principle. Because the company will cover many such properties, each of which has independent risk, perhaps the large one-policy risk (relative to the risk premium) can be brought down to a "satisfactory" level. Before we proceed, however, we pause for a digression on why this discussion is relevant to understanding portfolio risk. It is because the analogy between the insurance principle and portfolio diversification is essential to understanding risk in the long run.

Risk Pooling and the Insurance Principle

Suppose the insurance company sells 10,000 of these uncorrelated policies. In the context of portfolio diversification, one might think that 10,000 uncorrelated assets would diversify away practically all risk. The expected rate of return on each of the 10,000 identical, independent policies is .026%, and this is the rate of return of the collection of policies as well. To find the standard deviation of the rate of return we use Equation 7.20. Because the covariance between any two policies is zero and σ is the same for each policy, the variance and standard deviation of the rate of return on the 10,000-policy portfolio are

$$\sigma_P^2 = \frac{1}{n}\sigma^2$$
$$\sigma_P = \frac{\sigma}{\sqrt{n}} = \frac{3.16\%}{\sqrt{10{,}000}} = .0316\% \tag{7.22}$$

Now the standard deviation is of the same order as the risk premium, and in fact could be further decreased by selling even more policies. This is the insurance principle.

It seems that as the firm sells more policies, its risk continues to fall. The standard deviation of the rate of return on equity capital falls relative to the expected return, and the probability of loss with it. Sooner or later, it appears, the firm will earn a risk-free risk premium. Sound too good to be true? It is.

This line of reasoning might remind you of the familiar argument that investing in stocks for the long run reduces risk. In both cases, scaling up the bet (either by adding more policies or extending the investment to longer periods) appears to reduce risk. And, in fact, the flaw in this argument is the same as the one that we encountered when we looked at the claim that stock investments become less risky in the long run. We saw then that the probability of loss is an inadequate measure of risk, as it does not account for the magnitude of the possible loss. In the insurance application, the maximum possible loss is 10,000 × $100,000 = $1 billion, and hence a comparison with a one-policy "portfolio" (with a maximum loss of $100,000) cannot be made on the basis of means and standard deviations of rates of return.

This claim may be surprising. After all, the profits from many policies are normally distributed,[14] so the distribution is symmetric and the standard deviation should be an

[14]This argument for normality is similar to that of the newsstand example in Chapter 5. With many policies, the most likely outcomes for total payout are near the expected value. Deviations in either direction are less likely, and the probability distribution of payouts approaches the familiar bell-shaped curve.

appropriate measure of risk. Accordingly, it would seem that the steady decline of the portfolio standard deviation faithfully reflects risk reduction.

The problem with the argument is that increasing the size of the bundle of policies does not make for diversification! Diversifying a portfolio means dividing *a fixed investment budget* across more assets. If an investment of $100,000 in Microsoft is to be diversified, the same $100,000 must be divided between shares of Microsoft and shares of Wal-Mart and other firms. In contrast, an investor who currently has $100,000 invested in Microsoft does *not* reduce total risk by adding another $100,000 investment in Wal-Mart.

An investment of $200,000 divided equally between Microsoft and Wal-Mart, cannot be compared to an investment of $100,000 in Microsoft alone using *rate of return* statistics. This is because the scales of the investments are different. Put differently, if we wish to compare these two investments, the distribution of the rate of return is not reliable. We must compare the distribution of *dollar profits* from the two investments.[15]

When we combine n uncorrelated insurance policies, each with an expected profit of $\$\pi$, both expected total profit *and* standard deviation (SD) grow in direct proportion to n. This is so because

$$\begin{aligned} E(n\pi) &= nE(\pi) \\ \text{Var}(n\pi) &= n^2\text{Var}(\pi) = n^2\,\sigma^2 \\ \text{SD}(n\pi) &= n\sigma \end{aligned}$$

The ratio of mean to standard deviation does not change when n increases. The risk–return trade-off therefore does not improve with the assumption of additional policies. Ironically, the economics of the insurance industry has little to do with what is commonly called the insurance principle. Before we turn to the principle that does drive the industry, let's first turn back to see what this example suggests about risk in the long run.

Consider the investor with a $100,000 portfolio. Keeping the $100,000 in the risky portfolio for a second year does not diversify the risk associated with the first year investment. Keeping $100,000 in a risky investment for an additional year is analogous to the insurance company selling an additional $100,000 policy. Average rates of return cannot be used to meaningfully compare a 2-year investment in the risky portfolio with a 1-year investment in the same risky portfolio. Instead, we must compare the distribution of *terminal values* (or 2-year HPRs) of alternative *2-year* investments: 2 years in the risky portfolio versus 1 year in the risky portfolio *and* 1 year in a risk-free investment.

Risk Sharing

If risk *pooling* (the sale of additional independent policies) does not explain the insurance industry, then what does? The answer is risk *sharing,* the distribution of a fixed amount of risk among many investors.

The birth of the insurance industry is believed to have taken place in Edward Lloyd's coffee house in the late 1600s. The economic model underlying Lloyd's underwriters today is quite similar to insurance underwriting when the firm was founded. Suppose a U.S. corporation desires to insure the launch of a satellite valued at $100 million. It can contact one of Lloyd's independent underwriters. That underwriter will contact other underwriters who each will take a piece of the action—each will choose to insure a *fraction* of the project risk. When the lead underwriter successfully puts together a consortium that is

[15]Think back to your corporate finance class and you will see the analogy to ranking mutually exclusive projects of different magnitude. The rate of return, or IRR of two investments, can incorrectly rank the projects because it ignores size; only the net present value criterion can be relied on to correctly rank competing projects. This is so because NPV accounts for the dollar magnitude of the investment and subsequent cash flows.

willing to cover 100% of the risk, a proposal is made to the launch company. Notice that each underwriter has a *fixed amount* of equity capital. The underwriter diversifies its risk by allocating its investment budget across many projects that are not perfectly correlated, which is why one underwriter will decline to underwrite too large a fraction of any single project. In other words, the underwriters engage in risk sharing. They limit their exposure to any single source of risk by sharing that risk with other underwriters. Each one diversifies a largely fixed portfolio across many projects, and the risk of each project is shared with many other underwriters. This is the proper use of risk pooling: pooling many sources of risk in a portfolio of *given* size.[16]

Let's return to the property insurance. Suppose an insurance entrepreneur can market every year 10,000 policies of the type we discussed (each with $100,000 of coverage), for $1 billion of total coverage. With such prowess, this entrepreneur can go public and sell shares in the enterprise. Let's say 10,000 investors purchase one share of the billion-dollar company and share equally in the risk premium. If a particular policy pays off, each investor is at risk for only $100,000/10,000 = $10. There is minimal risk from any single policy.

Moreover, even if the insurance company has not pooled many policies, individual investors can still limit their risk by diversifying their own holdings. Shareholders of corporations do not look for the corporation to reduce their portfolio risk. Rather, they diversify their investment portfolios by divvying them up across stocks of many companies.

Keeping with the assumption that all policies are truly independent, it actually makes no difference how many separate insurance companies cover a given number of policies currently outstanding in an insurance market. Suppose that instead of the billion-dollar company, shares of two $500-million insurance companies trade, each with a "portfolio" of 5,000 policies. The distribution of the aggregate profit of the two companies is identical to that of the billion-dollar company. Therefore, buying one share in the large company provides the same diversification value as buying one share in each of the two smaller firms.

The bottom line is that portfolio risk management is about the allocation of a fixed investment budget to assets that are not perfectly correlated. In this environment, rate of return statistics, that is, expected returns, variances, and covariances, are sufficient to optimize the investment portfolio. Choices among alternative investments of a different magnitude require that we abandon rates of return in favor of dollar profits. This applies as well to investments for the long run.

[16]Underwriters that, through successful marketing and efficient administration, can underwrite profitable risks beyond the capacity of their own equity capital may turn to reinsurance companies to cover a fraction of the risk of a large venture. Competition in the reinsurance market keeps rates low and allows the underwriter to keep a good share of the profits of the reinsured risks. This is how insurers can leverage their equity capital.

SUMMARY

1. The expected return of a portfolio is the weighted average of the component security expected returns with the investment proportions as weights.
2. The variance of a portfolio is the weighted sum of the elements of the covariance matrix with the product of the investment proportions as weights. Thus the variance of each asset is weighted by the square of its investment proportion. The covariance of each pair of assets appears twice in the covariance matrix; thus the portfolio variance includes twice each covariance weighted by the product of the investment proportions in each of the two assets.
3. Even if the covariances are positive, the portfolio standard deviation is less than the weighted average of the component standard deviations, as long as the assets are not perfectly positively correlated. Thus portfolio diversification is of value as long as assets are less than perfectly correlated.

4. The greater an asset's covariance with the other assets in the portfolio, the more it contributes to portfolio variance. An asset that is perfectly negatively correlated with a portfolio can serve as a perfect hedge. The perfect hedge asset can reduce the portfolio variance to zero.

5. The efficient frontier is the graphical representation of a set of portfolios that maximize expected return for each level of portfolio risk. Rational investors will choose a portfolio on the efficient frontier.

6. A portfolio manager identifies the efficient frontier by first establishing estimates for asset expected returns and the covariance matrix. This input list is then fed into an optimization program that reports as outputs the investment proportions, expected returns, and standard deviations of the portfolios on the efficient frontier.

7. In general, portfolio managers will arrive at different efficient portfolios because of differences in methods and quality of security analysis. Managers compete on the quality of their security analysis relative to their management fees.

8. If a risk-free asset is available and input lists are identical, all investors will choose the same portfolio on the efficient frontier of risky assets: the portfolio tangent to the CAL. All investors with identical input lists will hold an identical risky portfolio, differing only in how much each allocates to this optimal portfolio and to the risk-free asset. This result is characterized as the separation principle of portfolio construction.

9. Diversification is based on the allocation of a *fixed* portfolio across several assets, limiting the exposure to any one source of risk. Adding additional risky assets to a portfolio, thereby increasing the total amounts invested, does not reduce dollar risk, even if it makes the rate of return more predictable. This is because that uncertainty is applied to a larger investment base. Nor does investing over longer horizons reduce risk. Increasing the investment horizon is analogous to investing in more assets. It increases total risk. Analogously, the key to the insurance industry is risk sharing—the spreading of risk across many investors, each of whom takes on only a small exposure to any given source of risk. Risk pooling—the assumption of ever-more sources of risk—may increase rate of return predictability, but not the predictability of total dollar returns.

Related Web sites for this chapter are available at **www.mhhe.com/bkm**

KEY TERMS

diversification
insurance principle
market risk
systematic risk
nondiversifiable risk
unique risk
firm-specific risk
nonsystematic risk
diversifiable risk
minimum-variance portfolio
portfolio opportunity set
reward-to-volatility ratio
optimal risky portfolio
minimum-variance frontier
efficient frontier of risky assets
input list
separation property

PROBLEM SETS

Quiz

1. Which of the following factors reflect *pure* market risk for a given corporation?
   a. Increased short-term interest rates.
   b. Fire in the corporate warehouse.
   c. Increased insurance costs.
   d. Death of the CEO.
   e. Increased labor costs.
2. When adding real estate to an asset allocation program that currently includes only stocks, bonds, and cash, which of the properties of real estate returns affect portfolio *risk?* Explain.
   a. Standard deviation.
   b. Expected return.
   c. Correlation with returns of the other asset classes.
3. Which of the following statements about the minimum variance portfolio of all risky securities are valid? (Assume short sales are allowed.) Explain.
   a. Its variance must be lower than those of all other securities or portfolios.
   b. Its expected return can be lower than the risk-free rate.

c. It may be the optimal risky portfolio.
d. It must include all individual securities.

Problems

The following data apply to Problems 4 through 10: A pension fund manager is considering three mutual funds. The first is a stock fund, the second is a long-term government and corporate bond fund, and the third is a T-bill money market fund that yields a rate of 8%. The probability distribution of the risky funds is as follows:

	Expected Return	Standard Deviation
Stock fund (*S*)	20%	30%
Bond fund (*B*)	12	15

The correlation between the fund returns is .10.

4. What are the investment proportions in the minimum-variance portfolio of the two risky funds, and what is the expected value and standard deviation of its rate of return?
5. Tabulate and draw the investment opportunity set of the two risky funds. Use investment proportions for the stock fund of zero to 100% in increments of 20%.
6. Draw a tangent from the risk-free rate to the opportunity set. What does your graph show for the expected return and standard deviation of the optimal portfolio?
7. Solve numerically for the proportions of each asset and for the expected return and standard deviation of the optimal risky portfolio.
8. What is the reward-to-volatility ratio of the best feasible CAL?
9. You require that your portfolio yield an expected return of 14%, and that it be efficient, on the best feasible CAL.
   a. What is the standard deviation of your portfolio?
   b. What is the proportion invested in the T-bill fund and each of the two risky funds?
10. If you were to use only the two risky funds, and still require an expected return of 14%, what would be the investment proportions of your portfolio? Compare its standard deviation to that of the optimized portfolio in Problem 9. What do you conclude?
11. Stocks offer an expected rate of return of 18%, with a standard deviation of 22%. Gold offers an expected return of 10% with a standard deviation of 30%.
    a. In light of the apparent inferiority of gold with respect to both mean return and volatility, would anyone hold gold? If so, demonstrate graphically why one would do so.
    b. Given the data above, reanswer (*a*) with the additional assumption that the correlation coefficient between gold and stocks equals 1. Draw a graph illustrating why one would or would not hold gold in one's portfolio. Could this set of assumptions for expected returns, standard deviations, and correlation represent an equilibrium for the security market?
12. Suppose that there are many stocks in the security market and that the characteristics of Stocks *A* and *B* are given as follows:

Stock	Expected Return	Standard Deviation
A	10%	5%
B	15	10

Correlation = –1

Suppose that it is possible to borrow at the risk-free rate, r_f. What must be the value of the risk-free rate? (*Hint:* Think about constructing a risk-free portfolio from stocks *A* and *B*.)

13. Assume that expected returns and standard deviations for all securities (including the risk-free rate for borrowing and lending) are known. In this case all investors will have the same optimal risky portfolio. (True or false?)
14. The standard deviation of the portfolio is always equal to the weighted average of the standard deviations of the assets in the portfolio. (True or false?)

15. Suppose you have a project that has a .7 chance of doubling your investment in a year and a .3 chance of halving your investment in a year. What is the standard deviation of the rate of return on this investment?

16. Suppose that you have $1 million and the following two opportunities from which to construct a portfolio:
    a. Risk-free asset earning 12% per year.
    b. Risky asset with expected return of 30% per year and standard deviation of 40%.

    If you construct a portfolio with a standard deviation of 30%, what is its expected rate of return?

The following data are for Problems 17 through 19: The correlation coefficients between pairs of stocks are as follows: Corr(*A*,*B*) = .85; Corr(*A*,*C*) = .60; Corr(*A*,*D*) = .45. Each stock has an expected return of 8% and a standard deviation of 20%.

17. If your entire portfolio is now composed of stock *A* and you can add some of only one stock to your portfolio, would you choose (explain your choice):
    a. *B.*
    b. *C.*
    c. *D.*
    d. Need more data.

18. Would the answer to Problem 17 change for more risk-averse or risk-tolerant investors? Explain.

19. Suppose that in addition to investing in one more stock you can invest in T-bills as well. Would you change your answers to Problems 17 and 18 if the T-bill rate is 8%?

Challenge Problems

The following table of compound annual returns by decade applies to Challenge Problems 20 and 21.

	1920s*	1930s	1940s	1950s	1960s	1970s	1980s	1990s
Small-company stocks	−3.72%	7.28%	20.63%	19.01%	13.72%	8.75%	12.46%	13.84%
Large-company stocks	18.36	−1.25	9.11	19.41	7.84	5.90	17.60	18.20
Long-term government	3.98	4.60	3.59	0.25	1.14	6.63	11.50	8.60
Intermediate-term government	3.77	3.91	1.70	1.11	3.41	6.11	12.01	7.74
Treasury bills	3.56	0.30	0.37	1.87	3.89	6.29	9.00	5.02
Inflation	−1.00	−2.04	5.36	2.22	2.52	7.36	5.10	2.93

*Based on the period 1926–1929.

20. Input the data from the table into a spreadsheet. Compute the serial correlation in decade returns for each asset class and for inflation. Also find the correlation between the returns of various asset classes. What do the data indicate?

21. Convert the asset returns by decade presented in the table into real rates. Repeat the analysis of Challenge Problem 20 for the real rates of return.

CFA® PROBLEMS

The following data apply to CFA Problems 1 through 3: Hennessy & Associates manages a $30 million equity portfolio for the multimanager Wilstead Pension Fund. Jason Jones, financial vice president of Wilstead, noted that Hennessy had rather consistently achieved the best record among the Wilstead's six equity managers. Performance of the Hennessy portfolio had been clearly superior to that of the S&P 500 in 4 of the past 5 years. In the one less-favorable year, the shortfall was trivial.

Hennessy is a "bottom-up" manager. The firm largely avoids any attempt to "time the market." It also focuses on selection of individual stocks, rather than the weighting of favored industries.

There is no apparent conformity of style among the six equity managers. The five managers, other than Hennessy, manage portfolios aggregating $250 million made up of more than 150 individual issues.

Jones is convinced that Hennessy is able to apply superior skill to stock selection, but the favorable returns are limited by the high degree of diversification in the portfolio. Over the years, the portfolio generally held 40–50 stocks, with about 2%–3% of total funds committed to each issue. The reason Hennessy seemed to do well most years was that the firm was able to identify each year 10 or 12 issues that registered particularly large gains.

Based on this overview, Jones outlined the following plan to the Wilstead pension committee:

> Let's tell Hennessy to limit the portfolio to no more than 20 stocks. Hennessy will double the commitments to the stocks that it really favors, and eliminate the remainder. Except for this one new restriction, Hennessy should be free to manage the portfolio exactly as before.

All the members of the pension committee generally supported Jones's proposal because all agreed that Hennessy had seemed to demonstrate superior skill in selecting stocks. Yet the proposal was a considerable departure from previous practice, and several committee members raised questions. Respond to each of the following questions.

1. *a.* Will the limitation to 20 stocks likely increase or decrease the risk of the portfolio? Explain.
   b. Is there any way Hennessy could reduce the number of issues from 40 to 20 without significantly affecting risk? Explain.
2. One committee member was particularly enthusiastic concerning Jones's proposal. He suggested that Hennessy's performance might benefit further from reduction in the number of issues to 10. If the reduction to 20 could be expected to be advantageous, explain why reduction to 10 might be less likely to be advantageous. (Assume that Wilstead will evaluate the Hennessy portfolio independently of the other portfolios in the fund.)
3. Another committee member suggested that, rather than evaluate each managed portfolio independently of other portfolios, it might be better to consider the effects of a change in the Hennessy portfolio on the total fund. Explain how this broader point of view could affect the committee decision to limit the holdings in the Hennessy portfolio to either 10 or 20 issues.
4. Which one of the following portfolios cannot lie on the efficient frontier as described by Markowitz?

	Portfolio	Expected Return (%)	Standard Deviation (%)
a.	*W*	15	36
b.	*X*	12	15
c.	*Z*	5	7
d.	*Y*	9	21

5. Which statement about portfolio diversification is correct?
   a. Proper diversification can reduce or eliminate systematic risk.
   b. Diversification reduces the portfolio's expected return because it reduces a portfolio's total risk.
   c. As more securities are added to a portfolio, total risk typically would be expected to fall at a decreasing rate.
   d. The risk-reducing benefits of diversification do not occur meaningfully until at least 30 individual securities are included in the portfolio.
6. The measure of risk for a security held in a diversified portfolio is:
   a. Specific risk.
   b. Standard deviation of returns.
   c. Reinvestment risk.
   d. Covariance.
7. Portfolio theory as described by Markowitz is most concerned with:
   a. The elimination of systematic risk.
   b. The effect of diversification on portfolio risk.

c. The identification of unsystematic risk.
d. Active portfolio management to enhance return.

8. Assume that a risk-averse investor owning stock in Miller Corporation decides to add the stock of either Mac or Green Corporation to her portfolio. All three stocks offer the same expected return and total variability. The covariance of return between Miller and Mac is −.05 and between Miller and Green is +.05. Portfolio risk is expected to:
a. Decline more when the investor buys Mac.
b. Decline more when the investor buys Green.
c. Increase when either Mac or Green is bought.
d. Decline or increase, depending on other factors.

9. Stocks *A*, *B*, and *C* have the same expected return and standard deviation. The following table shows the correlations between the returns on these stocks.

	Stock *A*	Stock *B*	Stock *C*
Stock *A*	+1.0		
Stock *B*	+0.9	+1.0	
Stock *C*	+0.1	−0.4	+1.0

Given these correlations, the portfolio constructed from these stocks having the lowest risk is a portfolio:
a. Equally invested in stocks *A* and *B*.
b. Equally invested in stocks *A* and *C*.
c. Equally invested in stocks *B* and *C*.
d. Totally invested in stock *C*.

10. Statistics for three stocks, *A*, *B*, and *C*, are shown in the following tables.

Standard Deviations of Returns

Stock:	*A*	*B*	*C*
Standard deviation (%):	40	20	40

Correlations of Returns

Stock	*A*	*B*	*C*
A	1.00	0.90	0.50
B		1.00	0.10
C			1.00

Based *only* on the information provided in the tables, and given a choice between a portfolio made up of equal amounts of stocks *A* and *B* *or* a portfolio made up of equal amounts of stocks *B* and *C*, which portfolio would you recommend? Justify your choice.

11. George Stephenson's current portfolio of $2 million is invested as follows:

Summary of Stephenson's Current Portfolio

	Value	Percent of Total	Expected Annual Return	Annual Standard Deviation
Short-term bonds	$ 200,000	10%	4.6%	1.6%
Domestic large-cap equities	600,000	30%	12.4%	19.5%
Domestic small-cap equities	1,200,000	60%	16.0%	29.9%
Total portfolio	$2,000,000	100%	13.8%	23.1%

Stephenson soon expects to receive an additional $2 million and plans to invest the entire amount in an index fund that best complements the current portfolio. Stephanie Coppa, CFA, is evaluating the four index funds shown in the following table for their ability to produce a portfolio that will meet two criteria relative to the current portfolio: (1) maintain or enhance expected return and (2) maintain or reduce volatility.

Each fund is invested in an asset class that is not substantially represented in the current portfolio.

Index Fund Characteristics

Index Fund	Expected Annual Return	Expected Annual Standard Deviation	Correlation of Returns with Current Portfolio
Fund *A*	15%	25%	+0.80
Fund *B*	11	22	+0.60
Fund *C*	16	25	+0.90
Fund *D*	14	22	+0.65

State which fund Coppa should recommend to Stephenson. Justify your choice by describing how your chosen fund *best* meets both of Stephenson's criteria. No calculations are required.

12. Abigail Grace has a $900,000 fully diversified portfolio. She subsequently inherits ABC Company common stock worth $100,000. Her financial adviser provided her with the following forecast information:

Risk and Return Characteristics

	Expected Monthly Returns	Standard Deviation of Monthly Returns
Original Portfolio	0.67%	2.37%
ABC Company	1.25	2.95

The correlation coefficient of ABC stock returns with the original portfolio returns is .40.

a. The inheritance changes Grace's overall portfolio and she is deciding whether to keep the ABC stock. Assuming Grace keeps the ABC stock, calculate the:

i. Expected return of her new portfolio which includes the ABC stock.
ii. Covariance of ABC stock returns with the original portfolio returns.
iii. Standard deviation of her new portfolio which includes the ABC stock.

b. If Grace sells the ABC stock, she will invest the proceeds in risk-free government securities yielding .42% monthly. Assuming Grace sells the ABC stock and replaces it with the government securities, calculate the

i. Expected return of her new portfolio, which includes the government securities.
ii. Covariance of the government security returns with the original portfolio returns.
iii. Standard deviation of her new portfolio, which includes the government securities.

c. Determine whether the systematic risk of her new portfolio, which includes the government securities, will be higher or lower than that of her original portfolio.

d. Based on conversations with her husband, Grace is considering selling the $100,000 of ABC stock and acquiring $100,000 of XYZ Company common stock instead. XYZ stock has the same expected return and standard deviation as ABC stock. Her husband comments, "It doesn't matter whether you keep all of the ABC stock or replace it with $100,000 of XYZ stock." State whether her husband's comment is correct or incorrect. Justify your response.

e. In a recent discussion with her financial adviser, Grace commented, "If I just don't lose money in my portfolio, I will be satisfied." She went on to say, "I am more afraid of losing money than I am concerned about achieving high returns."

i. Describe *one* weakness of using standard deviation of returns as a risk measure for Grace.
ii. Identify an alternate risk measure that is more appropriate under the circumstances.

13. Dudley Trudy, CFA, recently met with one of his clients. Trudy typically invests in a master list of 30 equities drawn from several industries. As the meeting concluded, the client made the following statement: "I trust your stock-picking ability and believe that you should invest my funds in your five best ideas. Why invest in 30 companies when you obviously have stronger opinions on a few of them?" Trudy plans to respond to his client within the context of Modern Portfolio Theory.
   a. Contrast the concepts of systematic risk and firm-specific risk, and give an example of *each* type of risk.
   b. Critique the client's suggestion. Discuss how both systematic and firm-specific risk change as the number of securities in a portfolio is increased.

E-Investments

Diversification

Go to the **www.investopedia.com/articles/basics/03/050203.asp** Web site to learn more about diversification, the factors that influence investors' risk preferences, and the types of investments that fit into each of the risk categories. Then check out **www.investopedia.com/articles/pf/05/061505.asp** for asset allocation guidelines for various types of portfolios from conservative to very aggressive. What do you conclude about your own risk preferences and the best portfolio type for you? What would you expect to happen to your attitude toward risk as you get older? How might your portfolio composition change?

SOLUTIONS TO CONCEPT CHECKS

1. *a.* The first term will be $w_D \times w_D \times \sigma_D^2$, because this is the element in the top corner of the matrix (σ_D^2) times the term on the column border (w_D) times the term on the row border (w_D). Applying this rule to each term of the covariance matrix results in the sum $w_D^2\sigma_D^2 + w_D w_E \text{Cov}(r_E, r_D) + w_E w_D \text{Cov}(r_D, r_E) + w_E^2\sigma_E^2$, which is the same as Equation 7.3, because $\text{Cov}(r_E, r_D) = \text{Cov}(r_D, r_E)$.
   b. The bordered covariance matrix is

	w_X	w_Y	w_Z
w_X	σ_X^2	$\text{Cov}(r_X, r_Y)$	$\text{Cov}(r_X, r_Z)$
w_Y	$\text{Cov}(r_Y, r_X)$	σ_Y^2	$\text{Cov}(r_Y, r_Z)$
w_Z	$\text{Cov}(r_Z, r_X)$	$\text{Cov}(r_Z, r_Y)$	σ_Z^2

There are nine terms in the covariance matrix. Portfolio variance is calculated from these nine terms:

$$\begin{aligned}
\sigma_P^2 &= w_X^2\sigma_X^2 + w_Y^2\sigma_Y^2 + w_Z^2\sigma_Z^2 \\
&\quad + w_X w_Y \text{Cov}(r_X, r_Y) + w_Y w_X \text{Cov}(r_Y, r_X) \\
&\quad + w_X w_Z \text{Cov}(r_X, r_Z) + w_Z w_X \text{Cov}(r_Z, r_X) \\
&\quad + w_Y w_Z \text{Cov}(r_Y, r_Z) + w_Z w_Y \text{Cov}(r_Z, r_Y) \\
&= w_X^2\sigma_X^2 + w_Y^2\sigma_Y^2 + w_Z^2\sigma_Z^2 \\
&\quad + 2w_X w_Y \text{Cov}(r_X, r_Y) + 2w_X w_Z \text{Cov}(r_X, r_Z) + 2w_Y w_Z \text{Cov}(r_Y, r_Z)
\end{aligned}$$

2. The parameters of the opportunity set are $E(r_D) = 8\%$, $E(r_E) = 13\%$, $\sigma_D = 12\%$, $\sigma_E = 20\%$, and $\rho(D,E) = .25$. From the standard deviations and the correlation coefficient we generate the covariance matrix:

Fund	*D*	*E*
D	144	60
E	60	400

The *global minimum-variance* portfolio is constructed so that

$$w_D = \frac{\sigma_E^2 - \text{Cov}(r_D, r_E)}{\sigma_D^2 + \sigma_E^2 - 2\,\text{Cov}(r_D, r_E)}$$

$$= \frac{400 - 60}{(144 + 400) - (2 \times 60)} = .8019$$

$$w_E = 1 - w_D = .1981$$

Its expected return and standard deviation are

$$E(r_P) = (.8019 \times 8) + (.1981 \times 13) = 8.99\%$$

$$\sigma_P = [w_D^2\sigma_D^2 + w_E^2\sigma_E^2 + 2w_D w_E \text{Cov}(r_D, r_E)]^{1/2}$$

$$= [(.8019^2 \times 144) + (.1981^2 \times 400) + (2 \times .8019 \times .1981 \times 60)]^{1/2}$$

$$= 11.29\%$$

For the other points we simply increase w_D from .10 to .90 in increments of .10; accordingly, w_E ranges from .90 to .10 in the same increments. We substitute these portfolio proportions in the formulas for expected return and standard deviation. Note that when $w_E = 1.0$, the portfolio parameters equal those of the stock fund; when $w_D = 1$, the portfolio parameters equal those of the debt fund.

We then generate the following table:

w_E	w_D	*E(r)*	σ
0.0	1.0	8.0	12.00
0.1	0.9	8.5	11.46
0.2	0.8	9.0	11.29
0.3	0.7	9.5	11.48
0.4	0.6	10.0	12.03
0.5	0.5	10.5	12.88
0.6	0.4	11.0	13.99
0.7	0.3	11.5	15.30
0.8	0.2	12.0	16.76
0.9	0.1	12.5	18.34
1.0	0.0	13.0	20.00
0.1981	0.8019	8.99	11.29 minimum variance portfolio

You can now draw your graph.

3. *a.* The computations of the opportunity set of the stock and risky bond funds are like those of Question 2 and will not be shown here. You should perform these computations, however, in order to give a graphical solution to part *a*. Note that the covariance between the funds is

$$\text{Cov}(r_A, r_B) = \rho(A, B) \times \sigma_A \times \sigma_B$$

$$= -.2 \times 20 \times 60 = -240$$

b. The proportions in the optimal risky portfolio are given by

$$w_A = \frac{(10-5)60^2 - (30-5)(-240)}{(10-5)60^2 + (30-5)20^2 - 30(-240)}$$
$$= .6818$$
$$w_B = 1 - w_A = .3182$$

The expected return and standard deviation of the optimal risky portfolio are

$$E(r_P) = (.6818 \times 10) + (.3182 \times 30) = 16.36\%$$
$$\sigma_P = \{(.6818^2 \times 20^2) + (.3182^2 \times 60^2) + [2 \times .6818 \times .3182(-240)]\}^{1/2}$$
$$= 21.13\%$$

Note that in this case the standard deviation of the optimal risky portfolio is smaller than the standard deviation of stock *A*. Note also that portfolio *P* is not the global minimum-variance portfolio. The proportions of the latter are given by

$$w_A = \frac{60^2 - (-240)}{60^2 + 20^2 - 2(-240)} = .8571$$
$$w_B = 1 - w_A = .1429$$

With these proportions, the standard deviation of the minimum-variance portfolio is

$$\sigma(\min) = \{(.8571^2 \times 20^2) + (.1429^2 \times 60^2) + [2 \times .8571 \times .1429 \times (-240)]\}^{1/2}$$
$$= 17.57\%$$

which is less than that of the optimal risky portfolio.

c. The CAL is the line from the risk-free rate through the optimal risky portfolio. This line represents all efficient portfolios that combine T-bills with the optimal risky portfolio. The slope of the CAL is

$$S = \frac{E(r_P) - r_f}{\sigma_P} = \frac{16.36 - 5}{21.13} = .5376$$

d. Given a degree of risk aversion, *A*, an investor will choose a proportion, *y*, in the optimal risky portfolio of (remember to express returns as decimals when using *A*):

$$y = \frac{E(r_P) - r_f}{A\sigma_P^2} = \frac{.1636 - .05}{5 \times .2113^2} = .5089$$

This means that the optimal risky portfolio, with the given data, is attractive enough for an investor with $A = 5$ to invest 50.89% of his or her wealth in it. Because stock *A* makes up 68.18% of the risky portfolio and stock *B* makes up 31.82%, the investment proportions for this investor are

Stock *A*:	.5089 × 68.18 =	34.70%
Stock *B*:	.5089 × 31.82 =	16.19%
Total		50.89%

4. Efficient frontiers derived by portfolio managers depend on forecasts of the rates of return on various securities and estimates of risk, that is, the covariance matrix. The forecasts themselves do not control outcomes. Thus preferring managers with rosier forecasts (northwesterly frontiers) is tantamount to rewarding the bearers of good news and punishing the bearers of bad news. What we should do is reward bearers of *accurate* news. Thus if you get a glimpse of the frontiers (forecasts) of portfolio managers on a regular basis, what you want to do is develop the track record of their forecasting accuracy and steer your advisees toward the more accurate forecaster. Their portfolio choices will, in the long run, outperform the field.
5. The parameters are $E(r) = 15$, $\sigma = 60$, and the correlation between any pair of stocks is $\rho = .5$.

a. The portfolio expected return is invariant to the size of the portfolio because all stocks have identical expected returns. The standard deviation of a portfolio with $n = 25$ stocks is

$$\sigma_P = [\sigma^2/n + \rho \times \sigma^2(n-1)/n]^{1/2}$$
$$= [60^2/25 + .5 \times 60^2 \times 24/25]^{1/2} = 43.27\%$$

b. Because the stocks are identical, efficient portfolios are equally weighted. To obtain a standard deviation of 43%, we need to solve for *n:*

$$43^2 = \frac{60^2}{n} + .5 \times \frac{60^2(n-1)}{n}$$
$$1{,}849n = 3{,}600 + 1{,}800n - 1{,}800$$
$$n = \frac{1{,}800}{49} = 36.73$$

Thus we need 37 stocks and will come in with volatility slightly under the target.

c. As n gets very large, the variance of an efficient (equally weighted) portfolio diminishes, leaving only the variance that comes from the covariances among stocks, that is

$$\sigma_P = \sqrt{\rho \times \sigma^2} = \sqrt{.5 \times 60^2} = 42.43\%$$

Note that with 25 stocks we came within .84% of the systematic risk, that is, the nonsystematic risk of a portfolio of 25 stocks is only .84%. With 37 stocks the standard deviation is 43%, of which nonsystematic risk is .57%.

d. If the risk-free is 10%, then the risk premium on any size portfolio is 15 − 10 = 5%. The standard deviation of a well-diversified portfolio is (practically) 42.43%; hence the slope of the CAL is

$$S = 5/42.43 = .1178$$

APPENDIX A: A Spreadsheet Model for Efficient Diversification

Several software packages can be used to generate the efficient frontier. We will demonstrate the method using Microsoft Excel. Excel is far from the best program for this purpose and is limited in the number of assets it can handle, but working through a simple portfolio optimizer in Excel can illustrate concretely the nature of the calculations used in more sophisticated "black-box" programs. You will find that even in Excel, the computation of the efficient frontier is fairly easy.

We apply the Markowitz portfolio optimization program to a practical problem of international diversification. We take the perspective of a portfolio manager serving U.S. clients, who wishes to construct for the next year an optimal risky portfolio of large stocks in the U.S and six developed capital markets (Japan, Germany, U.K., France, Canada, and Australia). First we describe the input list: forecasts of risk premiums and the covariance matrix. Next, we describe Excel's Solver, and finally we show the solution to the manager's problem.

The Covariance Matrix

To capture recent risk parameters the manager compiles an array of 60 recent monthly (annualized) rates of return, as well as the monthly T-bill rates for the same period.

The standard deviations of excess returns are shown in Table 7A.1 (column C). They range from 14.93% (U.K. large stocks) to 22.7% (Germany). For perspective on how these parameters can change over time, standard deviations for the period 1991–2000 are also shown (column B). In addition, we present the correlation coefficient between large stocks in the six foreign markets with U.S. large stocks for the same two periods. Here we see that correlations are higher in the more recent period, consistent with the process of globalization.

The covariance matrix shown in Table 7A.2 was estimated from the array of 60 returns of the seven countries using the COVARIANCE function from the dialog box of *Data Analysis* in Excel's Tools menu. Due to a quirk in the Excel software, the covariance matrix is not corrected for degrees-of-freedom bias; hence, each of the elements in the matrix was multiplied by 60/59 to eliminate downward bias.

Expected Returns

While estimation of the risk parameters (the covariance matrix) from excess returns is a simple technical matter, estimating the risk premium (the expected excess return) is a daunting task. As we discussed in Chapter 5, estimating expected returns using historical data is unreliable. Consider, for example, the negative average excess returns on U.S. large stocks over the period 2001–2005 (cell G6) and, more generally, the big differences in average returns between the 1991–2000 and 2001–2005 periods, as demonstrated in columns F and G.

In this example, we simply present the manager's forecasts of future returns as shown in column H. In Chapter 8 we will establish a framework that makes the forecasting process more explicit.

The Bordered Covariance Matrix and Portfolio Variance

The covariance matrix in Table 7A.2 is bordered by the portfolio weights, as explained in Section 7.2 and Table 7.2. The values in cells A18–A24, to the left of the covariance matrix, will be selected by the optimization program. For now, we arbitrarily input 1.0 for the U.S. and zero for the others. Cells A16–I16, above the covariance matrix, must be set equal to the column of weights on the left, so that they will change in tandem as the column weights are changed by Excel's Solver. Cell A25 sums the column weights and is used to force the optimization program to set the sum of portfolio weights to 1.0.

Cells C25–I25, below the covariance matrix, are used to compute the portfolio variance for any set of weights that appears in the borders. Each cell accumulates the contribution to portfolio variance from the column above it. It uses the function SUMPRODUCT to accomplish this task. For example, row 33 shows the formula used to derive the value that appears in cell C25.

Finally, the short column A26–A28 below the bordered covariance matrix presents portfolio statistics computed from the bordered covariance matrix. First is the portfolio risk premium in cell A26, with formula shown in row 35, which multiplies the column of portfolio weights by the column of forecasts (H6–H12) from Table 7A.1. Next is the portfolio standard deviation in cell A27. The variance is given by the sum of cells C25–I25 below the bordered covariance matrix. Cell A27 takes the square root of this sum to produce the standard deviation. The last statistic is the portfolio Sharpe ratio, cell A28, which is the slope of the CAL (capital allocation line) that runs through the portfolio constructed using the column weights (the value in cell A28 equals cell A26 divided by cell A27). The optimal risky portfolio is the one that maximizes the Sharpe ratio.

Using the Excel Solver

Excel's Solver is a user-friendly, but quite powerful, optimizer. It has three parts: (1) an objective function, (2) decision variables, and (3) constraints. Figure 7A.1 shows three pictures of the Solver. For the current discussion we refer to picture A.

	A	B	C	D	E	F	G	H
1								
2								
3	**7A.1 Country Index Statistics and Forecasts of Excess Returns**							
4		**Standard Deviation**		**Correlation with the U.S.**		**Average Excess Return**		**Forecast**
5	**Country**	1991–2000	2001–2005	1991–2000	2001–2005	1991–2000	2001–2005	2006
6	US	0.1295	0.1495	1	1	0.1108	−0.0148	0.0600
7	UK	0.1466	0.1493	0.64	0.83	0.0536	0.0094	0.0530
8	France	0.1741	0.2008	0.54	0.83	0.0837	0.0247	0.0700
9	Germany	0.1538	0.2270	0.53	0.85	0.0473	0.0209	0.0800
10	Australia	0.1808	0.1617	0.52	0.81	0.0468	0.1225	0.0580
11	Japan	0.2432	0.1878	0.41	0.43	−0.0177	0.0398	0.0450
12	Canada	0.1687	0.1727	0.72	0.79	0.0727	0.1009	0.0590

	A	B	C	D	E	F	G	H	I
13									
14	**7A.2 The Bordered Covariance Matrix**								
15									
16	**Portfolio Weights** →		**1.0000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**
17	↓		**US**	**UK**	**France**	**Germany**	**Australia**	**Japan**	**Canada**
18	1.0000	US	0.0224	0.0184	0.0250	0.0288	0.0195	0.0121	0.0205
19	0.0000	UK	0.0184	0.0223	0.0275	0.0299	0.0204	0.0124	0.0206
20	0.0000	France	0.0250	0.0275	0.0403	0.0438	0.0259	0.0177	0.0273
21	0.0000	Germany	0.0288	0.0299	0.0438	0.0515	0.0301	0.0183	0.0305
22	0.0000	Australia	0.0195	0.0204	0.0259	0.0301	0.0261	0.0147	0.0234
23	0.0000	Japan	0.0121	0.0124	0.0177	0.0183	0.0147	0.0353	0.0158
24	0.0000	Canada	0.0205	0.0206	0.0273	0.0305	0.0234	0.0158	0.0298
25	1.0000		0.0224	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
26	0.0600	Mean							
27	0.1495	SD							
28	0.4013	Slope							
29									
30	Cell A18 - A24		A18 is set arbitrarily to 1 while A19 to A24 are set to 0						
31	Formula in cell	C16	=A18	...	Formula in cell I16	= A24			
32	Formula in cell	A25	=SUM(A18:A24)						
33	Formula in cell	C25	=C16*SUMPRODUCT($A$18:$A$24,C18:C24)						
34	Formula in cell	D25-I25	Copied from C25 (note the absolute addresses)						
35	Formula in cell	A26	=SUMPRODUCT($A$18:$A$24,H6:H12)						
36	Formula in cell	A27	=SUM(C25:I25)^0.5						
37	Formula in cell	A28	=A26/A27						
38									

	A	B	C	D	E	F	G	H	I	J	K	L
39	**7A.3 The Efficient Frontier**											
40												
41	Cell to store constraint on risk premium				0.0400							
42												
43			**Min Var**					**Optimum**				
44	**Mean**		0.0383	0.0400	0.0450	0.0500	0.0550	0.0564	0.0575	0.0600	0.0700	0.0800
45	**SD**	0.1	0.1132	0.1135	0.1168	0.1238	0.1340	0.1374	0.1401	0.1466	0.1771	0.2119
46	**Slope**		0.3386	0.3525	0.3853	0.4037	0.4104	0.4107	0.4106	0.4092	0.3953	0.3774
47	US		0.6112	0.6195	0.6446	0.6696	0.6947	0.7018	0.7073	0.7198	0.7699	0.8201
48	UK		0.8778	0.8083	0.5992	0.3900	0.1809	0.1214	0.0758	−0.0283	−0.4465	−0.8648
49	France		−0.2140	−0.2029	−0.1693	−0.1357	−0.1021	−0.0926	−0.0852	−0.0685	−0.0014	0.0658
50	Germany		−0.5097	−0.4610	−0.3144	−0.1679	−0.0213	0.0205	0.0524	0.1253	0.4185	0.7117
51	Australia		0.0695	0.0748	0.0907	0.1067	0.1226	0.1271	0.1306	0.1385	0.1704	0.2023
52	Japan		0.2055	0.1987	0.1781	0.1575	0.1369	0.1311	0.1266	0.1164	0.0752	0.0341
53	Canada		−0.0402	−0.0374	−0.0288	−0.0203	−0.0118	−0.0093	−0.0075	−0.0032	0.0139	0.0309
54	CAL*	0.0411	0.0465	0.0466	0.0480	0.0509	0.0550	0.0564	0.0575	0.0602	0.0727	0.0871
55	*Risk premium on CAL = SD × slope of optimal risky portfolio											

TABLE 7A.1, 7A.2, 7A.3

Spreadsheet model for international diversification

The top panel of the Solver lets you choose a target cell for the "objective function," that is, the variable you are trying to optimize. In picture A, the target cell is A27, the portfolio standard deviation. Below the target cell, you can choose whether your objective is to maximize, minimize, or set your objective function equal to a value that you specify. Here we choose to minimize the portfolio standard deviation.

The next panel contains the decision variables. These are cells that the Solver can change in order to optimize the objective function in the target cell. Here, we input cells A18–A24, the portfolio weights that we select to minimize portfolio volatility.

The bottom panel of the Solver can include any number of constraints. One constraint that must always appear in portfolio optimization is the "feasibility constraint," namely, that portfolio weights sum to 1.0. When we bring up the constraint dialogue box, we specify that cell A25 (the sum of weights) be set equal to 1.0.

Finding the Minimum Variance Portfolio

It is helpful to begin by identifying the global minimum variance portfolio (*G*). This provides the starting point of the efficient part of the frontier. Once we input the target cell, the decision variable cells, and the feasibility constraint, as in picture A, we can select "solve" and the Solver returns portfolio *G*. We copy the portfolio statistics and weights to our output Table 7A.3. Column C in Table 7A.3 shows that the lowest standard deviation (SD) that can be achieved with our input list is 11.32%. Notice that the SD of portfolio *G* is considerably lower than even the lowest SD of the individual indexes. From the risk premium of portfolio *G* (3.83%) we begin building the efficient frontier with ever-larger risk premiums.

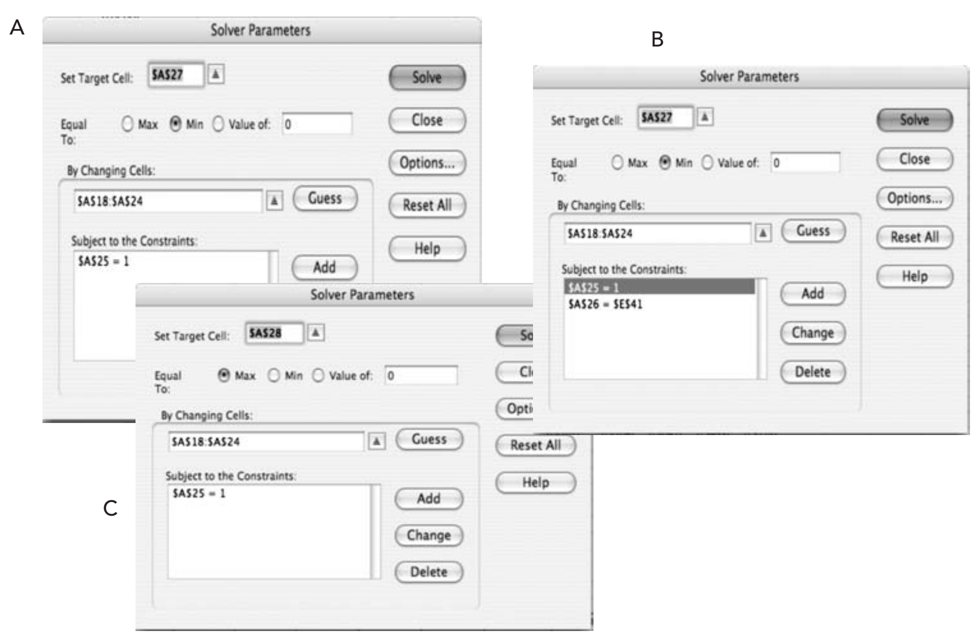


FIGURE 7A.1 Solver dialog box

Charting the Efficient Frontier of Risky Portfolios

We determine the desired risk premiums (points on the efficient frontier) that we wish to use to construct the graph of the efficient frontier. It is good practice to choose more points in the neighborhood of portfolio *G* because the frontier has the greatest curvature in that region. It is sufficient to choose for the highest point the highest risk premium from the input list (here, 8% for Germany). You can produce the entire efficient frontier in minutes following this procedure.

1. Input to the Solver a constraint that says: Cell A26 (the portfolio risk premium) must equal the value in cell E41. The Solver at this point is shown in picture B of Figure 7A.1. Cell E41 will be used to change the required risk premium and thus generate different points along the frontier.
2. For each additional point on the frontier, you input a different desired risk premium into cell E41, and ask the Solver to solve again.
3. Every time the Solver gives you a solution to the request in (2), copy the results into Table 7A.3, which tabulates the collection of points along the efficient frontier. For the next step, change cell E41 and repeat from step 2.

Finding the Optimal Risky Portfolio on the Efficient Frontier

Now that we have an efficient frontier, we look for the portfolio with the highest Sharpe ratio (i.e., reward-to-volatility ratio). This is the efficient frontier portfolio that is tangent to the CAL. To find it, we just need to make two changes to the Solver. First, change the target cell from cell A27 to cell A28, the Sharpe ratio of the portfolio, and request that the value in this cell be maximized. Next, eliminate the constraint on the risk premium that may be left over from the last time you used the Solver. At this point the Solver looks like picture C in Figure 7A.1.

The Solver now yields the optimal risky portfolio. Copy the statistics for the optimal portfolio and its weights to your Table 7A.3. In order to get a clean graph, place the column of the optimal portfolio in Table 7A.3 so that the risk premiums of all portfolios in the table are steadily increasing from the risk premium of portfolio *G* (3.83%) all the way up to 8%.

The efficient frontier is graphed using the data in cells C45–I45 (the horizontal or *x*-axis is portfolio standard deviation) and C44–I44 (the vertical or *y*-axis is portfolio risk premium). The resulting graph appears in Figure 7A.2.

The Optimal CAL

It is instructive to superimpose on the graph of the efficient frontier in Figure 7A.2 the CAL that identifies the optimal risky portfolio. This CAL has a slope equal to the Sharpe ratio of the optimal risky portfolio. Therefore, we add at the bottom of Table 7A.3 a row with entries obtained by multiplying the SD of each column's portfolio by the Sharpe ratio of the optimal risky portfolio from cell H46. This results in the risk premium for each portfolio along the CAL efficient frontier. We now add a series to the graph with the standard deviations in B45–I45 as the *x*-axis and cells B54–I54 as the *y*-axis. You can see this CAL in Figure 7A.2.

The Optimal Risky Portfolio and the Short-Sales Constraint

With the input list used by the portfolio manager, the optimal risky portfolio calls for significant short positions in the stocks of France and Canada (see column H of Table 7A.3). In many cases the portfolio manager is prohibited from taking short positions. If so, we need to amend the program to preclude short sales.

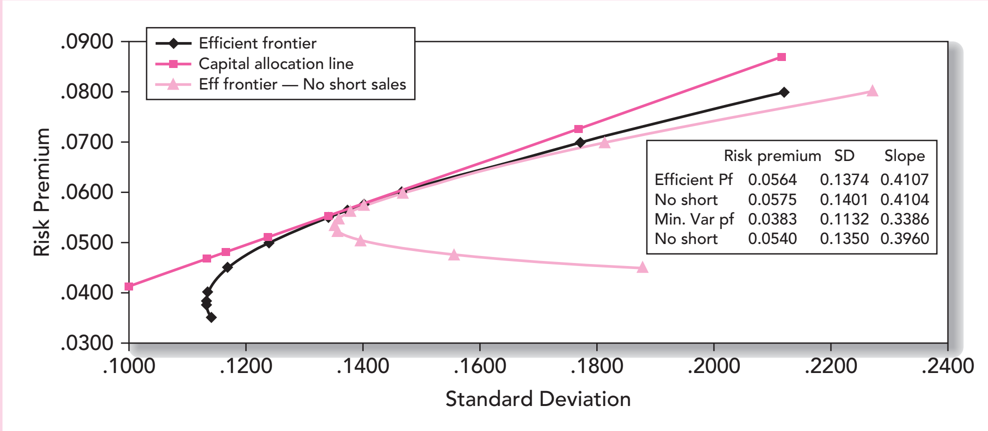


FIGURE 7A.2 Efficient frontier and CAL for country stock indexes

To accomplish this task, we repeat the exercise, but with one change. We add to the Solver the following constraint: Each element in the column of portfolio weights, A18–A24, must be greater than or equal to zero. You should try to produce the short-sale constrained efficient frontier in your own spreadsheet. The graph of the constrained frontier is also shown in Figure 7A.2.

APPENDIX B: Review of Portfolio Statistics

We base this review of scenario analysis on a two-asset portfolio. We denote the assets *D* and *E* (which you may think of as debt and equity), but the risk and return parameters we use in this appendix are not necessarily consistent with those used in Section 7.2.

Expected Returns

We use "expected value" and "mean" interchangeably. For an analysis with n scenarios, where the rate of return in scenario i is $r(i)$ with probability $p(i)$, the expected return is

$$E(r) = \sum_{i=1}^{n} p(i)r(i) \qquad \textbf{(7B.1)}$$

If you were to increase the rate of return assumed for each scenario by some amount Δ, then the mean return will increase by Δ. If you multiply the rate in each scenario by a factor w, the new mean will be multiplied by that factor:

$$\sum_{i=1}^{n} p(i)\times[r(i)+\Delta] = \sum_{i=1}^{n} p(i)\times r(i) + \Delta\sum_{i=1}^{n} p(i) = E(r)+\Delta \qquad \textbf{(7B.2)}$$

$$\sum_{i=1}^{n} p(i)\times[wr(i)] = w\sum_{i=1}^{n} p(i)\times r(i) = wE(r)$$

	A	B	C	D	E	F	G
1							
2			Scenario rates of return				
3	Scenario	Probability	$r_D(i)$	$r_D(i)$ + 0.03	0.4*$r_D(i)$		
4	1	0.14	–0.10	–0.07	–0.040		
5	2	0.36	0.00	0.03	0.000		
6	3	0.30	0.10	0.13	0.040		
7	4	0.20	0.32	0.35	0.128		
8		Mean	0.080	0.110	0.032		
9		Cell C8	=SUMPRODUCT($B$4:$B$7,C4:C7)				
10							
11							
12							

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TABLE 7B.1

Scenario analysis for bonds

EXAMPLE 7B.1 Expected Rates of Return

Column C of Table 7B.1 shows scenario rates of return for debt, *D*. In column D we add 3% to each scenario return and in column E we multiply each rate by .4. The table shows how we compute the expected return for columns C, D, and E. It is evident that the mean increases by 3% (from .08 to .11) in column D and is multiplied by .4 (from .08 to 0.032) in column E.

Now let's construct a portfolio that invests a fraction of the investment budget, $w(D)$, in bonds and the fraction $w(E)$ in stocks. The portfolio's rate of return in each scenario and its expected return are given by

$$r_P(i) = w_D r_D(i) + w_E r_E(i) \tag{7B.3}$$

$$\begin{aligned} E(r_P) &= \sum p(i)[w_D r_D(i) + w_E r_E(i)] = \sum p(i) w_D r_D(i) + \sum p(i) w_E r_E(i) \\ &= w_D E(r_D) + w_E E(r_E) \end{aligned}$$

The rate of return on the portfolio in each scenario is the weighted average of the component rates. The weights are the fractions invested in these assets, that is, the portfolio weights. The expected return on the portfolio is the weighted average of the asset means.

EXAMPLE 7B.2 Portfolio Rate of Return

Table 7B.2 lays out rates of return for both stocks and bonds. Using assumed weights of .4 for debt and .6 for equity, the portfolio return in each scenario appears in column L. Cell L8 shows the portfolio expected return as .1040, obtained using the SUMPRODUCT function, which multiplies each scenario return (column L) by the scenario probability (column I) and sums the results.

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	H	I	J	K	L
1					
2			Scenario rates of return		Portfolio return
3	Scenario	Probability	r_D(i)	r_E(i)	0.4*r_D(i)+0.6*r_E(i)
4	1	0.14	–0.10	–0.35	–0.2500
5	2	0.36	0.00	0.20	0.1200
6	3	0.30	0.10	0.45	0.3100
7	4	0.20	0.32	–0.19	0.0140
8		Mean	0.08	0.12	0.1040
9		Cell L4	=0.4*J4+0.6*K4		
10		Cell L8	=SUMPRODUCT($I$4:$I$7,L4:L7)		
11					
12					

TABLE 7B.2

Scenario analysis for bonds and stocks

Variance and Standard Deviation

The variance and standard deviation of the rate of return on an asset from a scenario analysis are given by[17]

$$\sigma^2(r) = \sum_{i=1}^{n} p(i)[r(i) - E(r)]^2 \tag{7B.4}$$

$$\sigma(r) = \sqrt{\sigma^2(r)}$$

Notice that the unit of variance is percent squared. In contrast, standard deviation, the square root of variance, has the same dimension as the original returns, and therefore is easier to interpret as a measure of return variability.

When you add a fixed incremental return, Δ, to each scenario return, you increase the mean return by that same increment. Therefore, the deviation of the realized return in each scenario from the mean return is unaffected, and both variance and SD are unchanged. In contrast, when you multiply the return in each scenario by a factor *w*, the variance is multiplied by the square of that factor (and the SD is multiplied by *w*):

$$\text{Var}(wr) = \sum_{i=1}^{n} p(i) \times [wr(i) - E(wr)]^2 = w^2 \sum_{i=1}^{n} p(i)[r(i) - E(r)]^2 = w^2\sigma^2 \tag{7B.5}$$

$$\text{SD}(wr) = \sqrt{w^2\sigma^2} = w\sigma(r)$$

Excel does not have a direct function to compute variance and standard deviation for a scenario analysis. Its STDEV and VAR functions are designed for time series. We need to calculate the probability-weighted squared deviations directly. To avoid having

[17]Variance (here, of an asset rate of return) is not the only possible choice to quantify variability. An alternative would be to use the *absolute* deviation from the mean instead of the *squared* deviation. Thus, the mean absolute deviation (MAD) is sometimes used as a measure of variability. The variance is the preferred measure for several reasons. First, it is mathematically more difficult to work with absolute deviations. Second, squaring deviations gives more weight to larger deviations. In investments, giving more weight to large deviations (hence, losses) is compatible with risk aversion. Third, when returns are normally distributed, the variance is one of the two parameters that fully characterize the distribution.

	A	B	C	D	E	F	G
13							
14			Scenario rates of return				
15	Scenario	Probability	$r_D(i)$	$r_D(i)$ + 0.03	0.4*$r_D(i)$		
16	1	0.14	–0.10	–0.07	–0.040		
17	2	0.36	0.00	0.03	0.000		
18	3	0.30	0.10	0.13	0.040		
19	4	0.20	0.32	0.35	0.128		
20		Mean	0.0800	0.1100	0.0240		
21		Variance	0.0185	0.0185	0.0034		
22		SD	0.1359	0.1359	0.0584		
23	Cell C21	=SUMPRODUCT($B$16:$B$19,C16:C19,C16:C19)–C20^2					
24	Cell C22	=C21^0.5					

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TABLE 7B.3

Scenario analysis for bonds

to first compute columns of squared deviations from the mean, however, we can simplify our problem by expressing the variance as a difference between two easily computable terms:

$$\begin{aligned}\sigma^2(r) &= E[r - E(r)]^2 = E\{r^2 + [E(r)]^2 - 2rE(r)\} \\ &= E(r^2) + [E(r)]^2 - 2E(r)E(r) \\ &= E(r^2) - [E(r)]^2 = \sum_{i=1}^{n} p(i)r(i)^2 - \left[\sum_{i=1}^{n} p(i)r(i)\right]^2\end{aligned} \quad \textbf{(7B.6)}$$

EXAMPLE 7B.3 Calculating the Variance of a Risky Asset in Excel

You can compute the first expression, $E(r^2)$, in Equation 7B.6 using Excel's SUMPRODUCT function. For example, in Table 7B.3, $E(r^2)$ is first calculated in cell C21 by using SUMPRODUCT to multiply the scenario probability times the asset return times the asset return again. Then $[E(r)]^2$ is subtracted (notice the subtraction of C20^2 in cell C21), to arrive at variance.

The variance of a *portfolio* return is not as simple to compute as the mean. The portfolio variance is *not* the weighted average of the asset variances. The deviation of the portfolio rate of return in any scenario from its mean return is

$$\begin{aligned}r_P - E(r_P) &= w_D r_D(i) + w_E r_E(i) - [w_D E(r_D) + w_E E(r_E)] \\ &= w_D[r_D(i) - E(r_D)] + w_E[r_E(i) - E(r_E)] \\ &= w_D d(i) + w_E e(i)\end{aligned} \quad \textbf{(7B.7)}$$

where the lowercase variables denote deviations from the mean:

$$d(i) = r_D(i) - E(r_D)$$
$$e(i) = r_E(i) - E(r_E)$$

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We express the variance of the portfolio return in terms of these deviations from the mean in Equation 7B.7:

$$\begin{aligned}\sigma_P^2 &= \sum_{i=1}^{n} p(i)[r_P - E(r_P)]^2 = \sum_{i=1}^{n} p(i)[w_D d(i) + w_E e(i)]^2 \\ &= \sum_{i=1}^{n} p(i)[w_D^2 d(i)^2 + w_E^2 e(i)^2 + 2w_D w_E d(i)e(i)] \\ &= w_D^2 \sum_{i=1}^{n} p(i)d(i)^2 + w_E^2 \sum_{i=1}^{n} p(i)e(i)^2 + 2w_D w_E \sum_{i=1}^{n} p(i)d(i)e(i) \\ &= w_D^2\sigma_D^2 + w_E^2\sigma_E^2 + 2w_D w_E \sum_{i=1}^{n} p(i)d(i)e(i)\end{aligned} \quad \textbf{(7B.8)}$$

The last line in Equation 7B.8 tells us that the variance of a portfolio is the weighted sum of portfolio variances (notice that the weights are the squares of the portfolio weights), plus an additional term that, as we will soon see, makes all the difference.

Notice also that $d(i) \times e(i)$ is the product of the deviations of the scenario returns of the two assets from their respective means. The probability-weighted average of this product is its expected value, which is called *covariance* and is denoted $\text{Cov}(r_D, r_E)$. The covariance between the two assets can have a big impact on the variance of a portfolio.

Covariance

The covariance between two variables equals

$$\begin{aligned}\text{Cov}(r_D, r_E) &= E(d \times e) = E\{[r_D - E(r_D)][r_E - E(r_E)]\} \\ &= E(r_D r_E) - E(r_D)E(r_E)\end{aligned} \quad \textbf{(7B.9)}$$

The covariance is an elegant way to quantify the covariation of two variables. This is easiest seen through a numerical example.

Imagine a three-scenario analysis of stocks and bonds as given in Table 7B.4. In scenario 1, bonds go down (negative deviation) while stocks go up (positive deviation). In scenario 3, bonds are up, but stocks are down. When the rates move in opposite directions, as in this case, the product of the deviations is negative; conversely, if the rates moved in the same direction, the sign of the product would be positive. The magnitude of the product shows the extent of the opposite or common movement in that scenario. The probability-weighted average of these products therefore summarizes the *average* tendency for the variables to co-vary across scenarios. In the last line of the spreadsheet, we see that the covariance is −80 (cell H6).

Suppose our scenario analysis had envisioned stocks generally moving in the same direction as bonds. To be concrete, let's switch the forecast rates on stocks in the first and

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	A	B	C	D	E	F	G	H
1		Rates of Return			Deviation from Mean			Product of
2	Probability	Bonds	Stocks		Bonds	Stocks		Deviations
3	0.25	–2	30		–8	20		–160
4	0.50	6	10		0	0		0
5	0.25	14	–10		8	–20		–160
6	Mean:	6	10		0	0		–80

TABLE 7B.4

Three-scenario analysis for stocks and bonds third scenarios, that is, let the stock return be −10% in the first scenario and 30% in the third. In this case, the absolute value of both products of these scenarios remains the same, but the signs are positive, and thus the covariance is positive, at +80, reflecting the tendency for both asset returns to vary in tandem. If the levels of the scenario returns change, the intensity of the covariation also may change, as reflected by the magnitude of the product of deviations. The change in the magnitude of the covariance quantifies the change in both direction and intensity of the covariation.

If there is no comovement at all, because positive and negative products are equally likely, the covariance is zero. Also, if one of the assets is risk-free, its covariance with any risky asset is zero, because its deviations from its mean are identically zero.

The computation of covariance using Excel can be made easy by using the last line in Equation 7B.9. The first term, $E(r_D \times r_E)$, can be computed in one stroke using Excel's SUMPRODUCT function. Specifically, in Table 7B.4, SUMPRODUCT(A3:A5, B3:B5, C3:C5) multiplies the probability times the return on debt times the return on equity in each scenario and then sums those three products.

Notice that adding Δ to each rate would not change the covariance because deviations from the mean would remain unchanged. But if you *multiply* either of the variables by a fixed factor, the covariance will increase by that factor. Multiplying both variables results in a covariance multiplied by the products of the factors because

$$\begin{aligned}\text{Cov}(w_D r_D, w_E r_E) &= E\{[w_D r_D - w_D E(r_D)][w_E r_E - w_E E(r_E)]\} \\ &= w_D w_E \text{Cov}(r_D, r_E)\end{aligned} \qquad \textbf{(7B.10)}$$

The covariance in Equation 7B.10 is actually the term that we add (twice) in the last line of the equation for portfolio variance, Equation 7B.8. So we find that portfolio variance is the weighted sum (not average) of the individual asset variances, *plus* twice their covariance weighted by the two portfolio weights ($w_D \times w_E$).

Like variance, the dimension (unit) of covariance is percent squared. But here we cannot get to a more easily interpreted dimension by taking the square root, because the average product of deviations can be negative, as it was in Table 7B.4. The solution in this case is to scale the covariance by the standard deviations of the two variables, producing the *correlation coefficient.*

Correlation Coefficient

Dividing the covariance by the product of the standard deviations of the variables will generate a pure number called *correlation.* We define correlation as follows:

$$\text{Corr}(r_D, r_E) = \frac{\text{Cov}(r_D, r_E)}{\sigma_D \sigma_E} \qquad \textbf{(7B.11)}$$

The correlation coefficient must fall within the range [−1, 1]. This can be explained as follows. What two variables should have the highest degree comovement? Logic says a variable with itself, so let's check it out.

$$\begin{aligned}\text{Cov}(r_D, r_D) &= E\{[r_D - E(r_D)] \times [r_D - E(r_D)]\} \\ &= E[r_D - E(r_D)]^2 = \sigma_D^2 \\ \text{Corr}(r_D, r_D) &= \frac{\text{Cov}(r_D, r_D)}{\sigma_D \sigma_D} = \frac{\sigma_D^2}{\sigma_D^2} = 1\end{aligned} \qquad \textbf{(7B.12)}$$

Similarly, the lowest (most negative) value of the correlation coefficient is −1. (Check this for yourself by finding the correlation of a variable with its own negative.)

An important property of the correlation coefficient is that it is unaffected by both addition and multiplication. Suppose we start with a return on debt, r_D, multiply it by a constant, w_D, and then add a fixed amount Δ. The correlation with equity is unaffected:

$$\begin{aligned}\text{Corr}(\Delta + w_D r_D, r_E) &= \frac{\text{Cov}(\Delta + w_D r_D, r_E)}{\sqrt{\text{Var}(\Delta + w_D r_D)} \times \sigma_E} \\ &= \frac{w_D \text{Cov}(r_D, r_E)}{\sqrt{w_D^2 \sigma_D^2} \times \sigma_E} = \frac{w_D \text{Cov}(r_D, r_E)}{w_D \sigma_D \times \sigma_E} \\ &= \text{Corr}(r_D, r_E)\end{aligned} \tag{7B.13}$$

Because the correlation coefficient gives more intuition about the relationship between rates of return, we sometimes express the covariance in terms of the correlation coefficient. Rearranging Equation 7B.11, we can write covariance as

$$\text{Cov}(r_D, r_E) = \sigma_D \sigma_E \text{Corr}(r_D, r_E) \tag{7B.14}$$

EXAMPLE 7B.4 Calculating Covariance and Correlation

Table 7B.5 shows the covariance and correlation between stocks and bonds using the same scenario analysis as in the other examples in this appendix. Covariance is calculated using Equation 7B.9. The SUMPRODUCT function used in cell J22 gives us $E(r_D \times r_E)$, from which we subtract $E(r_D) \times E(r_E)$ (i.e., we subtract J20 × K20). Then we calculate correlation in cell J23 by dividing covariance by the product of the asset standard deviations.

Portfolio Variance

We have seen in Equation 7B.8, with the help of Equation 7B.10, that the variance of a two-asset portfolio is the sum of the individual variances multiplied by the square of the portfolio weights, plus twice the covariance between the rates, multiplied by the product of the portfolio weights:

$$\begin{aligned}\sigma_P^2 &= w_D^2 \sigma_D^2 + w_E^2 \sigma_E^2 + 2 w_D w_E \text{Cov}(r_D, r_E) \\ &= w_D^2 \sigma_D^2 + w_E^2 \sigma_E^2 + 2 w_D w_E \sigma_D \sigma_E \text{Corr}(r_D, r_E)\end{aligned} \tag{7B.15}$$

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	H	I	J	K	L	M
13						
14			Scenario rates of return			
15	Scenario	Probability	$r_D(i)$	$r_E(i)$		
16	1	0.14	–0.10	–0.35		
17	2	0.36	0.00	0.20		
18	3	0.30	0.10	0.45		
19	4	0.20	0.32	–0.19		
20		Mean	0.08	0.12		
21		SD	0.1359	0.2918		
22		Covariance	–0.0034			
23		Correlation	–0.0847			
24	Cell J22	=SUMPRODUCT(I16:I19,J16:J19,K16:K19)–J20*K20				
25	Cell J23	=J22/(J21*K21)				

TABLE 7B.5

Scenario analysis for bonds and stocks

	A	B	C	D	E	F	G
25							
26							
27							
28			Scenario rates of return		Portfolio return		
29	Scenario	Probability	$r_D(i)$	$r_E(i)$	$0.4*r_D(i)+0.6r_E(i)$		
30	1	0.14	–0.10	–0.35	–0.25		
31	2	0.36	0.00	0.20	0.12		
32	3	0.30	0.10	0.45	0.31		
33	4	0.20	0.32	–0.19	0.014		
34		Mean	0.08	0.12	0.1040		
35		SD	0.1359	0.2918	0.1788		
36		Covariance	–0.0034		SD: 0.1788		
37		Correlation	–0.0847				
38	Cell E35	=SUMPRODUCT(B30:B33,E30:E33,E30:E33)–E34^2)^0.5					
39	Cell E36	=(0.4*C35)^2+(0.6*D35)^2+2*0.4*0.6*C36)^0.5					

TABLE 7B.6

Scenario analysis for bonds and stocks

EXAMPLE 7B.5 Calculating Portfolio Variance

We calculate portfolio variance in Table 7B.6. Notice there that we calculate the portfolio standard deviation in two ways: once from the scenario portfolio returns (cell E35) and again (in cell E36) using the first line of Equation 7B.15. The two approaches yield the same result. You should try to repeat the second calculation using the correlation coefficient from the second line in Equation 7B.15 instead of covariance in the formula for portfolio variance.

Suppose that one of the assets, say, *E*, is replaced with a money market instrument, that is, a risk-free asset. The variance of *E* is then zero, as is the covariance with *D*. In that case, as seen from Equation 7B.15, the portfolio standard deviation is just $w_D\ \sigma_D$. In other words, when we mix a risky portfolio with the risk-free asset, portfolio standard deviation equals the risky asset's standard deviation times the weight invested in that asset. This result was used extensively in Chapter 6.

THE CAPITAL ASSET PRICING MODEL

THE CAPITAL ASSET pricing model, almost always referred to as the CAPM, is a centerpiece of modern financial economics. The model gives us a precise prediction of the relationship that we should observe between the risk of an asset and its expected return. This relationship serves two vital functions. First, it provides a benchmark rate of return for evaluating possible investments. For example, if we are analyzing securities, we might be interested in whether the expected return we forecast for a stock is more or less than its "fair" return given its risk. Second, the model helps us to make an educated guess as to the expected return on assets that have not yet been traded in the marketplace. For example, how do we price an initial public offering of stock? How will a major new investment project affect the return investors require on a company's stock? Although the CAPM does not fully withstand empirical tests, it is widely used because of the insight it offers and because its accuracy is deemed acceptable for important applications.

9.1 THE CAPITAL ASSET PRICING MODEL

The capital asset pricing model is a set of predictions concerning equilibrium expected returns on risky assets. Harry Markowitz laid down the foundation of modern portfolio management in 1952. The CAPM was developed 12 years later in articles by William Sharpe,[1] John Lintner,[2] and Jan Mossin.[3] The time for this gestation indicates that the leap from Markowitz's portfolio selection model to the CAPM is not trivial.

We will approach the CAPM by posing the question "what if," where the "if" part refers to a simplified world. Positing an admittedly unrealistic world allows a relatively easy leap to the "then" part. Once we accomplish this, we can add complexity to the hypothesized

[1]William Sharpe, "Capital Asset Prices: A Theory of Market Equilibrium," *Journal of Finance*, September 1964.

[2]John Lintner, "The Valuation of Risk Assets and the Selection of Risky Investments in Stock Portfolios and Capital Budgets," *Review of Economics and Statistics*, February 1965.

[3]Jan Mossin, "Equilibrium in a Capital Asset Market," *Econometrica*, October 1966.

environment one step at a time and see how the conclusions must be amended. This process allows us to derive a reasonably realistic and comprehensible model.

We summarize the simplifying assumptions that lead to the basic version of the CAPM in the following list. The thrust of these assumptions is that we try to ensure that individuals are as alike as possible, with the notable exceptions of initial wealth and risk aversion. We will see that conformity of investor behavior vastly simplifies our analysis.

1. There are many investors, each with an endowment (wealth) that is small compared to the total endowment of all investors. Investors are price-takers, in that they act as though security prices are unaffected by their own trades. This is the usual perfect competition assumption of microeconomics.
2. All investors plan for one identical holding period. This behavior is myopic (short-sighted) in that it ignores everything that might happen after the end of the single-period horizon. Myopic behavior is, in general, suboptimal.
3. Investments are limited to a universe of publicly traded financial assets, such as stocks and bonds, and to risk-free borrowing or lending arrangements. This assumption rules out investment in nontraded assets such as education (human capital), private enterprises, and governmentally funded assets such as town halls and international airports. It is assumed also that investors may borrow or lend any amount at a fixed, risk-free rate.
4. Investors pay no taxes on returns and no transaction costs (commissions and service charges) on trades in securities. In reality, of course, we know that investors are in different tax brackets and that this may govern the type of assets in which they invest. For example, tax implications may differ depending on whether the income is from interest, dividends, or capital gains. Furthermore, actual trading is costly, and commissions and fees depend on the size of the trade and the good standing of the individual investor.
5. All investors are rational mean-variance optimizers, meaning that they all use the Markowitz portfolio selection model.
6. All investors analyze securities in the same way and share the same economic view of the world. The result is identical estimates of the probability distribution of future cash flows from investing in the available securities; that is, for any set of security prices, they all derive the same input list to feed into the Markowitz model. Given a set of security prices and the risk-free interest rate, all investors use the same expected returns and covariance matrix of security returns to generate the efficient frontier and the unique optimal risky portfolio. This assumption is often referred to as **homogeneous expectations** or beliefs.

These assumptions represent the "if" of our "what if" analysis. Obviously, they ignore many real-world complexities. With these assumptions, however, we can gain some powerful insights into the nature of equilibrium in security markets.

We can summarize the equilibrium that will prevail in this hypothetical world of securities and investors briefly. The rest of the chapter explains and elaborates on these implications.

1. All investors will choose to hold a portfolio of risky assets in proportions that duplicate representation of the assets in the **market portfolio** (*M*), which includes all traded assets. For simplicity, we generally refer to all risky assets as *stocks.* The proportion of each stock in the market portfolio equals the market value of the stock

(price per share multiplied by the number of shares outstanding) divided by the total market value of all stocks.

2. Not only will the market portfolio be on the efficient frontier, but it also will be the tangency portfolio to the optimal capital allocation line (CAL) derived by each and every investor. As a result, the *capital market line* (CML), the line from the risk-free rate through the market portfolio, *M*, is also the best attainable capital allocation line. All investors hold *M* as their optimal risky portfolio, differing only in the amount invested in it versus in the risk-free asset.
3. The risk premium on the market portfolio will be proportional to its risk and the degree of risk aversion of the representative investor. Mathematically,

$$E(r_M) - r_f = \overline{A}\sigma_M^2$$

where σ_M^2 is the variance of the market portfolio and $\overline{A}$ is the average degree of risk aversion across investors. Note that because *M* is the optimal portfolio, which is efficiently diversified across all stocks, σ_M^2 is the systematic risk of this universe.
4. The risk premium on *individual* assets will be proportional to the risk premium on the market portfolio, *M*, and the *beta coefficient* of the security relative to the market portfolio. Beta measures the extent to which returns on the stock and the market move together. Formally, beta is defined as

$$\beta_i = \frac{\text{Cov}(r_i, r_M)}{\sigma_M^2}$$

and the risk premium on individual securities is

$$E(r_i) - r_f = \frac{\text{Cov}(r_i, r_M)}{\sigma_M^2}[E(r_M) - r_f] = \beta_i[E(r_M) - r_f]$$

Why Do All Investors Hold the Market Portfolio?

What is the market portfolio? When we sum over, or aggregate, the portfolios of all individual investors, lending and borrowing will cancel out (because each lender has a corresponding borrower), and the value of the aggregate risky portfolio will equal the entire wealth of the economy. This is the market portfolio, *M*. The proportion of each stock in this portfolio equals the market value of the stock (price per share times number of shares outstanding) divided by the sum of the market values of all stocks.[4] The CAPM implies that as individuals attempt to optimize their personal portfolios, they each arrive at the same portfolio, with weights on each asset equal to those of the market portfolio.

Given the assumptions of the previous section, it is easy to see that all investors will desire to hold identical risky portfolios. If all investors use identical Markowitz analysis (Assumption 5) applied to the same universe of securities (Assumption 3) for the same time horizon (Assumption 2) and use the same input list (Assumption 6), they all must arrive at the same composition of the optimal risky portfolio, the portfolio on the efficient frontier identified by the tangency line from T-bills to that frontier, as in Figure 9.1. This

[4]As noted previously, we use the term "stock" for convenience; the market portfolio properly includes all assets in the economy.

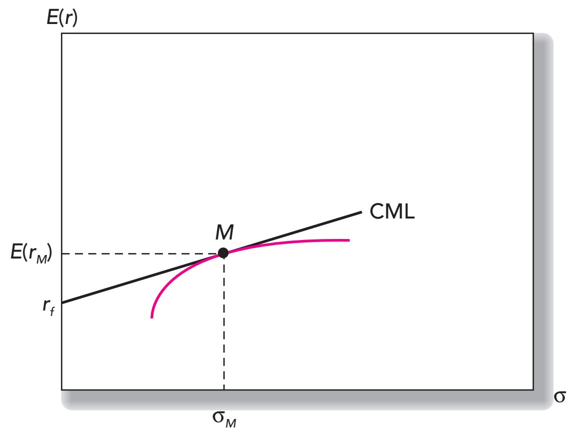


FIGURE 9.1 The efficient frontier and the capital market line

implies that if the weight of GE stock, for example, in each common risky portfolio is 1%, then GE also will comprise 1% of the market portfolio. The same principle applies to the proportion of any stock in each investor's risky portfolio. As a result, the optimal risky portfolio of all investors is simply a share of the market portfolio in Figure 9.1.

Now suppose that the optimal portfolio of our investors does not include the stock of some company, such as Delta Airlines. When all investors avoid Delta stock, the demand is zero, and Delta's price takes a free fall. As Delta stock gets progressively cheaper, it becomes ever more attractive and other stocks look relatively less attractive. Ultimately, Delta reaches a price where it is attractive enough to include in the optimal stock portfolio.

Such a price adjustment process guarantees that all stocks will be included in the optimal portfolio. It shows that *all* assets have to be included in the market portfolio. The only issue is the price at which investors will be willing to include a stock in their optimal risky portfolio.

This may seem a roundabout way to derive a simple result: If all investors hold an identical risky portfolio, this portfolio has to be *M*, the market portfolio. Our intention, however, is to demonstrate a connection between this result and its underpinnings, the equilibrating process that is fundamental to security market operation.

The Passive Strategy Is Efficient

In Chapter 6 we defined the CML (capital market line) as the CAL (capital allocation line) that is constructed from a money market account (or T-bills) and the market portfolio. Perhaps now you can fully appreciate why the CML is an interesting CAL. In the simple world of the CAPM, *M* is the optimal tangency portfolio on the efficient frontier, as shown in Figure 9.1.

In this scenario, the market portfolio held by all investors is based on the common input list, thereby incorporating all relevant information about the universe of securities. This means that investors can skip the trouble of doing security analysis and obtain an efficient portfolio simply by holding the market portfolio. (Of course, if everyone were to follow this strategy, no one would perform security analysis and this result would no longer hold. We discuss this issue in greater depth in Chapter 11 on market efficiency.)

Thus the passive strategy of investing in a market index portfolio is efficient. For this reason, we sometimes call this result a **mutual fund theorem.** The mutual fund theorem is another incarnation of the separation property discussed in Chapter 7. Assuming that all investors choose to hold a market index mutual fund, we can separate portfolio selection into two components—a technical problem, creation of mutual funds by professional managers—and a personal problem that depends on an investor's risk aversion, allocation of the *complete* portfolio between the mutual fund and risk-free assets.

In reality, different investment managers do create risky portfolios that differ from the market index. We attribute this in part to the use of different input lists in the formation of the optimal risky portfolio. Nevertheless, the practical significance of the mutual fund theorem is that a passive investor may view the market index as a reasonable first approximation to an efficient risky portfolio.

THE PARABLE OF THE MONEY MANAGERS

WORDS FROM THE STREET

Some years ago, in a land called Indicia, revolution led to the overthrow of a socialist regime and the restoration of a system of private property. Former government enterprises were reformed as corporations, which then issued stocks and bonds. These securities were given to a central agency, which offered them for sale to individuals, pension funds, and the like (all armed with newly printed money).

Almost immediately a group of money managers came forth to assist these investors. Recalling the words of a venerated elder, uttered before the previous revolution ("Invest in Corporate Indicia"), they invited clients to give them money, with which they would buy a cross-section of all the newly issued securities. Investors considered this a reasonable idea, and soon everyone held a piece of Corporate Indicia.

Before long the money managers became bored because there was little for them to do. Soon they fell into the habit of gathering at a beachfront casino where they passed the time playing roulette, craps, and similar games, for low stakes, with their own money.

After a while, the owner of the casino suggested a new idea. He would furnish an impressive set of rooms which would be designated the Money Managers' Club. There the members could place bets with one another about the fortunes of various corporations, industries, the level of the Gross National Product, foreign trade, etc. To make the betting more exciting, the casino owner suggested that the managers use their clients' money for this purpose.

The offer was immediately accepted, and soon the money managers were betting eagerly with one another. At the end of each week, some found that they had won money for their clients, while others found that they had lost. But the losses always exceeded the gains, for a certain amount was deducted from each bet to cover the costs of the elegant surroundings in which the gambling took place.

Before long a group of professors from Indicia U. suggested that investors were not well served by the activities being conducted at the Money Managers' Club. "Why pay people to gamble with your money? Why not just hold your own piece of Corporate Indicia?" they said.

This argument seemed sensible to some of the investors, and they raised the issue with their money managers. A few capitulated, announcing that they would henceforth stay away from the casino and use their clients' money only to buy proportionate shares of all the stocks and bonds issued by corporations.

The converts, who became known as managers of Indicia funds, were initially shunned by those who continued to frequent the Money Managers' Club, but in time, grudging acceptance replaced outright hostility. The wave of puritan reform some had predicted failed to materialize, and gambling remained legal. Many managers continued to make their daily pilgrimage to the casino. But they exercised more restraint than before, placed smaller bets, and generally behaved in a manner consonant with their responsibilities. Even the members of the Lawyers' Club found it difficult to object to the small amount of gambling that still went on.

And everyone but the casino owner lived happily ever after.

Source: William F. Sharpe, "The Parable of the Money Managers," *The Financial Analysts' Journal* 32 (July/August 1976), p. 4.

The nearby box contains a parable illustrating the argument for indexing. If the passive strategy is efficient, then attempts to beat it simply generate trading and research costs with no offsetting benefit, and ultimately inferior results.

CONCEPT CHECK 1

If there are only a few investors who perform security analysis, and all others hold the market portfolio, *M,* would the CML still be the efficient CAL for investors who do not engage in security analysis? Why or why not?

The Risk Premium of the Market Portfolio

In Chapter 6 we discussed how individual investors go about deciding how much to invest in the risky portfolio. Returning now to the decision of how much to invest in portfolio *M* versus in the risk-free asset, what can we deduce about the equilibrium risk premium of portfolio *M?*

We asserted earlier that the equilibrium risk premium on the market portfolio, $E(r_M) - r_f$, will be proportional to the average degree of risk aversion of the investor population and the risk of the market portfolio, σ_M^2. Now we can explain this result.

Recall that each individual investor chooses a proportion *y*, allocated to the optimal portfolio *M*, such that

$$y = \frac{E(r_M) - r_f}{A\sigma_M^2} \tag{9.1}$$

In the simplified CAPM economy, risk-free investments involve borrowing and lending among investors. Any borrowing position must be offset by the lending position of the creditor. This means that net borrowing and lending across all investors must be zero, and in consequence, substituting the representative investor's risk aversion, $\overline{A}$, for *A*, the average position in the risky portfolio is 100%, or $\overline{y} = 1$. Setting $y = 1$ in Equation 9.1 and rearranging, we find that the risk premium on the market portfolio is related to its variance by the average degree of risk aversion:

$$E(r_M) - r_f = \overline{A}\sigma_M^2 \tag{9.2}$$

CONCEPT CHECK 2

Data from the last eight decades (see Table 5.3) for the S&P 500 index yield the following statistics: average excess return, 8.4%; standard deviation, 20.3%.

a. To the extent that these averages approximated investor expectations for the period, what must have been the average coefficient of risk aversion?

b. If the coefficient of risk aversion were actually 3.5, what risk premium would have been consistent with the market's historical standard deviation?

Expected Returns on Individual Securities

The CAPM is built on the insight that the appropriate risk premium on an asset will be determined by its contribution to the risk of investors' overall portfolios. Portfolio risk is what matters to investors and is what governs the risk premiums they demand.

Remember that all investors use the same input list, that is, the same estimates of expected returns, variances, and covariances. We saw in Chapter 7 that these covariances can be arranged in a covariance matrix, so that the entry in the fifth row and third column, for example, would be the covariance between the rates of return on the fifth and third securities. Each diagonal entry of the matrix is the covariance of one security's return with itself, which is simply the variance of that security.

Suppose, for example, that we want to gauge the portfolio risk of GE stock. We measure the contribution to the risk of the overall portfolio from holding GE stock by its covariance with the market portfolio. To see why this is so, let us look again at the way the variance of the market portfolio is calculated. To calculate the variance of the market portfolio, we use the bordered covariance matrix with the market portfolio weights, as discussed in Chapter 7. We highlight GE in this depiction of the *n* stocks in the market portfolio.

Portfolio Weights	w_1	w_2	. . .	w_{GE}	. . .	w_n
w_1	Cov(r_1, r_1)	Cov(r_1, r_2)	. . .	Cov(r_1, r_{GE})	. . .	Cov(r_1, r_n)
w_2	Cov(r_2, r_1)	Cov(r_2, r_2)	. . .	Cov(r_2, r_{GE})	. . .	Cov(r_2, r_n)
⋮	⋮	⋮		⋮		⋮
w_{GE}	Cov(r_{GE}, r_1)	Cov(r_{GE}, r_2)	. . .	Cov(r_{GE}, r_{GE})	. . .	Cov(r_{GE}, r_n)
⋮	⋮	⋮		⋮		⋮
w_n	Cov(r_n, r_1)	Cov(r_n, r_2)	. . .	Cov(r_n, r_{GE})	. . .	Cov(r_n, r_n)

Recall that we calculate the variance of the portfolio by summing over all the elements of the covariance matrix, first multiplying each element by the portfolio weights from the row and the column. The contribution of one stock to portfolio variance therefore can be expressed as the sum of all the covariance terms in the column corresponding to the stock, where each covariance is first multiplied by both the stock's weight from its row and the weight from its column.[5]

For example, the contribution of GE's stock to the variance of the market portfolio is

$$w_{GE}[w_1\text{Cov}(r_1, r_{GE}) + w_2\text{Cov}(r_2, r_{GE}) + \ldots + w_{GE}\text{Cov}(r_{GE}, r_{GE}) + \ldots + w_n\text{Cov}(r_n, r_{GE})] \qquad \textbf{(9.3)}$$

Equation 9.3 provides a clue about the respective roles of variance and covariance in determining asset risk. When there are many stocks in the economy, there will be many more covariance terms than variance terms. Consequently, the covariance of a particular stock with all other stocks will dominate that stock's contribution to total portfolio risk. Notice that the sum inside the square brackets in Equation 9.3 is the covariance of GE with the market portfolio. In other words, we can best measure the stock's contribution to the risk of the market portfolio by its covariance with that portfolio:

$$\text{GE's contribution to variance} = w_{GE}\text{Cov}(r_{GE}, r_M)$$

This should not surprise us. For example, if the covariance between GE and the rest of the market is negative, then GE makes a "negative contribution" to portfolio risk: By providing returns that move inversely with the rest of the market, GE stabilizes the return on the overall portfolio. If the covariance is positive, GE makes a positive contribution to overall portfolio risk because its returns reinforce swings in the rest of the portfolio.

To demonstrate this more rigorously, note that the rate of return on the market portfolio may be written as

$$r_M = \sum_{k=1}^{n} w_k r_k$$

[5]An alternative approach would be to measure GE's contribution to market variance as the sum of the elements in the row *and* the column corresponding to GE. In this case, GE's contribution would be twice the sum in Equation 9.3. The approach that we take in the text allocates contributions to portfolio risk among securities in a convenient manner in that the sum of the contributions of each stock equals the total portfolio variance, whereas the alternative measure of contribution would sum to twice the portfolio variance. This results from a type of double-counting, because adding both the rows and the columns for each stock would result in each entry in the matrix being added twice.

Therefore, the covariance of the return on GE with the market portfolio is

$$\text{Cov}(r_{GE}, r_M) = \text{Cov}\left(r_{GE}, \sum_{k=1}^{n} w_k r_k\right) = \sum_{k=1}^{n} w_k \text{Cov}(r_k, r_{GE}) \tag{9.4}$$

Notice that the last term of Equation 9.4 is precisely the same as the term in brackets in Equation 9.3. Therefore, Equation 9.3, which is the contribution of GE to the variance of the market portfolio, may be simplified to $w_{GE}\,\text{Cov}(r_{GE}, r_M)$. We also observe that the contribution of our holding of GE to the risk premium of the market portfolio is $w_{GE}\,[E(r_{GE}) - r_f]$.

Therefore, the reward-to-risk ratio for investments in GE can be expressed as

$$\frac{\text{GE's contribution to risk premium}}{\text{GE's contribution to variance}} = \frac{w_{GE}[E(r_{GE}) - r_f]}{w_{GE}\text{Cov}(r_{GE}, r_M)} = \frac{E(r_{GE}) - r_f}{\text{Cov}(r_{GE}, r_M)}$$

The market portfolio is the tangency (efficient mean-variance) portfolio. The reward-to-risk ratio for investment in the market portfolio is

$$\frac{\text{Market risk premium}}{\text{Market variance}} = \frac{E(r_M) - r_f}{\sigma_M^2} \tag{9.5}$$

The ratio in Equation 9.5 is often called the **market price of risk**[6] because it quantifies the extra return that investors demand to bear portfolio risk. Notice that for *components* of the efficient portfolio, such as shares of GE, we measure risk as the *contribution* to portfolio variance (which depends on its *covariance* with the market). In contrast, for the efficient portfolio itself, its variance is the appropriate measure of risk.

A basic principle of equilibrium is that all investments should offer the same reward-to-risk ratio. If the ratio were better for one investment than another, investors would rearrange their portfolios, tilting toward the alternative with the better trade-off and shying away from the other. Such activity would impart pressure on security prices until the ratios were equalized. Therefore we conclude that the reward-to-risk ratios of GE and the market portfolio should be equal:

$$\frac{E(r_{GE}) - r_f}{\text{Cov}(r_{GE}, r_M)} = \frac{E(r_M) - r_f}{\sigma_M^2} \tag{9.6}$$

To determine the fair risk premium of GE stock, we rearrange Equation 9.6 slightly to obtain

$$E(r_{GE}) - r_f = \frac{\text{Cov}(r_{GE}, r_M)}{\sigma_M^2}[E(r_M) - r_f] \tag{9.7}$$

[6]We open ourselves to ambiguity in using this term, because the market portfolio's reward-to-volatility ratio

$$\frac{E(r_M) - r_f}{\sigma_M}$$

sometimes is referred to as the market price of risk. Note that because the appropriate risk measure of GE is its covariance with the market portfolio (its contribution to the variance of the market portfolio), this risk is measured in percent squared. Accordingly, the price of this risk, $[E(r_M) - r_f]/\sigma^2$, is defined as the percentage expected return per percent square of variance.

The ratio $\text{Cov}(r_{\text{GE}}, r_M)/\sigma_M^2$ measures the contribution of GE stock to the variance of the market portfolio as a fraction of the total variance of the market portfolio. The ratio is called **beta** and is denoted by β. Using this measure, we can restate Equation 9.7 as

$$E(r_{\text{GE}}) = r_f + \beta_{\text{GE}}[E(r_M) - r_f] \quad \textbf{(9.8)}$$

This **expected return–beta relationship** is the most familiar expression of the CAPM to practitioners. We will have a lot more to say about the expected return–beta relationship shortly.

We see now why the assumptions that made individuals act similarly are so useful. If everyone holds an identical risky portfolio, then everyone will find that the beta of each asset with the market portfolio equals the asset's beta with his or her own risky portfolio. Hence everyone will agree on the appropriate risk premium for each asset.

Does the fact that few real-life investors actually hold the market portfolio imply that the CAPM is of no practical importance? Not necessarily. Recall from Chapter 7 that reasonably well-diversified portfolios shed firm-specific risk and are left with mostly systematic or market risk. Even if one does not hold the precise market portfolio, a well-diversified portfolio will be so very highly correlated with the market that a stock's beta relative to the market will still be a useful risk measure.

In fact, several authors have shown that modified versions of the CAPM will hold true even if we consider differences among individuals leading them to hold different portfolios. For example, Brennan[7] examined the impact of differences in investors' personal tax rates on market equilibrium, and Mayers[8] looked at the impact of nontraded assets such as human capital (earning power). Both found that although the market portfolio is no longer each investor's optimal risky portfolio, the expected return–beta relationship should still hold in a somewhat modified form.

If the expected return–beta relationship holds for any individual asset, it must hold for any combination of assets. Suppose that some portfolio *P* has weight w_k for stock *k*, where *k* takes on values 1, . . . , *n*. Writing out the CAPM Equation 9.8 for each stock, and multiplying each equation by the weight of the stock in the portfolio, we obtain these equations, one for each stock:

$$\begin{array}{rcl} w_1E(r_1) & = & w_1r_f + w_1\beta_1[E(r_M) - r_f] \\ +w_2E(r_2) & = & w_2r_f + w_2\beta_2[E(r_M) - r_f] \\ + \quad \cdots & = & \cdots \\ \underline{+w_nE(r_n)} & \underline{=} & \underline{w_nr_f + w_n\beta_n[E(r_M) - r_f]} \\ E(r_P) & = & r_f + \beta_P[E(r_M) - r_f] \end{array}$$

Summing each column shows that the CAPM holds for the overall portfolio because $E(r_P) = \sum_k w_k E(r_k)$ is the expected return on the portfolio, and $\beta_P = \sum_k w_k \beta_k$ is the portfolio beta. Incidentally, this result has to be true for the market portfolio itself,

$$E(r_M) = r_f + \beta_M[E(r_M) - r_f]$$

[7]Michael J. Brennan, "Taxes, Market Valuation, and Corporate Finance Policy," *National Tax Journal,* December 1973.

[8]David Mayers, "Nonmarketable Assets and Capital Market Equilibrium under Uncertainty," in *Studies in the Theory of Capital Markets,* ed. M. C. Jensen (New York: Praeger, 1972). We will look at this model more closely later in the chapter.

Indeed, this is a tautology because $\beta_M = 1$, as we can verify by noting that

$$\beta_M = \frac{\text{Cov}(r_M, r_M)}{\sigma_M^2} = \frac{\sigma_M^2}{\sigma_M^2}$$

This also establishes 1 as the weighted-average value of beta across all assets. If the market beta is 1, and the market is a portfolio of all assets in the economy, the weighted-average beta of all assets must be 1. Hence betas greater than 1 are considered aggressive in that investment in high-beta stocks entails above-average sensitivity to market swings. Betas below 1 can be described as defensive.

A word of caution: We are all accustomed to hearing that well-managed firms will provide high rates of return. We agree this is true if one measures the *firm's* return on investments in plant and equipment. The CAPM, however, predicts returns on investments in the *securities* of the firm.

Let us say that everyone knows a firm is well run. Its stock price will therefore be bid up, and consequently returns to stockholders who buy at those high prices will not be excessive. Security prices, in other words, already reflect public information about a firm's prospects; therefore only the risk of the company (as measured by beta in the context of the CAPM) should affect expected returns. In an efficient market investors receive high expected returns only if they are willing to bear risk.

Of course, investors do not directly observe or determine expected returns on securities. Rather, they observe security prices and bid those prices up or down. Expected rates of return are determined by the prices investors must pay compared to the cash flows those investments might garner.

CONCEPT CHECK 3

Suppose that the risk premium on the market portfolio is estimated at 8% with a standard deviation of 22%. What is the risk premium on a portfolio invested 25% in GM and 75% in Ford, if they have betas of 1.10 and 1.25, respectively?

The Security Market Line

We can view the expected return–beta relationship as a reward–risk equation. The beta of a security is the appropriate measure of its risk because beta is proportional to the risk that the security contributes to the optimal risky portfolio.

Risk-averse investors measure the risk of the optimal risky portfolio by its variance. In this world we would expect the reward, or the risk premium on individual assets, to depend on the *contribution* of the individual asset to the risk of the portfolio. The beta of a stock measures its contribution to the variance of the market portfolio. Hence we expect, for any asset or portfolio, the required risk premium to be a function of beta. The CAPM confirms this intuition, stating further that the security's risk premium is directly proportional to both the beta and the risk premium of the market portfolio; that is, the risk premium equals $\beta[E(r_M) - r_f]$.

The expected return–beta relationship can be portrayed graphically as the **security market line (SML)** in Figure 9.2. Because the market's beta is 1, the slope is the risk premium of the market portfolio. At the point on the horizontal axis where $\beta = 1$, we can read off the vertical axis the expected return on the market portfolio.

It is useful to compare the security market line to the capital market line. The CML graphs the risk premiums of *efficient* portfolios (i.e., portfolios composed of the market and the risk-free asset) as a function of portfolio standard deviation. This is appropriate because standard deviation is a valid measure of risk for efficiently diversified portfolios that are candidates for an investor's overall portfolio. The SML, in contrast, graphs *individual asset* risk premiums as a function of asset risk. The relevant measure of risk for individual assets held as parts of well-diversified portfolios is not the asset's standard deviation or variance; it is, instead, the contribution of the asset to the portfolio variance, which we measure by the asset's beta. The SML is valid for both efficient portfolios and individual assets.

The security market line provides a benchmark for the evaluation of investment performance. Given the risk of an investment, as measured by its beta, the SML provides the required rate of return necessary to compensate investors for both risk as well as the time value of money.

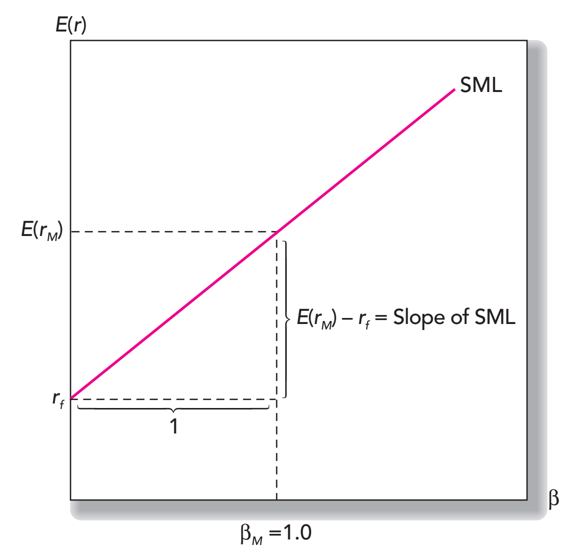


FIGURE 9.2 The security market line

Because the security market line is the graphic representation of the expected return–beta relationship, "fairly priced" assets plot exactly on the SML; that is, their expected returns are commensurate with their risk. Given the assumptions we made at the start of this section, all securities must lie on the SML in market equilibrium. Nevertheless, we see here how the CAPM may be of use in the money-management industry. Suppose that the SML relation is used as a benchmark to assess the fair expected return on a risky asset. Then security analysis is performed to calculate the return actually expected. (Notice that we depart here from the simple CAPM world in that some investors now apply their own unique analysis to derive an "input list" that may differ from their competitors'.) If a stock is perceived to be a good buy, or underpriced, it will provide an expected return in excess of the fair return stipulated by the SML. Underpriced stocks therefore plot above the SML: Given their betas, their expected returns are greater than dictated by the CAPM. Overpriced stocks plot below the SML.

The difference between the fair and actually expected rates of return on a stock is called the stock's **alpha,** denoted by α. For example, if the market return is expected to be 14%, a stock has a beta of 1.2, and the T-bill rate is 6%, the SML would predict an expected return on the stock of $6 + 1.2(14 - 6) = 15.6\%$. If one believed the stock would provide an expected return of 17%, the implied alpha would be 1.4% (see Figure 9.3).

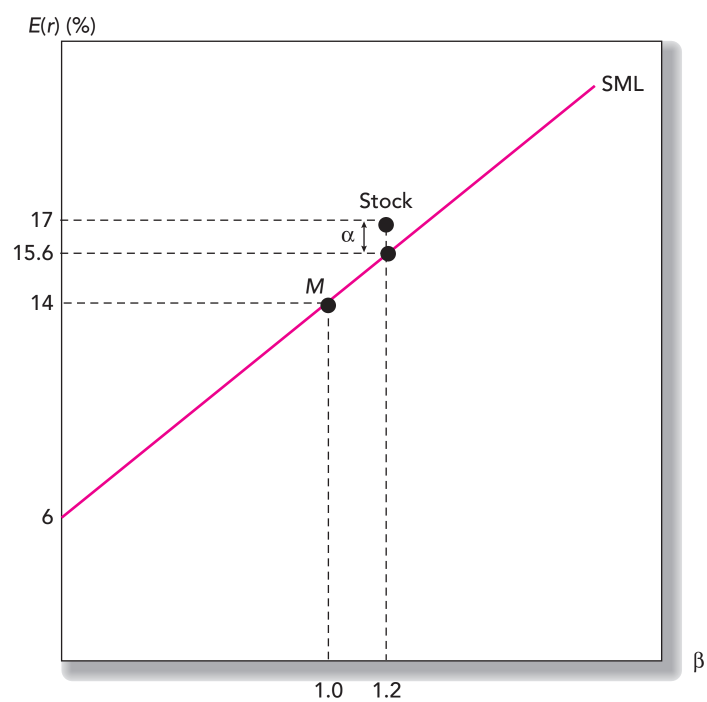


FIGURE 9.3 The SML and a positive-alpha stock

One might say that security analysis (which we treat in Part Five) is about uncovering securities with nonzero alphas. This analysis suggests that the starting point of

TALES FROM THE FAR SIDE

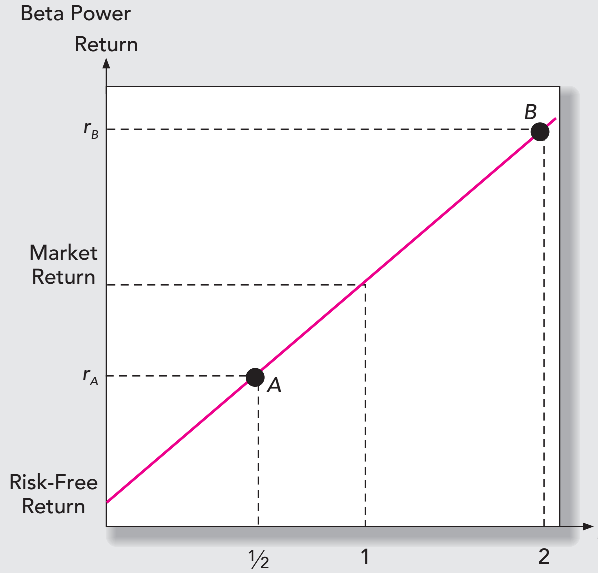
Financial markets' evaluation of risk determines the way firms invest. What if the markets are wrong? Investors are rarely praised for their good sense. But for the past two decades a growing number of firms have based their decisions on a model which assumes that people are perfectly rational. If they are irrational, are businesses making the wrong choices?

The model, known as the "capital-asset pricing model," or CAPM, has come to dominate modern finance. Almost any manager who wants to defend a project—be it a brand, a factory or a corporate merger—must justify his decision partly based on the CAPM. The reason is that the model tells a firm how to calculate the return that its investors demand. If shareholders are to benefit, the returns from any project must clear this "hurdle rate."

Although the CAPM is complicated, it can be reduced to five simple ideas:

- Investors can eliminate some risks—such as the risk that workers will strike, or that a firm's boss will quit—by diversifying across many regions and sectors.
- Some risks, such as that of a global recession, cannot be eliminated through diversification. So even a basket of all of the stocks in a stock market will still be risky.
- People must be rewarded for investing in such a risky basket by earning returns above those that they can get on safer assets, such as Treasury bills.
- The rewards on a specific investment depend only on the extent to which it affects the market basket's risk.
- Conveniently, that contribution to the market basket's risk can be captured by a single measure—dubbed "beta"—which expresses the relationship between the investment's risk and the market's.

Beta is what makes the CAPM so powerful. Although an investment may face many risks, diversified



investors should care only about those that are related to the market basket. Beta not only tells managers how to measure those risks, but it also allows them to translate them directly into a hurdle rate. If the future profits from a project will not exceed that rate, it is not worth shareholders' money.

The diagram shows how the CAPM works. Safe investments, such as Treasury bills, have a beta of zero. Riskier investments should earn a premium over the risk-free rate which increases with beta. Those whose risks roughly match the market's have a beta of one, by definition, and should earn the market return.

So suppose that a firm is considering two projects, *A* and *B*. Project *A* has a beta of ½: when the market rises or falls by 10%, its returns tend to rise or fall by 5%. So its risk premium is only half that of the market. Project *B*'s risk premium is twice that of the

portfolio management can be a passive market-index portfolio. The portfolio manager will then increase the weights of securities with positive alphas and decrease the weights of securities with negative alphas. We showed one strategy for adjusting the portfolio weights in such a manner in Chapter 8.

The CAPM is also useful in capital budgeting decisions. For a firm considering a new project, the CAPM can provide the *required rate of return* that the project needs to yield, based on its beta, to be acceptable to investors. Managers can use the CAPM to obtain this cutoff internal rate of return (IRR), or "hurdle rate" for the project.

The nearby box describes how the CAPM can be used in capital budgeting. It also discusses some empirical anomalies concerning the model, which we address in detail in Chapters 11–13. The article asks whether the CAPM is useful for capital budgeting in light of these shortcomings; it concludes that even given the anomalies cited, the model still can be useful to managers who wish to increase the fundamental value of their firms.

market, so it must earn a higher return to justify the expenditure.

NEVER KNOWINGLY UNDERPRICED

But there is one small problem with the CAPM: Financial economists have found that beta is not much use for explaining rates of return on firms' shares. Worse, there appears to be another measure which explains these returns quite well.

That measure is the ratio of a firm's book value (the value of its assets at the time they entered the balance sheet) to its market value. Several studies have found that, on average, companies that have high book-to-market ratios tend to earn excess returns over long periods, even after adjusting for the risks that are associated with beta.

The discovery of this book-to-market effect has sparked a fierce debate among financial economists. All of them agree that some risks ought to carry greater rewards. But they are now deeply divided over how risk should be measured. Some argue that since investors are rational, the book-to-market effect must be capturing an extra risk factor. They conclude, therefore, that managers should incorporate the book-to-market effect into their hurdle rates. They have labeled this alternative hurdle rate the "new estimator of expected return," or NEER.

Other financial economists, however, dispute this approach. Since there is no obvious extra risk associated with a high book-to-market ratio, they say, investors must be mistaken. Put simply, they are under-pricing high book-to-market stocks, causing them to earn abnormally high returns. If managers of such firms try to exceed those inflated hurdle rates, they will forgo many profitable investments. With economists now at odds, what is a conscientious manager to do?

Jeremy Stein, an economist at the Massachusetts Institute of Technology's business school, offers a paradoxical answer.* If investors are rational, then beta cannot be the only measure of risk, so managers should stop using it. Conversely, if investors are irrational, then beta is still the right measure in many cases. Mr. Stein argues that if beta captures an asset's fundamental risk—that is, its contribution to the market basket's risk—then it will often make sense for managers to pay attention to it, even if investors are somehow failing to.

Often, but not always. At the heart of Mr. Stein's argument lies a crucial distinction—that between (*a*) boosting a firm's long-term value and (*b*) trying to raise its share price. If investors are rational, these are the same thing: any decision that raises long-term value will instantly increase the share price as well. But if investors are making predictable mistakes, a manager must choose.

For instance, if he wants to increase today's share price—perhaps because he wants to sell his shares, or to fend off a takeover attempt—he must usually stick with the NEER approach, accommodating investors' misperceptions. But if he is interested in long-term value, he should usually continue to use beta. Showing a flair for marketing, Mr. Stein labels this far-sighted alternative to NEER the "fundamental asset risk"—or FAR—approach.

Mr. Stein's conclusions will no doubt irritate many company bosses, who are fond of denouncing their investors' myopia. They have resented the way in which CAPM—with its assumption of investor infallibility—has come to play an important role in boardroom decision-making. But it now appears that if they are right, and their investors are wrong, then those same far-sighted managers ought to be the CAPM's biggest fans.

*Jeremy Stein, "Rational Capital Budgeting in an Irrational World," *The Journal of Business,* October 1996.

Source: "Tales from the FAR Side," *The Economist Group,* Inc. November 16, 1996, p. 8.

EXAMPLE 9.1 Using the CAPM

Yet another use of the CAPM is in utility rate-making cases.[9] In this case the issue is the rate of return that a regulated utility should be allowed to earn on its investment in plant and equipment. Suppose that the equityholders have invested $100 million in the firm and that the beta of the equity is .6. If the T-bill rate is 6% and the market risk premium is 8%, then the fair profits to the firm would be assessed as $6 + .6 \times 8 = 10.8\%$ of the $100 million investment, or $10.8 million. The firm would be allowed to set prices at a level expected to generate these profits.

[9]This application is fast disappearing, as many states are in the process of deregulating their public utilities and allowing a far greater degree of free market pricing. Nevertheless, a considerable amount of rate setting still takes place.

CONCEPT CHECK 4 and 5

Stock XYZ has an expected return of 12% and risk of $\beta = 1$. Stock ABC has expected return of 13% and $\beta = 1.5$. The market's expected return is 11%, and $r_f = 5\%$.

a. According to the CAPM, which stock is a better buy?

b. What is the alpha of each stock? Plot the SML and each stock's risk–return point on one graph. Show the alphas graphically.

The risk-free rate is 8% and the expected return on the market portfolio is 16%. A firm considers a project that is expected to have a beta of 1.3.

a. What is the required rate of return on the project?

b. If the expected IRR of the project is 19%, should it be accepted?

9.2 THE CAPM AND THE INDEX MODEL

Actual Returns versus Expected Returns

The CAPM is an elegant model. The question is whether it has real-world value—whether its implications are borne out by experience. Chapter 13 provides a range of empirical evidence on this point, but for now we focus briefly on a more basic issue: Is the CAPM testable even in principle?

For starters, one central prediction of the CAPM is that the market portfolio is a mean-variance efficient portfolio. Consider that the CAPM treats all traded risky assets. To test the efficiency of the CAPM market portfolio, we would need to construct a value-weighted portfolio of a huge size and test its efficiency. So far, this task has not been feasible. An even more difficult problem, however, is that the CAPM implies relationships among *expected* returns, whereas all we can observe are actual or realized holding-period returns, and these need not equal prior expectations. Even supposing we could construct a portfolio to represent the CAPM market portfolio satisfactorily, how would we test its mean-variance efficiency? We would have to show that the reward-to-volatility ratio of the market portfolio is higher than that of any other portfolio. However, this reward-to-volatility ratio is set in terms of expectations, and we have no way to observe these expectations directly.

The problem of measuring expectations haunts us as well when we try to establish the validity of the second central set of CAPM predictions, the expected return–beta relationship. This relationship is also defined in terms of expected returns $E(r_i)$ and $E(r_M)$:

$$E(r_i) = r_f + \beta_i[E(r_M) - r_f] \quad \textbf{(9.9)}$$

The upshot is that, as elegant and insightful as the CAPM is, we must make additional assumptions to make it implementable and testable.

The Index Model and Realized Returns

We have said that the CAPM is a statement about ex ante or expected returns, whereas in practice all anyone can observe directly are ex post or realized returns. To make the leap from expected to realized returns, we can employ the index model, which we will use in excess return form as

$$R_i = \alpha_i + \beta_i R_M + e_i \tag{9.10}$$

We saw in Chapter 8 how to apply standard regression analysis to estimate Equation 9.10 using observable realized returns over some sample period. Let us now see how this framework for statistically decomposing actual stock returns meshes with the CAPM.

We start by deriving the covariance between the returns on stock *i* and the market index. By definition, the firm-specific or nonsystematic component is independent of the market wide or systematic component, that is, $\text{Cov}(R_M, e_i) = 0$. From this relationship, it follows that the covariance of the excess rate of return on security *i* with that of the market index is

$$\begin{aligned}\text{Cov}(R_i, R_M) &= \text{Cov}(\beta_i R_M + e_i, R_M) \\ &= \beta_i \text{Cov}(R_M, R_M) + \text{Cov}(e_i, R_M) \\ &= \beta_i \sigma_M^2\end{aligned}$$

Note that we can drop α_i from the covariance terms because α_i is a constant and thus has zero covariance with all variables.

Because $\text{Cov}(R_i, R_M) = \beta_i \sigma_M^2$, the sensitivity coefficient, β_i, in Equation 9.10, which is the slope of the regression line representing the index model, equals

$$\beta_i = \frac{\text{Cov}(R_i, R_M)}{\sigma_M^2}$$

The index model beta coefficient turns out to be the same beta as that of the CAPM expected return–beta relationship, except that we replace the (theoretical) market portfolio of the CAPM with the well-specified and observable market index.

The Index Model and the Expected Return–Beta Relationship

Recall that the CAPM expected return–beta relationship is, for any asset *i* and the (theoretical) market portfolio,

$$E(r_i) - r_f = \beta_i [E(r_M) - r_f]$$

where $\beta_i = \text{Cov}(R_i, R_M)/\sigma_M^2$. This is a statement about the mean or expected excess returns of assets relative to the mean excess return of the (theoretical) market portfolio.

If the index *M* in Equation 9.10 represents the true market portfolio, we can take the expectation of each side of the equation to show that the index model specification is

$$E(r_i) - r_f = \alpha_i + \beta_i [E(r_M) - r_f]$$

A comparison of the index model relationship to the CAPM expected return–beta relationship (Equation 9.9) shows that the CAPM predicts that α_i should be zero for all assets. The alpha of a stock is its expected return in excess of (or below) the fair expected return as predicted by the CAPM. If the stock is fairly priced, its alpha must be zero.

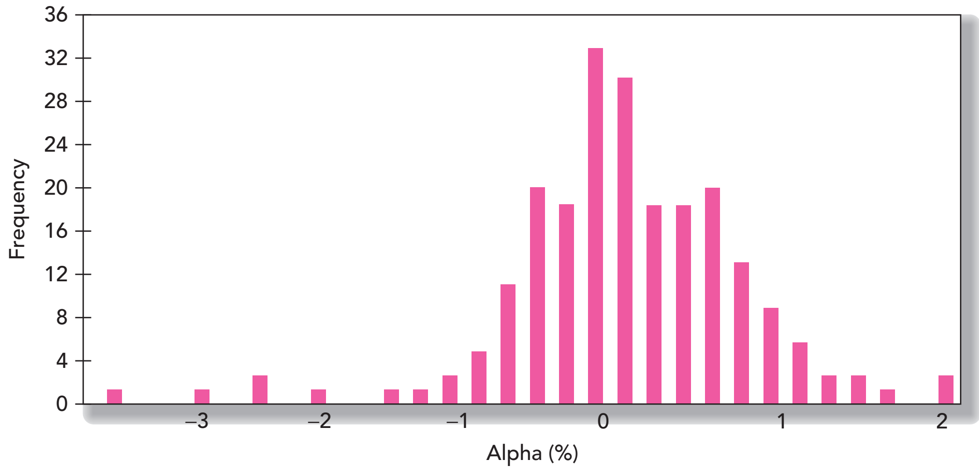


FIGURE 9.4 Estimates of individual mutual fund alphas, 1972–1991

This is a plot of the frequency distribution of estimated alphas for all-equity mutual funds with 10-year continuous records.

Source: Burton G. Malkiel, "Returns from Investing in Equity Mutual Funds 1971–1991," *Journal of Finance* 50 (June 1995), pp. 549–72. Reprinted by permission of the publisher, Blackwell Publishing, Inc.

We emphasize again that this is a statement about *expected* returns on a security. After the fact, of course, some securities will do better or worse than expected and will have returns higher or lower than predicted by the CAPM; that is, they will exhibit positive or negative alphas over a sample period. But this superior or inferior performance could not have been forecast in advance.

Therefore, if we estimate the index model for several firms, using Equation 9.10 as a regression equation, we should find that the ex post or realized alphas (the regression intercepts) for the firms in our sample center around zero. If the initial expectation for alpha were zero, as many firms would be expected to have a positive as a negative alpha for some sample period. The CAPM states that the *expected* value of alpha is zero for all securities, whereas the index model representation of the CAPM holds that the *realized* value of alpha should average out to zero for a sample of historical observed returns. Just as important, the sample alphas should be unpredictable, that is, independent from one sample period to the next.

Indirect evidence on the efficiency of the market portfolio can be found in a study by Burton Malkiel,[10] who estimates alpha values for a large sample of equity mutual funds. The results, which appear in Figure 9.4, show that the distribution of alphas is roughly bell shaped, with a mean that is slightly negative but statistically indistinguishable from zero. On average, it does not appear that mutual funds outperform the market index (the S&P 500) on a risk-adjusted basis.[11]

[10]Burton G. Malkiel, "Returns from Investing in Equity Mutual Funds 1971–1991," *Journal of Finance* 50 (June 1995), pp. 549–72.

[11]Notice that the study included all mutual funds with at least 10 years of continuous data. This suggests the average alpha from this sample would be upward biased because funds that failed after less than 10 years were ignored and omitted from the left tail of the distribution. This *survivorship bias* makes the finding that the average fund underperformed the index even more telling. We discuss survivorship bias further in Chapter 11.

This result is quite meaningful. While we might expect realized alpha values of individual securities to center around zero, professionally managed mutual funds might be expected to demonstrate average positive alphas. Funds with superior performance (and we do expect this set to be non-empty) should tilt the sample average to a positive value. The small impact of superior funds on this distribution suggests the difficulty in beating the passive strategy that the CAPM deems to be optimal.

There is yet another applicable variation on the intuition of the index model, the **market model.** Formally, the market model states that the return "surprise" of any security is proportional to the return surprise of the market, plus a firm-specific surprise:

$$r_i - E(r_i) = \beta_i[r_M - E(r_M)] + e_i$$

This equation divides returns into firm-specific and systematic components somewhat differently from the index model. If the CAPM is valid, however, you can confirm that, substituting for $E(r_i)$ from Equation 9.9, the market model equation becomes identical to the index model. For this reason the terms "index model" and "market model" often are used interchangeably.

CONCEPT CHECK 6

Can you sort out the nuances of the following maze of models?

a. CAPM
b. Single-factor model
c. Single-index model
d. Market model

9.3 IS THE CAPM PRACTICAL?

To discuss the role of the CAPM in real-life investments we have to answer two questions. First, even if we all agreed that the CAPM were the best available theoretical model to explain rates of return on risky assets, how would this affect practical investment policy? Second, how can we determine whether the CAPM is in fact the best available model to explain rates of return on risky assets?

Notice the wording of the first question. We don't pose it as: "Suppose the CAPM perfectly explains the rates of return on risky assets. . . ." All models, whether in economics or science, are based on simplifications that enable us to come to grips with a complicated reality, which means that perfection is an unreasonable and unusable standard. In our context, we must clarify what "perfectly explains" would mean. From the previous section we know that if the CAPM were valid, a single-index model in which the index includes all traded securities (i.e., all risky securities in the investable universe as in Assumption 3) also would be valid. In this case, "perfectly explains" would mean that all alpha values in security risk premiums would be identically zero.

The notion that all alpha values can be identically zero is feasible in principle, but such a configuration cannot be expected to emerge in real markets. This was demonstrated by Grossman and Stiglitz, who showed that such an equilibrium may be one that the real economy can approach, but not necessarily reach.[12] Their basic idea is that the

[12]Sanford J. Grossman and Joseph E. Stiglitz, "On the Impossibility of Informationally Efficient Markets," *American Economic Review* 70 (June 1981).

actions of security analysts are the forces that drive security prices to "proper" levels at which alpha is zero. But if all alphas were identically zero, there would be no incentive to engage in such security analysis. Instead, the market equilibrium will be characterized by prices hovering "near" their proper values, at which alphas are almost zero, but with enough slippage (and therefore reward for superior insight) to induce analysts to continue their efforts.

A more reasonable standard, that the CAPM is the "best available model to explain rates of return on risky assets," means that in the absence of security analysis, one should take security alphas as zero. A security is mispriced if and only if its alpha is nonzero—underpriced if alpha is positive and overpriced if alpha is negative—and positive or negative alphas are revealed only by superior security analysis. Absent the investment of significant resources in such analysis, an investor would obtain the best investment portfolio on the assumption that all alpha values are zero. This definition of the superiority of the CAPM over any other model also determines its role in real-life investments.

Under the assumption that the CAPM is the best available model, investors willing to expend resources to construct a superior portfolio must (1) identify a practical index to work with and (2) deploy macro analysis to obtain good forecasts for the index and security analysis to identify mispriced securities. This procedure was described in Chapter 8 and is further elaborated on in Part Five (Security Analysis) and Part Seven (Applied Portfolio Management).

We will examine several tests of the CAPM in Chapter 13. But it is important to explain the results of these tests and their implications.

Is the CAPM Testable?

Let us consider for a moment what testability means. A model consists of (i) a set of assumptions, (ii) logical/mathematical development of the model through manipulation of those assumptions, and (iii) a set of predictions. Assuming the logical/mathematical manipulations are free of errors, we can test a model in two ways, *normative* and *positive.* Normative tests examine the assumptions of the model, while positive tests examine the predictions.

If a model's assumptions are valid, and the development is error-free, then the predictions of the model must be true. In this case, testing the assumptions is synonymous with testing the model. But few, if any, models can pass the normative test. In most cases, as with the CAPM, the assumptions are admittedly invalid—we recognize that we have simplified reality, and therefore to this extent are relying on "untrue" assumptions. The motivation for invoking unrealistic assumptions is clear; we simply cannot solve a model that is perfectly consistent with the full complexity of real-life markets. As we've noted, the need to use simplifying assumptions is not peculiar to economics—it characterizes all of science.

Assumptions are chosen first and foremost to render the model solvable. But we prefer assumptions to which the model is "robust." A model is robust with respect to an assumption if its predictions are not highly sensitive to violation of the assumption. If we use only assumptions to which the model is robust, the model's predictions will be reasonably accurate despite its shortcomings. The upshot of all this is that tests of models are almost always positive—we judge a model on the success of its empirical predictions. This standard brings statistics into any science and requires us to take a stand on what are acceptable levels of significance and power.[13] Because the nonrealism of the assumptions precludes a normative test, the positive test is really a test of the robustness of the model to its assumptions.

The CAPM implications are embedded in two predictions: (1) the market portfolio is efficient, and (2) the security market line (the expected return–beta relationship) accurately describes the risk–return trade-off, that is, alpha values are zero. In fact, the second implication can be derived from the first, and therefore both stand or fall together in a test that the market portfolio is mean-variance efficient. The central problem in testing this prediction is that the hypothesized market portfolio is unobservable. The "market portfolio" includes *all* risky assets that can be held by investors. This is far more extensive than an equity index. It would include bonds, real estate, foreign assets, privately held businesses, and human capital. These assets are often traded thinly or (for example, in the case of human capital) not traded at all. It is difficult to test the efficiency of an observable portfolio, let alone an unobservable one. These problems alone make adequate testing of the model infeasible.[14] Moreover, even small departures from efficiency in the market portfolio can lead to large departures from the expected return–beta relationship of the SML, which would negate the practical usefulness of the model.

The CAPM Fails Empirical Tests

Because the market portfolio cannot be observed, tests of the CAPM revolve around the expected return–beta relationship. The tests use proxies such as the S&P 500 index to stand in for the true market portfolio. These tests therefore appeal to robustness of the assumption that the market proxy is sufficiently close to the true, unobservable market portfolio. The CAPM fails these tests, that is, the data reject the hypothesis that alpha values are uniformly zero at acceptable levels of significance. For example, we find that, on average, low-beta securities have positive alphas and high-beta securities have negative alphas.

It is possible that this is a result of a failure of our data, the validity of the market proxy, or statistical method. If so, we would conclude the following: There is no better model out there, but we measure beta and alpha values with unsatisfactory precision. This situation

[13] To illustrate the meanings of significance and power, consider a test of the efficacy of a new drug. The agency testing the drug may make two possible errors. The drug may be useless (or even harmful), but the agency may conclude that it is useful. This is called a "Type I" error. The *significance level* of a test is the probability of a Type I error. Typical practice is to fix the level of significance at some low level, for example, 5%. In the case of drug testing, for example, the first goal is to avoid introducing ineffective or harmful treatments. The other possible error is that the drug is actually useful, but the testing procedure concludes it is not. This mistake, called "Type II" error, would lead us to discard a useful treatment. The *power* of the test is the probability of avoiding Type II error (i.e., one minus the probability of making such an error), that is, the probability of accepting the drug if it is indeed useful. We want tests that, at a given level of significance, have the most power, so we will admit effective drugs with high probability. In social sciences in particular, available tests often have low power, in which case they are susceptible to Type II error and will reject a correct model (a "useful drug") with high frequency. "The drug is useful" is analogous in the CAPM to alphas being zero. When the test data reject the hypothesis that observed alphas are zero at the desired level of significance, the CAPM fails. However, if the test has low power, the probability that we accept the model when true is not all that high.

[14] The best-known discussion of the difficulty in testing the CAPM is now called "Roll's critique." See Richard Roll, "A Critique of the Asset Pricing Theory's Tests: Part I: On Past and Potential Testability of the Theory," *Journal of Financial Economics* 4 (1977). The issue is developed further in Richard Roll and Stephen A. Ross, "On the Cross-Sectional Relation between Expected Return and Betas," *Journal of Finance* 50 (1995); and Schmuel Kandel and Robert F. Stambaugh, "Portfolio Inefficiency and the Cross-Section of Expected Returns," *Journal of Finance* 50 (1995).

would call for improved technique. But if the rejection of the model is not an artifact of statistical problems, then we must search for extensions to the CAPM, or substitute models. We will consider several extensions of the model later in the chapter.

The Economy and the Validity of the CAPM

For better or worse, some industries are regulated, with rate commissions either setting or approving prices. Imagine a commission pondering a rate case for a regulated utility. The rate commission must decide whether the rates charged by the company are sufficient to grant shareholders a fair rate of return on their investments. The normative framework of the typical rate hearing is that shareholders, who have made an investment in the firm, are entitled to earn a "fair" rate of return on their equity investment. The firm is therefore allowed to charge prices that are expected to generate a profit consistent with that fair rate of return.

The question of fairness of the rate of return to the company shareholders cannot be divorced from the level of risk of these returns. The CAPM provides the commission a clear criterion: If the rates under current regulation are too low, then the rate of return to equity investors would be less than commensurate with risk, and alpha would be negative. As we pointed out in Example 9.1, the commissioner's problem may now be organized around arguments about estimates of risk and the security market line.

Similar applications arise in many legal settings. For example, contracts with payoffs that are contingent on a fair rate of return can be based on the index rate of return and the beta of appropriate assets. Many disputes involving damages require that a stream of losses be discounted to a present value. The proper discount rate depends on risk, and disputes about fair compensation to litigants can be (and often are) set on the basis of the SML, using past data that differentiate systematic from firm-specific risk.

It may be surprising to find that the CAPM is an accepted norm in the U.S. and many other developed countries, despite its empirical shortcomings. We can offer a twofold explanation. First, the logic of the decomposition to systematic and firm-specific risk is compelling. Absent a better model to assess nonmarket components of risk premiums, we must use the best method available. As improved methods of generating equilibrium security returns become empirically validated, they gradually will be incorporated into institutional decision making. Such improvements may come either from extensions of the CAPM and its companion, arbitrage pricing theory (discussed in the next chapter), or from a yet- undiscovered new model.

Second, there is impressive, albeit less-formal, evidence that the central conclusion of the CAPM—the efficiency of the market portfolio—may not be all that far from being valid. Thousands of mutual funds within hundreds of investment companies compete for investor money. These mutual funds employ professional analysts and portfolio managers and expend considerable resources to construct superior portfolios. But the number of funds that consistently outperform a simple strategy of investing in passive market index portfolios is extremely small, suggesting that the single-index model with ex ante zero alpha values may be a reasonable working approximation for most investors.

The Investments Industry and the Validity of the CAPM

More than other practitioners, investment firms must take a stand on the validity of the CAPM. If they judge the CAPM invalid, they must turn to a substitute framework to guide them in constructing optimal portfolios.

For example, the CAPM provides discount rates that help security analysts assess the intrinsic value of a firm. If an analyst believes that some actual prices differ from intrinsic values, then those securities have nonzero alphas, and there is an opportunity to construct an active portfolio with a superior risk–return profile. But if the discount rate used to assess intrinsic value is incorrect because of a failure in the CAPM, the estimate of alpha will be biased, and both the Markowitz model of Chapter 7 and the index model of Chapter 8 will actually lead to inferior portfolios. When constructing their presumed optimal risky portfolios, practitioners must be satisfied that the passive index they use for that purpose is satisfactory and that the ratios of alpha to residual variance are appropriate measures of investment attractiveness. This would not be the case if the CAPM is invalid. Yet it appears many practitioners do use index models (albeit often with additional indexes) when assessing security prices. The curriculum of the CFA Institute also suggests a widespread acceptance of the CAPM, at least as a starting point for thinking about the risk–return relationship. An explanation similar to the one we offered in the previous subsection is equally valid here.

The central conclusion from our discussion so far is that, explicitly or implicitly, practitioners do use a CAPM. If they use a single-index model and derive optimal portfolios from ratios of alpha forecasts to residual variance, they behave as if the CAPM is valid.[15] If they use a multi-index model, then they use one of the extensions of the CAPM (discussed later in this chapter) or arbitrage pricing theory (discussed in the next chapter). Thus, theory and evidence on the CAPM should be of interest to all sophisticated practitioners.

9.4 ECONOMETRICS AND THE EXPECTED RETURN–BETA RELATIONSHIP

When assessing the empirical success of the CAPM, we must also consider our econometric technique. If our tests are poorly designed, we may mistakenly reject the model. Similarly, some empirical tests implicitly introduce additional assumptions that are not part of the CAPM, for example, that various parameters of the model such as beta or residual variance are constant over time. If these extraneous additional assumptions are too restrictive, we also may mistakenly reject the model.

To begin, notice that all the coefficients of a regression equation are estimated simultaneously, and these estimates are not independent. In particular, the estimate of the intercept (alpha) of a single- (independent) variable regression depends on the estimate of the slope coefficient. Hence, if the beta estimate is inefficient and/or biased, so will be the estimate of the intercept. Unfortunately, statistical bias is easily introduced.

An example of this hazard was pointed out in an early paper by Miller and Scholes,[16] who demonstrated how econometric problems could lead one to reject the CAPM even if it were perfectly valid. They considered a checklist of difficulties encountered in testing the model and showed how these problems potentially could bias conclusions. To prove the point, they simulated rates of return that were *constructed* to satisfy the predictions of the CAPM and used these rates to "test" the model with standard statistical techniques of the day. The result of these tests was a rejection of the model that looks surprisingly similar to what we find in tests of returns from actual data—this despite the fact that the "data" were constructed to satisfy the CAPM. Miller and Scholes thus demonstrated that econometric technique alone could be responsible for the rejection of the model in actual tests.

[15]We need to be a bit careful here. On its face, the CAPM asserts that alpha values will equal zero in security market equilibrium. But as we argued earlier, consistent with the vast amount of security analysis that actually takes place, a better way to interpret the CAPM is that equilibrium really means that alphas should be taken to be zero in the absence of security analysis. With private information or superior insight one presumably would be able to identify stocks that are mispriced by the market and thus offer nonzero alphas.

[16]Merton H. Miller and Myron Scholes, "Rates of Return in Relations to Risk: A Re-examination of Some Recent Findings," in *Studies in the Theory of Capital Markets*, Michael C. Jensen, ed. (New York: Praeger, 1972).

There are several potential problems with the estimation of beta coefficients. First, when residuals are correlated (as is common for firms in the same industry), standard beta estimates are not efficient. A simple approach to this problem would be to use statistical techniques designed for these complications. For example, we might replace OLS (ordinary least squares) regressions with GLS (generalized least squares) regressions, which account for correlation across residuals. Moreover, both coefficients, alpha and beta, as well as residual variance, are likely time varying. There is nothing in the CAPM that precludes such time variation, but standard regression techniques rule it out and thus may lead to false rejection of the model. There are now well-known techniques to account for time-varying parameters. In fact, Robert Engle won the Nobel Prize for his pioneering work on econometric techniques to deal with time-varying volatility, and a good portion of the applications of these new techniques have been in finance.[17] Moreover, betas may vary not purely randomly over time, but in response to changing economic conditions. A "conditional" CAPM allows risk and return to change with a set of "conditioning variables."[18]

As importantly, Campbell and Vuolteenaho[19] find that the beta of a security can be decomposed into two components, one of which measures sensitivity to changes in corporate profitability and another which measures sensitivity to changes in the market's discount rates. These are found to be quite different in many cases. Improved econometric techniques such as those proposed in this short survey may help resolve part of the empirical failure of the simple CAPM.

9.5 EXTENSIONS OF THE CAPM

The CAPM uses a number of simplifying assumptions. We can gain greater predictive accuracy at the expense of greater complexity by relaxing some of those assumptions. In this section, we will consider a few of the more important attempts to extend the model. This discussion is not meant to be exhaustive. Rather, it introduces a few extensions of the basic model to provide insight into the various attempts to improve empirical content.

The Zero-Beta Model

Efficient frontier portfolios have a number of interesting characteristics, independently derived by Merton and Roll.[20] Three of these are

1. Any portfolio that is a combination of two frontier portfolios is itself on the efficient frontier.

[17]Engle's work gave rise to the widespread use of so-called ARCH models. ARCH stands for autoregressive conditional heteroskedasticity, which is a fancy way of saying that volatility changes over time, and that recent levels of volatility can be used to form optimal estimates of future volatility.

[18]There is now a large literature on conditional models of security market equilibrium. Much of it derives from Ravi Jagannathan and Zhenyu Wang, "The Conditional CAPM and the Cross-Section of Expected Returns," *Journal of Finance* 51 (March 1996), vol pp. 3–53.

[19]John Campbell and Tuomo Vuolteenaho, "Bad Beta, Good Beta," *American Economic Review* 94 (December 2004), pp. 1249–75.

[20]Robert C. Merton, "An Analytic Derivation of the Efficient Portfolio Frontier," *Journal of Financial and Quantitative Analysis*, 1972. Roll, see footnote 14.

2. The expected return of any asset can be expressed as an exact linear function of the expected return on any two efficient-frontier portfolios P and Q according to the following equation:

$$E(r_i) - E(r_Q) = [E(r_P) - E(r_Q)]\frac{\text{Cov}(r_i, r_P) - \text{Cov}(r_P, r_Q)}{\sigma_P^2 - \text{Cov}(r_P, r_Q)} \qquad \textbf{(9.11)}$$

3. Every portfolio on the efficient frontier, except for the global minimum-variance portfolio, has a "companion" portfolio on the bottom (inefficient) half of the frontier with which it is uncorrelated. Because it is uncorrelated, the companion portfolio is referred to as the **zero-beta portfolio** of the efficient portfolio. If we choose the market portfolio M and its zero-beta companion portfolio Z, then Equation 9.11 simplifies to the CAPM-like equation

$$E(r_i) - E(r_Z) = [E(R_M) - E(R_Z)]\frac{\text{Cov}(r_i, r_M)}{\sigma_M^2} = \beta_i[E(r_M) - E(r_Z)] \qquad \textbf{(9.12)}$$

   Equation 9.12 resembles the SML of the CAPM, except that the risk-free rate is replaced with the expected return on the zero-beta companion of the market index portfolio.

Fischer Black used these properties to show that Equation 9.12 is the CAPM equation that results when investors face restrictions on borrowing and/or investment in the risk-free asset.[21] In this case, at least some investors will choose portfolios on the efficient frontier that are not necessarily the market index portfolio. Because average returns on the zero-beta portfolio are greater than observed T-bill rates, the zero-beta model can explain why average estimates of alpha values are positive for low-beta securities and negative for high-beta securities, contrary to the prediction of the CAPM. Despite this, the model is not sufficient to rescue the CAPM from empirical rejection.

Labor Income and Nontraded Assets

An important departure from realism is the CAPM assumption that all risky assets are traded. Two important asset classes that are *not* traded are human capital and privately held businesses. The discounted value of future labor income exceeds the total market value of traded assets. The market value of privately held corporations and businesses is of the same order of magnitude. Human capital and private enterprises are different types of assets with possibly different implications for equilibrium returns on traded securities.

Privately held business may be the lesser of the two sources of departures from the CAPM. Nontraded firms can be incorporated or sold at will, save for liquidity considerations that we discuss in the next section. Owners of private business also can borrow against their value, further diminishing the material difference between ownership of private and public business. Suppose that privately held business have similar risk characteristics as those of traded assets. In this case, individuals can partially offset the diversification problems posed by their nontraded entrepreneurial assets by reducing their portfolio demand for securities of similar, traded assets. Thus, the CAPM expected return–beta equation may not be greatly disrupted by the presence of entrepreneurial income.

To the extent that risk characteristics of private enterprises differ from those of traded securities, a portfolio of traded assets that best hedges the risk of typical private business

[21]Fischer Black, "Capital Market Equilibrium with Restricted Borrowing," *Journal of Business*, July 1972.

would enjoy excess demand from the population of private business owners. The price of assets in this portfolio will be bid up relative to the CAPM considerations, and the expected returns on these securities will be lower in relation to their systematic risk. Conversely, securities highly correlated with such risk will have high equilibrium risk premiums and may appear to exhibit positive alphas relative to the conventional SML. In fact, Heaton and Lucas show that adding proprietary income to a standard asset-pricing model improves its predictive performance.[22]

The size of labor income and its special nature is of greater concern for the validity of the CAPM. The possible effect of labor income on equilibrium returns can be appreciated from its important effect on personal portfolio choice. Despite the fact that an individual can borrow against labor income (via a home mortgage) and reduce some of the uncertainty about future labor income via life insurance, human capital is less "portable" across time and may be more difficult to hedge using traded securities than nontraded business. This may induce pressure on security prices and result in departures from the CAPM expected return–beta equation. For one example, surely an individual seeking diversification should avoid investing in his employer's stock and limit investments in the same industry. Thus, the demand for stocks of labor-intensive firms may be reduced, and these stocks may require a higher expected return than predicted by the CAPM.

Mayers[23] derives the equilibrium expected return–beta equation for an economy in which individuals are endowed with labor income of varying size relative to their nonlabor capital. The resultant SML equation is

$$E(R_i) = E(R_M)\frac{\text{Cov}(R_i, R_M) + \frac{P_H}{P_M}\text{Cov}(R_i, R_H)}{\sigma_M^2 + \frac{P_H}{P_M}\text{Cov}(R_M, R_H)} \qquad \textbf{(9.13)}$$

where

P_H = value of aggregate human capital,

P_M = market value of traded assets (market portfolio),

R_H = excess rate of return on aggregate human capital.

The CAPM measure of systematic risk, beta, is replaced in the extended model by an adjusted beta that also accounts for covariance with the portfolio of aggregate human capital. Notice that the ratio of human capital to market value of all traded assets, P_H/P_M, may well be greater than 1, and hence the effect of the covariance of a security with labor income, $\text{Cov}(R_i, R_H)$, relative to the average, $\text{Cov}(R_M, R_H)$, is likely to be economically significant. When $\text{Cov}(R_i, R_H)$ is positive, the adjusted beta is greater when the CAPM beta is smaller than 1, and vice versa. Because we expect $\text{Cov}(R_i, R_H)$ to be positive for the average security, the risk premium in this model will be greater, on average, than predicted by the CAPM for securities with beta less than 1, and smaller for securities with beta greater than 1. The model thus predicts a security market line that is less steep than that of the standard CAPM. This may help explain the average negative alpha of high-beta securities and positive alpha of low-beta securities that lead to the statistical failure of the CAPM equation. In Chapter 13 on empirical evidence we present additional results along these lines.

[22]John Heaton and Deborah Lucas, "Portfolio Choice and Asset Prices: The Importance of Entrepreneurial Risk, *Journal of Finance* 55 (June 2000). This paper offers evidence of the effect of entrepreneurial risk on both portfolio choice and the risk–return relationship.

[23]See footnote 8.

A Multiperiod Model and Hedge Portfolios

Robert C. Merton revolutionized financial economics by using continuous-time models to extend many of our models of asset pricing.[24] While his (Nobel Prize–winning) contributions to option-pricing theory and financial engineering (along with those of Fischer Black and Myron Scholes) may have had greater impact on the investment industry, his solo contribution to portfolio theory was equally important for our understanding of the risk–return relationship.

In his basic model, Merton relaxes the "single-period" myopic assumptions about investors. He envisions individuals who optimize a lifetime consumption/investment plan, and who continually adapt consumption/investment decisions to current wealth and planned retirement age. When uncertainty about portfolio returns is the only source of risk and investment opportunities remain unchanged through time, that is, there is no change in the probability distribution of the return on the market portfolio or individual securities, Merton's so-called intertemporal capital asset pricing model (ICAPM) predicts the same expected return–beta relationship as the single-period equation.[25]

But the situation changes when we include additional sources of risk. These extra risks are of two general kinds. One concerns changes in the parameters describing investment opportunities, such as future risk-free rates, expected returns, or the risk of the market portfolio. For example, suppose that the real interest rate may change over time. If it falls in some future period, one's level of wealth will now support a lower stream of real consumption. Future spending plans, for example, for retirement spending, may be put in jeopardy. To the extent that returns on some securities are correlated with changes in the risk-free rate, a portfolio can be formed to hedge such risk, and investors will bid up the price (and bid down the expected return) of those hedge assets. Investors will sacrifice some expected return if they can find assets whose returns will be higher when other parameters (in this case, the risk-free rate) change adversely.

The other additional source of risk concerns the prices of the consumption goods that can be purchased with any amount of wealth. Consider as an example inflation risk. In addition to the expected level and volatility of their nominal wealth, investors must be concerned about the cost of living—what those dollars can buy. Therefore, inflation risk is an important extramarket source of risk, and investors may be willing to sacrifice some expected return to purchase securities whose returns will be higher when the cost of living changes adversely. If so, hedging demands for securities that help to protect against inflation risk would affect portfolio choice and thus expected return. One can push this conclusion even further, arguing that empirically significant hedging demands may arise for important subsectors of consumer expenditures; for example, investors may bid up share prices of energy companies that will hedge energy price uncertainty. These sorts of effects may characterize any assets that hedge important extramarket sources of risk.

More generally, suppose we can identify K sources of extramarket risk and find K associated hedge portfolios. Then, Merton's ICAPM expected return–beta equation would generalize the SML to a multi-index version:

$$E(R_i) = \beta_{iM} E(R_M) + \sum_{k=1}^{K} \beta_{ik} E(R_k) \tag{9.14}$$

where β_{iM} is the familiar security beta on the market-index portfolio, and β_{ik} is the beta on the kth hedge portfolio.

[24]Merton's classic works are collected in *Continuous-Time Finance* (Oxford, U.K.: Basil Blackwell, 1992).

[25]Eugene F. Fama also made this point in "Multiperiod Consumption-Investment Decisions," *American Economic Review* 60 (1970).

Other multifactor models using additional factors that do not arise from extramarket sources of risk have been developed and lead to SMLs of a form identical to that of the ICAPM. These models also may be considered extensions of the CAPM in the broad sense. We examine these models in the next chapter.

A Consumption-Based CAPM

The logic of the CAPM together with the hedging demands noted in the previous subsection suggests that it might be useful to center the model directly on consumption. Such models were first proposed by Mark Rubinstein, Robert Lucas, and Douglas Breeden.[26]

In a lifetime consumption plan, the investor must in each period balance the allocation of current wealth between today's consumption and the savings and investment that will support future consumption. When optimized, the utility value from an additional dollar of consumption today must be equal to the utility value of the expected future consumption that can be financed by that additional dollar of wealth.[27] Future wealth will grow from labor income, as well as returns on that dollar when invested in the optimal complete portfolio.

Suppose risky assets are available and you wish to increase expected consumption growth by allocating some of your savings to a risky portfolio. How would we measure the risk of these assets? As a general rule, investors will value additional income more highly during difficult economic times (when consumption opportunities are scarce) than in affluent times (when consumption is already abundant). An asset will therefore be viewed as riskier in terms of consumption if it has positive covariance with consumption growth—in other words, if its payoff is higher when consumption is already high and lower when consumption is relatively restricted. Therefore, equilibrium risk premiums will be greater for assets that exhibit higher covariance with consumption growth. Developing this insight, we can write the risk premium on an asset as a function of its "consumption risk" as follows:

$$E(R_i) = \beta_{iC}\text{RP}_C \tag{9.15}$$

where portfolio C may be interpreted as a *consumption-tracking portfolio* (also called a *consumption-mimicking portfolio*), that is, the portfolio with the highest correlation with consumption growth; β_{iC} is the slope coefficient in the regression of asset i's excess returns, R_i, on those of the consumption-tracking portfolio; and, finally, RP_C is the risk premium associated with consumption uncertainty, which is measured by the expected excess return on the consumption-tracking portfolio:

$$\text{RP}_C = E(R_C) = E(r_C) - r_f \tag{9.16}$$

Notice how similar this conclusion is to the conventional CAPM. The consumption-tracking portfolio in the CCAPM plays the role of the market portfolio in the conventional CAPM. This is in accord with its focus on the risk of *consumption* opportunities rather than the risk and return of the *dollar* value of the portfolio. The excess return on the

[26]Mark Rubinstein, "The Valuation of Uncertain Income Streams and the Pricing of Options," *Bell Journal of Economics and Management Science* 7 (1976), pp. 407–25; Robert Lucas, "Asset Prices in an Exchange Economy," *Econometrica* 46 (1978), pp. 1429–45; Douglas Breeden, "An Intertemporal Asset Pricing Model with Stochastic Consumption and Investment Opportunities," *Journal of Financial Economics* 7 (1979), pp. 265–96.

[27]Wealth at each point in time equals the market value of assets in the balance sheet plus the present value of future labor income. These models of consumption and investment decisions are often made tractable by assuming investors exhibit constant relative risk aversion, or CRRA. CRRA implies that an individual invests a constant proportion of wealth in the optimal risky portfolio regardless of the level of wealth. You might recall that our prescription for optimal capital allocation in Chapter 6 also called for an optimal investment proportion in the risky portfolio regardless of the level of wealth. The utility function we employed there also exhibited CRRA.

consumption-tracking portfolio plays the role of the excess return on the market portfolio, *M*. Both approaches result in linear, single-factor models that differ mainly in the identity of the factor they use.

In contrast to the CAPM, the beta of the market portfolio on the market factor of the CCAPM is not necessarily 1. It is perfectly plausible and empirically evident that this beta is substantially greater than 1. This means that in the linear relationship between the market index risk premium and that of the consumption portfolio,

$$E(R_M) = \alpha_M + \beta_{MC} E(R_C) + \varepsilon_M \quad \textbf{(9.17)}$$

where α_M and ε_M allow for empirical deviation from the exact model in Equation 9.15, and β_{MC} is not necessarily equal to 1.

Because the CCAPM is so similar to the CAPM, one might wonder about its usefulness. Indeed, just as the CAPM is empirically flawed because not all assets are traded, so is the CCAPM. The attractiveness of this model is in that it compactly incorporates consumption hedging and possible changes in investment opportunities, that is, in the parameters of the return distributions in a single-factor framework. There is a price to pay for this compactness, however. Consumption growth figures are published infrequently (monthly at the most) compared with financial assets, and are measured with significant error. Nevertheless, recent empirical research[28] indicates that this model is more successful in explaining realized returns than the CAPM, which is a reason why students of investments should be familiar with it. We return to this issue, as well as empirical evidence concerning the CCAPM, in Chapter 13.

9.6 LIQUIDITY AND THE CAPM

Standard models of asset pricing (such as the CAPM) assume frictionless markets, meaning that securities can be traded costlessly. But these models actually have little to say about trading activity. For example, in the equilibrium of the CAPM, all investors share all available information and demand identical portfolios of risky assets. The awkward implication of this result is that there is no reason for trade. If all investors hold identical portfolios of risky assets, then when new (unexpected) information arrives, prices will change commensurately, but each investor will continue to hold a piece of the market portfolio, which requires no exchange of assets. How do we square this implication with the observation that on a typical day, more than 3 billion shares change hands on the New York Stock Exchange alone? One obvious answer is heterogeneous expectations, that is, beliefs not shared by the entire market. Such private information will give rise to trading as investors attempt to profit by rearranging portfolios in accordance with their now-heterogeneous demands. In reality, trading (and trading costs) will be of great importance to investors.

The **liquidity** of an asset is the ease and speed with which it can be sold at fair market value. Part of liquidity is the cost of engaging in a transaction, particularly the bid–ask spread. Another part is price impact—the adverse movement in price one would encounter when attempting to execute a larger trade. Yet another component is immediacy—the ability to sell the asset quickly without reverting to fire-sale prices. Conversely, **illiquidity** can be measured in part by the discount from fair market value a seller must accept if the asset is to be sold quickly. A perfectly liquid asset is one that would entail no illiquidity discount.

[28] Ravi Jagannathan and Yong Wang, "Lazy Investors, Discretionary Consumption, and the Cross-Section of Stock Returns," *Journal of Finance* 62 (August 2007), pp. 1633–61.

WORDS FROM THE STREET

STOCK INVESTORS PAY HIGH PRICE FOR LIQUIDITY

Given a choice between liquid and illiquid stocks, most investors, to the extent they think of it at all, opt for issues they know are easy to get in and out of.

But for long-term investors who don't trade often—which includes most individuals—that may be unnecessarily expensive. Recent studies of the performance of listed stocks show that, on average, less-liquid issues generate substantially higher returns—as much as several percentage points a year at the extremes.

ILLIQUIDITY PAYOFF

Among the academic studies that have attempted to quantify this illiquidity payoff is a recent work by two finance professors, Yakov Amihud of New York University and Tel Aviv University, and Haim Mendelson of the University of Rochester. Their study looks at New York Stock Exchange issues over the 1961–1980 period and defines liquidity in terms of bid–asked spreads as a percentage of overall share price.

Market makers use spreads in quoting stocks to define the difference between the price they'll bid to take stock off an investor's hands and the price they'll offer to sell stock to any willing buyer. The bid price is always somewhat lower because of the risk to the broker of tying up precious capital to hold stock in inventory until it can be resold.

If a stock is relatively illiquid, which means there's not a ready flow of orders from customers clamoring to buy it, there's more of a chance the broker will lose money on the trade. To hedge this risk, market makers demand an even bigger discount to service potential sellers, and the spread will widen further.

The study by Profs. Amihud and Mendelson shows that liquidity spreads—measured as a percentage discount from the stock's total price—ranged from less than 0.1%, for widely held International Business Machines Corp., to as much as 4% to 5%. The widest-spread group was dominated by smaller, low-priced stocks.

The study found that, overall, the least-liquid stocks averaged an 8.5 percent-a-year higher return than the most-liquid stocks over the 20-year period. On average, a one percentage point increase in the spread was associated with a 2.5% higher annual return for New York Stock Exchange stocks. The relationship held after results were adjusted for size and other risk factors.

An extension of the study of Big Board stocks done at *The Wall Street Journal*'s request produced similar findings. It shows that for the 1980–85 period, a one percentage-point-wider spread was associated with an extra average annual gain of 2.4%. Meanwhile, the least-liquid stocks outperformed the most-liquid stocks by almost six percentage points a year.

COST OF TRADING

Since the cost of the spread is incurred each time the stock is traded, illiquid stocks can quickly become prohibitively expensive for investors who trade frequently. On the other hand, long-term investors needn't worry so much about spreads, since they can amortize them over a longer period.

In terms of investment strategy, this suggests "that the small investor should tailor the types of stocks he or she buys to his expected holding period," Prof. Mendelson says. If the investor expects to sell within three months, he says, it's better to pay up for the liquidity and get the lowest spread. If the investor plans to hold the stock for a year or more, it makes sense to aim at stocks with spreads of 3% or more to capture the extra return.

Source: Barbara Donnelly, *The Wall Street Journal,* April 28, 1987, p. 37. Reprinted by permission of *The Wall Street Journal.*

Liquidity (or the lack of it) has long been recognized as an important characteristic that affects asset values. For example, in legal cases, courts have routinely applied very steep discounts to the values of businesses that cannot be publicly traded. But liquidity has not always been appreciated as an important factor in security markets, presumably due to the relatively small trading cost per transaction compared with the large costs of trading assets such as real estate. The breakthrough came in the work of Amihud and Mendelson[29] (see the nearby box) and today, liquidity is increasingly viewed as an important determinant of prices and expected returns. We supply only a brief synopsis of this important topic here and provide empirical evidence in Chapter 13.

[29] Yakov Amihud and Haim Mendelson, "Asset Pricing and the Bid–Ask Spread," *Journal of Financial Economics* 17(1986). A summary of the ensuing large body of literature on liquidity can be found in Yakov Amihud, Haim Mendelson, and Lasse Heje Pedersen, "Liquidity and Asset Prices," *Foundations and Trends in Finance* 1, no. 4 (2005).

Early models of liquidity focused on the inventory management problem faced by security dealers. Dealers in over-the-counter markets post prices at which they are willing to buy a security (the bid price) or sell it (the ask price). The willingness of security dealers to add to their inventory or sell shares from their inventory makes them crucial contributors to overall market liquidity. The fee they earn for supplying this liquidity is the bid–ask spread. Part of the bid–ask spread may be viewed as compensation for bearing the price risk involved in holding an inventory of securities and allowing their inventory levels to absorb the fluctuations in overall security demand. Assuming the fair price of the stock is the average of the bid and ask prices, an investor pays half the spread upon purchase and another half upon sale of the stock. A dealer on the other side of the transaction earns these spreads. The spread is one important component of liquidity—it is the cost of transacting in a security.

The advent of electronic trading has steadily diminished the role of dealers, but traders still must contend with a bid–ask spread. For example, in electronic markets, the limit-order book contains the "inside spread," that is, the difference between the highest price at which some investor will purchase any shares and the lowest price at which another investor is willing to sell. The effective bid–ask spread will also depend on the size of the desired transaction. Larger purchases will require a trader to move deeper into the limit-order book and accept less-attractive prices. While inside spreads on electronic markets often appear extremely low, effective spreads can be much larger, because the limit orders are good for only small numbers of shares.

Even without the inventory problems faced by traditional securities dealers, the importance of the spread persists. There is greater emphasis today on the component of the spread that is due to asymmetric information. By asymmetric information, we mean the potential for one trader to have private information about the value of the security that is not known to the trading partner. To see why such an asymmetry can affect the market, think about the problems facing someone buying a used car. The seller knows more about the car than the buyer, so the buyer naturally wonders if the seller is trying to get rid of the car because it is a "lemon." At the least, buyers worried about overpaying will shave the prices they are willing to pay for a car of uncertain quality. In extreme cases of asymmetric information, trading may cease altogether.[30] Similarly, traders who post offers to buy or sell at limit prices need to be worried about being picked off by better-informed traders who hit their limit prices only when they are out of line with the intrinsic value of the firm.

Broadly speaking, we may envision investors trading securities for two reasons. Some trades are driven by "noninformational" motives, for example, selling assets to raise cash for a big purchase, or even just for portfolio rebalancing. These sorts of trades, which are not motivated by private information that bears on the value of the traded security, are called *noise trades.* Security dealers will earn a profit from the bid–ask spread when transacting with noise traders (also called *liquidity traders* because their trades may derive from needs for liquidity, i.e., cash).

Other transactions are motivated by private information known only to the seller or buyer. These transactions are generated when traders believe they have come across information that a security is mispriced, and try to profit from that analysis. If an information trader identifies an advantageous opportunity, it must be disadvantageous to the other party in the transaction. If private information indicates a stock is overpriced, and the trader decides to sell it, a dealer who has posted a bid price or another trader who has posted a

[30]The problem of informational asymmetry in markets was introduced by the 2001 Nobel Laureate George A. Akerlof and has since become known as the *lemons problem.* A good introduction to Akerlof's contributions can be found in George A. Akerlof, *An Economic Theorist's Book of Tales* (Cambridge, U.K.: Cambridge University Press, 1984).

limit-buy order and ends up on the other side of the transaction will purchase the stock at what will later be revealed to have been an inflated price. Conversely, when private information results in a decision to buy, the price at which the security is traded will eventually be recognized as less than fair value.

Information traders impose a cost on both dealers and other investors who post limit orders. Although on average dealers make money from the bid–ask spread when transacting with liquidity traders, they will absorb losses from information traders. Similarly, any trader posting a limit order is at risk from information traders. The response is to increase limit-ask prices and decrease limit-bid orders—in other words, the spread must widen. The greater the relative importance of information traders, the greater the required spread to compensate for the potential losses from trading with them. In the end, therefore, liquidity traders absorb most of the cost of the information trades because the bid–ask spread that they must pay on their "innocent" trades widens when informational asymmetry is more severe.

The discount in a security price that results from illiquidity can be surprisingly large, far larger than the bid–ask spread. Consider a security with a bid–ask spread of 1%. Suppose it will change hands once a year for the next 3 years and then will be held forever by the third buyer. For the last trade, the investor will pay for the security 99.5% or .995 of its fair price; the price is reduced by half the spread that will be incurred when the stock is sold. The second buyer, knowing the security will be sold a year later for .995 of fair value, and having to absorb half the spread upon purchase, will be willing to pay $.995 - .005/1.05 = .9902$ (i.e., 99.02% of fair value), if the cost of trading is discounted at a rate of 5%. Finally, the current buyer, knowing the loss next year, when the stock will be sold for .9902 of fair value (a discount of .0098), will pay for the security only $.995 - .0098/1.05 = .9857$. Thus the discount has ballooned from .5% to 1.43%. In other words, the present values of all three future trading costs (spreads) are discounted into the current price.[31] To extend this logic, if the security will be traded once a year forever, its current illiquidity cost will equal immediate cost plus the present value of a perpetuity of .5%. At an annual discount rate of 5%, this sum equals $.005 + .005/.05 = .105$, or 10.5%! Obviously, liquidity is of potentially large value and should not be ignored in deriving the equilibrium value of securities.

Consider three stocks with equal bid–ask spreads of 1%. The first trades once a year, the second once every 2 years, and the third every 3 years. We have already calculated the price discount due to illiquidity as the present value of illiquidity costs for the first as 10.5%. The discount for the second security is .5% plus the present value of a biannual perpetuity of .5%, which at a discount rate of 5% amounts to $.5 + .5/(1.05^2 - 1) = 5.38\%$. Similarly, the cost for the security that trades only every 3 years is 3.67%. From this pattern of discounts—10.5%, 5.38%, and 3.67%—it seems that for any *given* spread, the price discount will increase almost in proportion to the frequency of trading. It also would appear that the discount should be proportional to the bid–ask spread. However, trading frequency may well vary inversely with the spread, and this will impede the response of the price discount to the spread.

An investor who plans to hold a security for a given period will calculate the impact of illiquidity costs on expected rate of return; liquidity costs will be amortized over the anticipated holding period. Investors who trade less frequently therefore will be less affected by high trading costs. The reduction in the rate of return due to trading costs is lower the longer the security is held. Hence in equilibrium, investors with long holding periods will,

[31] We will see another instance of such capitalization of trading costs in Chapter 13, where one explanation for large discounts on closed-end funds is the substantial present value of a *stream* of apparently small per-period expenses.

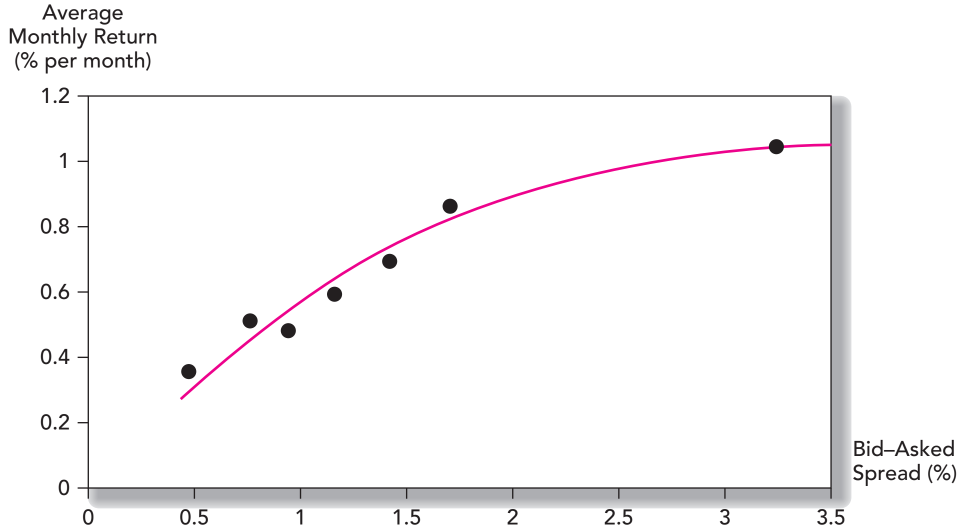


FIGURE 9.5 The relationship between illiquidity and average returns

Source: Derived from Yakov Amihud and Haim Mendelson, "Asset Pricing and the Bid–Ask Spread," *Journal of Financial Economics* 17 (1986), pp. 223–49.

on average, hold more of the illiquid securities, while short-horizon investors will more strongly prefer liquid securities. This "clientele effect" mitigates the effect of the bid–ask spread for illiquid securities. The end result is that the liquidity premium should increase with the bid–ask spread at a decreasing rate. Figure 9.5 confirms this prediction.

So far, we have shown that the expected level of liquidity can affect prices, and therefore expected rates of return. What about unanticipated changes in liquidity? Investors may also demand compensation for *liquidity risk.* The bid–ask spread of a security is not constant through time, nor is the ability to sell a security at a fair price with little notice. Both depend on overall conditions in security markets. If asset liquidity fails at times when it is most desired, then investors will require an additional price discount beyond that required for the expected cost of illiquidity.[32] In other words, there may be a *systematic* component to liquidity risk that affects the equilibrium rate of return and hence the expected return–beta relationship.

As a concrete example of such a model, Acharya and Pedersen[33] consider the impacts of both the level and the risk of liquidity on security pricing. They include three components to liquidity risk—each captures the extent to which liquidity varies systematically

[32] A good example of systematic effects in liquidity risk surrounds the demise of Long-Term Capital Management in the summer of 1998. Despite extensive analysis that indicated its portfolio was highly diversified, many of its assets went bad at the same time when Russia defaulted on its debt. The problem was that despite the fact that short and long positions were expected to balance price changes based on normal market fluctuations, a massive decline in the market liquidity and prices of some assets was not offset by increased prices of more liquid assets. As a supplier of liquidity to others, LTCM was a large holder of less-liquid securities and a liquidity shock of this magnitude was at that time an unimaginable event. While its portfolio may have been diversified in terms of exposure to traditional business condition shocks, it was undiversified in terms of exposure to liquidity shocks.

[33] V. V. Acharya and L. H. Pedersen, "Asset Pricing with Liquidity Risk," *Journal of Financial Economics* 77 (2005).

with other market conditions. They identify three relevant "liquidity betas," which measure in turn: (i) the extent to which the stock's illiquidity varies with market illiquidity; (ii) the extent to which the stock's return varies with market illiquidity; and (iii) the extent to which the stock illiquidity varies with the market return. Therefore, expected return depends on expected liquidity, as well as the conventional "CAPM beta" and three additional liquidity-related betas:

$$E(R_i) = kE(C_i) + \lambda(\beta + \beta_{L1} - \beta_{L2} - \beta_{L3}) \qquad \textbf{(9.18)}$$

where

$E(C_i)$ = expected cost of illiquidity,

k = adjustment for average holding period over all securities,

λ = market risk premium net of average market illiquidity cost, $E(R_M - C_M)$,

β = measure of systematic market risk,

$\beta_{L1}, \beta_{L2}, \beta_{L3}$ = liquidity betas.

Compared to the conventional CAPM, the expected return–beta equation now has a predicted firm-specific component that accounts for the effect of security liquidity. Such an effect would appear to be an alpha in the conventional index model.

The market risk premium itself is measured net of the average cost of illiquidity, that is, $\lambda = E(R_M - C_M)$, where C_M is the market-average cost of illiquidity.

The overall risk of each security now must account for the three elements of liquidity risk, which are defined as follows:[34]

$$\beta_{L1} = \frac{\text{Cov}(C_i, C_M)}{\text{Var}(R_M - C_M)}$$

Measures the sensitivity of the security's illiquidity to market illiquidity. Investors want additional compensation for holding a security that becomes illiquid when general liquidity is low.[34]

$$\beta_{L2} = \frac{\text{Cov}(R_i, C_M)}{\text{Var}(R_M - C_M)}$$

Measures the sensitivity of the stock's return to market illiquidity. This coefficient appears with a negative sign in Equation 9.18 because investors are willing to accept a lower average return on stocks that will provide higher returns when market illiquidity is greater.

$$\beta_{L3} = \frac{\text{Cov}(C_i, R_M)}{\text{Var}(R_M - C_M)}$$

Measures the sensitivity of security illiquidity to the market rate of return. This sensitivity also appears with a negative sign, because investors will be willing to accept a lower average return on securities that can be sold more easily (have low illiquidity costs) when the market declines.

A good number of variations on this model can be found in the current (and rapidly growing) literature on liquidity.[35] What is common to all liquidity variants is that they improve on the explanatory power of the CAPM equation and hence there is no doubt that, sooner or later, practitioner optimization models and, more important, security analysis will incorporate the empirical content of these models.

[34]Several papers have shown that there is important covariance across asset illiquidity. See for example, T. Chordia, R. Roll, and A. Subramanyam, "Commonality in Liquidity," *Journal of Financial Economics* 56 (2000), pp. 3–28 or J. Hasbrouck and D. H. Seppi "Common Factors in Prices, Order Flows and Liquidity," *Journal of Financial Economics* 59 (2001), pp. 383–411.

[35]Another influential study of liquidity risk and asset pricing is L. Pastor and R. Stambaugh, "Liquidity Risk and Expected Stock Returns," *Journal of Political Economy* 111 (2003), pp. 642–85.

SUMMARY

1. The CAPM assumes that investors are single-period planners who agree on a common input list from security analysis and seek mean-variance optimal portfolios.
2. The CAPM assumes that security markets are ideal in the sense that:
   a. They are large, and investors are price-takers.
   b. There are no taxes or transaction costs.
   c. All risky assets are publicly traded.
   d. Investors can borrow and lend any amount at a fixed risk-free rate.
3. With these assumptions, all investors hold identical risky portfolios. The CAPM holds that in equilibrium the market portfolio is the unique mean-variance efficient tangency portfolio. Thus a passive strategy is efficient.
4. The CAPM market portfolio is a value-weighted portfolio. Each security is held in a proportion equal to its market value divided by the total market value of all securities.
5. If the market portfolio is efficient and the average investor neither borrows nor lends, then the risk premium on the market portfolio is proportional to its variance, σ_M^2, and to the average coefficient of risk aversion across investors, *A:*

$$E(r_M) - r_f = \overline{A}\sigma_M^2$$

6. The CAPM implies that the risk premium on any individual asset or portfolio is the product of the risk premium on the market portfolio and the beta coefficient:

$$E(r_i) - r_f = \beta_i[E(r_M) - r_f]$$

where the beta coefficient is the covariance of the asset with the market portfolio as a fraction of the variance of the market portfolio

$$\beta_i = \frac{\text{Cov}(r_i, r_M)}{\sigma_M^2}$$

7. When risk-free investments are restricted but all other CAPM assumptions hold, then the simple version of the CAPM is replaced by its zero-beta version. Accordingly, the risk-free rate in the expected return–beta relationship is replaced by the zero-beta portfolio's expected rate of return:

$$E(r_i) = E[r_{Z(M)}] + \beta_i E[r_M - r_{Z(M)}]$$

8. The simple version of the CAPM assumes that investors are myopic. When investors are assumed to be concerned with lifetime consumption and bequest plans, but investors' tastes and security return distributions are stable over time, the market portfolio remains efficient and the simple version of the expected return–beta relationship holds. But if those distributions change unpredictably, or if investors seek to hedge nonmarket sources of risk to their consumption, the simple CAPM will give way to a multifactor version in which the security's exposure to these nonmarket sources of risk command risk premiums.
9. The consumption-based capital asset pricing model (CCAPM) is a single-factor model in which the market portfolio excess return is replaced by that of a consumption-tracking portfolio. By appealing directly to consumption, the model naturally incorporates consumption-hedging considerations and changing investment opportunities within a single-factor framework.
10. The Security Market Line of the CAPM must be modified to account for labor income and other significant nontraded assets.
11. Liquidity costs and liquidity risk can be incorporated into the CAPM relationship. Investors demand compensation for both expected costs of illiquidity as well as the risk surrounding those costs.

Related Web sites for this chapter are available at **www.mhhe.com/bkm**

KEY TERMS

homogeneous expectations
market portfolio
mutual fund theorem
market price of risk
beta
expected return–beta relationship
security market line (SML)
alpha
market model
zero-beta portfolio
liquidity
illiquidity

PROBLEM SETS

Quiz

1. What must be the beta of a portfolio with $E(r_P) = 18\%$, if $r_f = 6\%$ and $E(r_M) = 14\%$?
2. The market price of a security is \$50. Its expected rate of return is 14%. The risk-free rate is 6% and the market risk premium is 8.5%. What will be the market price of the security if its correlation coefficient with the market portfolio doubles (and all other variables remain unchanged)? Assume that the stock is expected to pay a constant dividend in perpetuity.
3. Are the following true or false? Explain.
   a. Stocks with a beta of zero offer an expected rate of return of zero.
   b. The CAPM implies that investors require a higher return to hold highly volatile securities.
   c. You can construct a portfolio with beta of .75 by investing .75 of the investment budget in T-bills and the remainder in the market portfolio.

Problems

4. You are a consultant to a large manufacturing corporation that is considering a project with the following net after-tax cash flows (in millions of dollars):

Years from Now	After-Tax Cash Flow
0	−40
1–10	15

The project's beta is 1.8. Assuming that $r_f = 8\%$ and $E(r_M) = 16\%$, what is the net present value of the project? What is the highest possible beta estimate for the project before its NPV becomes negative?

5. Consider the following table, which gives a security analyst's expected return on two stocks for two particular market returns:

Market Return	Aggressive Stock	Defensive Stock
5%	−2%	6%
25	38	12

   a. What are the betas of the two stocks?
   b. What is the expected rate of return on each stock if the market return is equally likely to be 5% or 25%?
   c. If the T-bill rate is 6% and the market return is equally likely to be 5% or 25%, draw the SML for this economy.
   d. Plot the two securities on the SML graph. What are the alphas of each?
   e. What hurdle rate should be used by the management of the aggressive firm for a project with the risk characteristics of the defensive firm's stock?

For Problems 6 to 12: If the simple CAPM is valid, which of the following situations are possible? Explain. Consider each situation independently.

6.

Portfolio	Expected Return	Beta
A	20	1.4
B	25	1.2

7.

Portfolio	Expected Return	Standard Deviation
A	30	35
B	40	25

8.

Portfolio	Expected Return	Standard Deviation
Risk-free	10	0
Market	18	24
A	16	12

9.

Portfolio	Expected Return	Standard Deviation
Risk-free	10	0
Market	18	24
A	20	22

10.

Portfolio	Expected Return	Beta
Risk-free	10	0
Market	18	1.0
A	16	1.5

11.

Portfolio	Expected Return	Beta
Risk-free	10	0
Market	18	1.0
A	16	0.9

12.

Portfolio	Expected Return	Standard Deviation
Risk-free	10	0
Market	18	24
A	16	22

For Problems 13 to 15 assume that the risk-free rate of interest is 6% and the expected rate of return on the market is 16%.

13. A share of stock sells for $50 today. It will pay a dividend of $6 per share at the end of the year. Its beta is 1.2. What do investors expect the stock to sell for at the end of the year?
14. I am buying a firm with an expected perpetual cash flow of $1,000 but am unsure of its risk. If I think the beta of the firm is .5, when in fact the beta is really 1, how much *more* will I offer for the firm than it is truly worth?
15. A stock has an expected rate of return of 4%. What is its beta?
16. Two investment advisers are comparing performance. One averaged a 19% rate of return and the other a 16% rate of return. However, the beta of the first investor was 1.5, whereas that of the second was 1.
    a. Can you tell which investor was a better selector of individual stocks (aside from the issue of general movements in the market)?
    b. If the T-bill rate were 6% and the market return during the period were 14%, which investor would be the superior stock selector?
    c. What if the T-bill rate were 3% and the market return were 15%?

17. Suppose the rate of return on short-term government securities (perceived to be risk-free) is about 5%. Suppose also that the expected rate of return required by the market for a portfolio with a beta of 1 is 12%. According to the capital asset pricing model:
    a. What is the expected rate of return on the market portfolio?
    b. What would be the expected rate of return on a stock with $\beta = 0$?
    c. Suppose you consider buying a share of stock at $40. The stock is expected to pay $3 dividends next year and you expect it to sell then for $41. The stock risk has been evaluated at $\beta = -.5$. Is the stock overpriced or underpriced?
18. Suppose that borrowing is restricted so that the zero-beta version of the CAPM holds. The expected return on the market portfolio is 17%, and on the zero-beta portfolio it is 8%. What is the expected return on a portfolio with a beta of .6?
19. *a.* A mutual fund with beta of .8 has an expected rate of return of 14%. If $r_f = 5\%$, and you expect the rate of return on the market portfolio to be 15%, should you invest in this fund? What is the fund's alpha?
    b. What passive portfolio comprised of a market-index portfolio and a money market account would have the same beta as the fund? Show that the difference between the expected rate of return on this passive portfolio and that of the fund equals the alpha from part (*a*).

Challenge Problem

20. Outline how you would incorporate the following into the CCAPM:
    a. Liquidity
    b. Nontraded assets (Do you have to worry about labor income?)

CFA® PROBLEMS

1. *a.* John Wilson is a portfolio manager at Austin & Associates. For all of his clients, Wilson manages portfolios that lie on the Markowitz efficient frontier. Wilson asks Mary Regan, CFA, a managing director at Austin, to review the portfolios of two of his clients, the Eagle Manufacturing Company and the Rainbow Life Insurance Co. The expected returns of the two portfolios are substantially different. Regan determines that the Rainbow portfolio is virtually identical to the market portfolio and concludes that the Rainbow portfolio must be superior to the Eagle portfolio. Do you agree or disagree with Regan's conclusion that the Rainbow portfolio is superior to the Eagle portfolio? Justify your response with reference to the capital market line.
    b. Wilson remarks that the Rainbow portfolio has a higher expected return because it has greater nonsystematic risk than Eagle's portfolio. Define nonsystematic risk and explain why you agree or disagree with Wilson's remark.
2. Wilson is now evaluating the expected performance of two common stocks, Furhman Labs Inc. and Garten Testing Inc. He has gathered the following information:
    - The risk-free rate is 5%.
    - The expected return on the market portfolio is 11.5%.
    - The beta of Furhman stock is 1.5.
    - The beta of Garten stock is .8.

    Based on his own analysis, Wilson's forecasts of the returns on the two stocks are 13.25% for Furhman stock and 11.25% for Garten stock. Calculate the required rate of return for Furhman Labs stock and for Garten Testing stock. Indicate whether each stock is undervalued, fairly valued, or overvalued.
3. The security market line depicts:
    a. A security's expected return as a function of its systematic risk.
    b. The market portfolio as the optimal portfolio of risky securities.

c. The relationship between a security's return and the return on an index.
d. The complete portfolio as a combination of the market portfolio and the risk-free asset.

4. Within the context of the capital asset pricing model (CAPM), assume:
   - Expected return on the market = 15%.
   - Risk-free rate = 8%.
   - Expected rate of return on XYZ security = 17%.
   - Beta of XYZ security = 1.25.

   Which one of the following is correct?

   a. XYZ is overpriced.
   b. XYZ is fairly priced.
   c. XYZ's alpha is −.25%.
   d. XYZ's alpha is .25%.

5. What is the expected return of a zero-beta security?

   a. Market rate of return.
   b. Zero rate of return.
   c. Negative rate of return.
   d. Risk-free rate of return.

6. Capital asset pricing theory asserts that portfolio returns are best explained by:

   a. Economic factors.
   b. Specific risk.
   c. Systematic risk.
   d. Diversification.

7. According to CAPM, the expected rate of return of a portfolio with a beta of 1.0 and an alpha of 0 is:

   a. Between r_M and r_f.
   b. The risk-free rate, r_f.
   c. $\beta (r_M - r_f)$.
   d. The expected return on the market, r_M.

The following table shows risk and return measures for two portfolios.

Portfolio	Average Annual Rate of Return	Standard Deviation	Beta
R	11%	10%	0.5
S&P 500	14%	12%	1.0

8. When plotting portfolio *R* on the preceding table relative to the SML, portfolio *R* lies:

   a. On the SML.
   b. Below the SML.
   c. Above the SML.
   d. Insufficient data given.

9. When plotting portfolio *R* relative to the capital market line, portfolio *R* lies:

   a. On the CML.
   b. Below the CML.
   c. Above the CML.
   d. Insufficient data given.

10. Briefly explain whether investors should expect a higher return from holding portfolio *A* versus portfolio *B* under capital asset pricing theory (CAPM). Assume that both portfolios are fully diversified.

	Portfolio A	Portfolio B
Systematic risk (beta)	1.0	1.0
Specific risk for each individual security	High	Low

11. Joan McKay is a portfolio manager for a bank trust department. McKay meets with two clients, Kevin Murray and Lisa York, to review their investment objectives. Each client expresses an interest in changing his or her individual investment objectives. Both clients currently hold well-diversified portfolios of risky assets.

    a. Murray wants to increase the expected return of his portfolio. State what action McKay should take to achieve Murray's objective. Justify your response in the context of the CML.

    b. York wants to reduce the risk exposure of her portfolio but does not want to engage in borrowing or lending activities to do so. State what action McKay should take to achieve York's objective. Justify your response in the context of the SML.

12. Karen Kay, a portfolio manager at Collins Asset Management, is using the capital asset pricing model for making recommendations to her clients. Her research department has developed the information shown in the following exhibit.

Forecast Returns, Standard Deviations, and Betas

	Forecast Return	Standard Deviation	Beta
Stock *X*	14.0%	36%	0.8
Stock *Y*	17.0	25	1.5
Market index	14.0	15	1.0
Risk-free rate	5.0		

a. Calculate expected return and alpha for each stock.

b. Identify and justify which stock would be more appropriate for an investor who wants to

i. add this stock to a well-diversified equity portfolio.

ii. hold this stock as a single-stock portfolio.

STANDARD &POOR'S

Go to **www.mhhe.com/edumarketinsight** and link to *Company*, then *Population.* Select a company of interest to you and link to the Company Research page. Look for the *Excel Analytics section*, and choose *Valuation Data*, then review the *Profitability* report. Find the row that shows the historical betas for your firm. Is beta stable from year to year? Go back to the *Company Research* page and look at the latest available *S&P Stock Report* for your firm. What beta does the report indicate for your firm? Why might this be different from the one in the *Profitability Report*? Based on current risk-free rates (available at **finance.yahoo.com**), and the historical risk premiums discussed in Chapter 5, estimate the expected rate of return on your company's stock by using the CAPM.

E-Investments

Beta and Security Returns

Fidelity provides data on the risk and return of its funds at **www.fidelity.com.** Click on the *Research* link, then choose *Mutual Funds* from the submenu. In the *Fund Evaluator* section, choose *Advanced Search*. Scroll down until you find the *Risk/ Volatility Measures* section and indicate that you want to screen for funds with betas less than or equal to .50. Click *Search Funds* to see the results. Click on the link that says *View All Matching Fidelity Funds*. Select five funds from the resulting list and click *Compare*. Rank the five funds according to their betas and then according to their standard deviations. Do both lists rank the funds in the same order? How would you explain any difference in the rankings? Note the 1-Year return for one of the funds (use the load-adjusted return if it is available). Repeat the exercise to compare five funds that have betas greater than or equal to 1.50.

SOLUTIONS TO CONCEPT CHECKS

1. We can characterize the entire population by two representative investors. One is the "uninformed" investor, who does not engage in security analysis and holds the market portfolio, whereas the other optimizes using the Markowitz algorithm with input from security analysis. The uninformed investor does not know what input the informed investor uses to make portfolio purchases. The uninformed investor knows, however, that if the other investor is informed, the market portfolio proportions will be optimal. Therefore, to depart from these proportions would constitute an uninformed bet, which will, on average, reduce the efficiency of diversification with no compensating improvement in expected returns.

2. *a.* Substituting the historical mean and standard deviation in Equation 9.2 yields a coefficient of risk aversion of

$$\bar{A} = \frac{E(r_M) - r_f}{\sigma_M^2} = \frac{.084}{.203^2} = 2.04$$

b. This relationship also tells us that for the historical standard deviation and a coefficient of risk aversion of 3.5 the risk premium would be

$$E(r_M) - r_f = \bar{A}\sigma_M^2 = 3.5 \times .203^2 = .144 = 14.4\%$$

3. For these investment proportions, w_{Ford}, w_{GM}, the portfolio β is

$$\begin{aligned}\beta_P &= w_{\text{Ford}}\beta_{\text{Ford}} + w_{\text{GM}}\beta_{\text{GM}} \\ &= (.75 \times 1.25) + (.25 \times 1.10) = 1.2125\end{aligned}$$

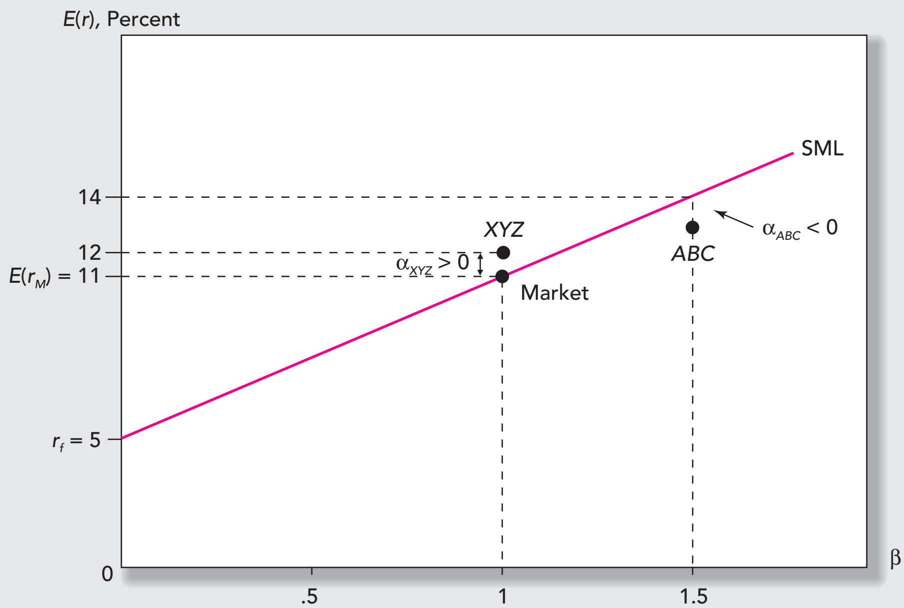
As the market risk premium, $E(r_M) - r_f$, is 8%, the portfolio risk premium will be

$$\begin{aligned}E(r_P) - r_f &= \beta_P[E(r_M) - r_f] \\ &= 1.2125 \times 8 = 9.7\%\end{aligned}$$

4. The alpha of a stock is its expected return in excess of that required by the CAPM.

$$\alpha = E(r) - \{r_f + \beta[E(r_M) - r_f]\}$$
$$\alpha_{XYZ} = 12 - [5 + 1.0(11 - 5)] = 1\%$$
$$\alpha_{ABC} = 13 - [5 + 1.5(11 - 5)] = -1\%$$

ABC plots below the SML, while *XYZ* plots above.



5. The project-specific required return is determined by the project beta coupled with the market risk premium and the risk-free rate. The CAPM tells us that an acceptable expected rate of return for the project is

$$r_f + \beta[E(r_M) - r_f] = 8 + 1.3(16 - 8) = 18.4\%$$

which becomes the project's hurdle rate. If the IRR of the project is 19%, then it is desirable. Any project with an IRR equal to or less than 18.4% should be rejected.

6. The CAPM is a model that relates expected rates of return to risk. It results in the expected return–beta relationship, where the expected risk premium on any asset is proportional to the expected risk premium on the market portfolio with beta as the proportionality constant. As such the model is impractical for two reasons: (i) expectations are unobservable, and (ii) the theoretical market portfolio includes every risky asset and is in practice unobservable. The next three models incorporate additional assumptions to overcome these problems.

The single-factor model assumes that one economic factor, denoted *F*, exerts the only common influence on security returns. Beyond it, security returns are driven by independent, firm-specific factors. Thus for any security, *i*,

$$r_i = E(r_i) + \beta_i F + e_i$$

The single-index model assumes that in the single-factor model, the factor *F* can be replaced by a broad-based index of securities that can proxy for the CAPM's theoretical market portfolio. The index model can be stated as $R_i = \alpha_i + \beta_i R_M + e_i$.

At this point it should be said that many interchange the meaning of the index and market models. The concept of the market model is that rate of return *surprises* on a stock are proportional to corresponding surprises on the market index portfolio, again with proportionality constant β.

CHAPTER FOURTEEN

14

PART IV

BOND PRICES AND YIELDS

IN THE PREVIOUS chapters on risk and return relationships, we have treated securities at a high level of abstraction. We assumed implicitly that a prior, detailed analysis of each security already had been performed, and that its risk and return features had been assessed.

We turn now to specific analyses of particular security markets. We examine valuation principles, determinants of risk and return, and portfolio strategies commonly used within and across the various markets.

We begin by analyzing **debt securities.** A debt security is a claim on a specified periodic stream of income. Debt securities are often called *fixed-income securities* because they promise either a fixed stream of income or a stream of income that is determined according to a specified formula. These securities have the advantage of being relatively easy to understand because the payment formulas are specified in advance. Risk considerations are minimal as long as the issuer of the security is sufficiently creditworthy. That makes these securities a convenient starting point for our analysis of the universe of potential investment vehicles.

The bond is the basic debt security, and this chapter starts with an overview of the universe of bond markets, including Treasury, corporate, and international bonds. We turn next to bond pricing, showing how bond prices are set in accordance with market interest rates and why bond prices change with those rates. Given this background, we can compare the myriad measures of bond returns such as yield to maturity, yield to call, holding-period return, or realized compound rate of return. We show how bond prices evolve over time, discuss certain tax rules that apply to debt securities, and show how to calculate after-tax returns. Finally, we consider the impact of default or credit risk on bond pricing and look at the determinants of credit risk and the default premium built into bond yields.

14.1 BOND CHARACTERISTICS

A **bond** is a security that is issued in connection with a borrowing arrangement. The borrower issues (i.e., sells) a bond to the lender for some amount of cash; the bond is the "IOU" of the borrower. The arrangement obligates the issuer to make specified payments to the bondholder on specified dates. A typical coupon bond obligates the issuer to make semiannual payments of interest to the bondholder for the life of the bond. These are called *coupon payments* because in precomputer days, most bonds had coupons that investors would clip off and present to claim the interest payment. When the bond matures, the issuer repays the debt by paying the bondholder the bond's **par value** (equivalently, its **face value**). The **coupon rate** of the bond serves to determine the interest payment: The annual payment is the coupon rate times the bond's par value. The coupon rate, maturity date, and par value of the bond are part of the **bond indenture,** which is the contract between the issuer and the bondholder.

To illustrate, a bond with par value of $1,000 and coupon rate of 8% might be sold to a buyer for $1,000. The bondholder is then entitled to a payment of 8% of $1,000, or $80 per year, for the stated life of the bond, say, 30 years. The $80 payment typically comes in two semiannual installments of $40 each. At the end of the 30-year life of the bond, the issuer also pays the $1,000 par value to the bondholder.

Bonds usually are issued with coupon rates set just high enough to induce investors to pay par value to buy the bond. Sometimes, however, **zero-coupon bonds** are issued that make no coupon payments. In this case, investors receive par value at the maturity date but receive no interest payments until then: The bond has a coupon rate of zero. These bonds are issued at prices considerably below par value, and the investor's return comes solely from the difference between issue price and the payment of par value at maturity. We will return to these bonds later.

Treasury Bonds and Notes

Figure 14.1 is an excerpt from the listing of Treasury issues. Treasury note maturities range up to 10 years, while Treasury bonds with maturities ranging from 10 to 30 years appear in the figure. Most bonds and notes are issued in denominations of $1,000 or more, but the minimum denomination was reduced to $100 in 2008. Both make semiannual coupon payments.

The highlighted bond in Figure 14.1 matures in January 2011. The *n* after 2011 denotes that this is a Treasury note, not a bond. Its coupon rate is 4.25%. Par value typically is $1,000; thus the bond pays interest of $42.50 per year in two semiannual payments of $21.25. Payments are made in January and July of each year. The bid and asked prices[1] are quoted in points plus fractions of $\frac{1}{32}$ of a point (the numbers after the colons are the fractions of a point). Although bonds usually are sold in denominations of $1,000, the prices are quoted as a percentage of par value. Therefore, the bid price of the bond is $98{:}07 = 98\frac{7}{32} = 98.219\%$ of par value, or $982.19, whereas the asked price is $99\frac{8}{32}\%$ of par, or $982.50.

The last column, labeled "Ask Yld," is the yield to maturity on the bond based on the asked price. The yield to maturity is a measure of the average rate of return to an investor who purchases the bond for the asked price and holds it until its maturity date. We will have much to say about yield to maturity below.[2]

[1]Recall that the bid price is the price at which you can sell the bond to a dealer. The asked price, which is slightly higher, is the price at which you can buy the bond from a dealer.

[2]Notice that some of the bonds in Figure 14.1 have the letter *i* after the maturity year and that these bonds have lower reported yields to maturity. These are inflation-indexed bonds, and their yields should be interpreted as after-inflation, or real returns. We discuss these bonds in detail later in the chapter.

U.S. Government Bonds and Notes

Representative Over-the-Counter quotation based on transactions of $1 million or more.
Treasury bond, note and bill quotes are from midafternoon. Colons in bond and note bid-and-asked quotes represent 32nds; 101:01 means 101 1/32. Net change in 32nds. n-Treasury Note. i-inflation-indexed issue. Treasury bill quotes in hundredths, quoted in terms of a rate of discount. Days to maturity calculated from settlement date. All yields are to maturity and based on the asked quote. For bonds callable prior to maturity, yields are computed to the earliest call date for issues quoted above par and to the maturity date for issues quoted below par.
*-When issued. Daily change expressed in basis points.

RATE	MATURITY MO/YR	BID	ASKED	CHG	ASK YLD
3.125	Jan 07n	99:29	99:30	....	4.83
2.250	Feb 07n	99:24	99:25	....	4.88
6.250	Feb 07n	100:02	100:03	....	4.88
3.375	Feb 07n	99:25	99:26	....	4.97
3.750	Mar 07n	99:23	99:24	+1	4.97
3.625	Apr 07n	99:18	99:19	....	4.99
5.750	Aug 10n	103:09	103:10	+2	4.73
4.125	Aug 10n	98:00	98:01	+3	4.73
3.875	Sep 10n	97:03	97:04	+3	4.73
4.250	Oct 10n	98:10	98:11	+3	4.73
4.500	Nov 10n	99:05	99:06	+3	4.73
4.375	Dec 10n	98:22	98:23	+3	4.74
4.250	Jan 11n	98:07	98:08	+3	4.74
3.500	Jan 11i	103:26	103:27	+3	2.48
5.000	Feb 11n	101:03	101:04	+3	4.69
4.500	Feb 11n	99:04	99:05	+3	4.73
4.750	Mar 11n	100:01	100:02	+3	4.73
2.375	Apr 11i	99:11	99:12	+2	2.53
4.875	Apr 11n	100:16	100:17	+3	4.73
4.875	May 11n	100:17	100:18	+3	4.73
5.125	Jun 11n	101:17	101:18	+4	4.73
4.875	Jul 11n	100:18	100:19	+4	4.73
4.625	Dec 11n	99:15	99:16	+4	4.74
3.375	Jan 12i	104:01	104:02	+3	2.50
4.875	Feb 12n	100:24	100:25	+4	4.70
3.000	Jul 12i	102:17	102:18	+2	2.49
4.375	Aug 12n	98:13	98:14	+4	4.69
4.000	Nov 12n	96:13	96:14	+4	4.71
10.375	Nov 12	104:11	104:12	+2	4.87
3.375	Feb 13n	95:17	95:18	+4	4.72
3.625	May 13n	94:02	94:03	+5	4.71
1.375	Jul 13i	96:09	96:10	+4	2.49
4.250	Aug 13n	97:10	97:11	+6	4.72
5.250	Nov 28	104:12	104:13	+8	4.92
5.250	Feb 29	104:14	104:15	+9	4.92
3.875	Apr 29i	124:17	124:18	+16	2.44
6.125	Aug 29	116:12	116:13	+9	4.91
6.250	May 30	118:19	118:20	+10	4.90
5.375	Feb 31	106:20	106:21	+8	4.90
3.375	Apr 32i	119:09	119:10	+16	2.35
4.500	Feb 36	94:19	94:20	+9	4.84

Treasury Bills

MATURITY	DAYS TO MAT	BID	ASKED	CHG	ASK YLD
Mar 22 07	64	4.96	4.95	+0.02	5.06
Mar 29 07	71	4.96	4.95	+0.01	5.07
Apr 05 07	78	4.96	4.95	+0.01	5.07
Apr 12 07	85	4.96	4.95	....	5.08
Apr 19 07	92	4.98	4.97	+0.02	5.10
Apr 26 07	99	4.96	4.95	....	5.09
May 03 07	106	4.96	4.95	+0.01	5.09
May 10 07	113	4.96	4.95	....	5.10
May 17 07	120	4.97	4.96	+0.01	5.11
May 24 07	127	4.97	4.96	+0.01	5.12
May 31 07	134	4.95	4.94	+0.01	5.10
Jun 07 07	141	4.94	4.93	+0.01	5.10
Jun 14 07	148	4.94	4.93	....	5.10
Jun 21 07	155	4.94	4.93	....	5.11
Jun 28 07	162	4.94	4.93	....	5.11
Jul 05 07	169	4.95	4.94	....	5.13
Jul 12 07	176	4.95	4.94	....	5.13

FIGURE 14.1 Listing of Treasury issues

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Accrued Interest and Quoted Bond Prices The bond prices that you see quoted in the financial pages are not actually the prices that investors pay for the bond. This is because the quoted price does not include the interest that accrues between coupon payment dates.

If a bond is purchased between coupon payments, the buyer must pay the seller for accrued interest, the prorated share of the upcoming semiannual coupon. For example, if 30 days have passed since the last coupon payment, and there are 182 days in the semiannual coupon period, the seller is entitled to a payment of accrued interest of 30/182 of the semiannual coupon. The sale, or *invoice*, price of the bond would equal the stated price plus the accrued interest.

In general, the formula for the amount of accrued interest between two dates is

$$\text{Accrued interest} = \frac{\text{Annual coupon payment}}{2} \times \frac{\text{Days since last coupon payment}}{\text{Days separating coupon payments}}$$

EXAMPLE 14.1 Accrued Interest

Suppose that the coupon rate is 8%. Then the annual coupon is \$80 and the semiannual coupon payment is \$40. Because 30 days have passed since the last coupon payment, the accrued interest on the bond is $\$40 \times (30/182) = \$6.59$. If the quoted price of the bond is \$990, then the invoice price will be $\$990 + \$6.59 = \$996.59$.

The practice of quoting bond prices net of accrued interest explains why the price of a maturing bond is listed at $1,000 rather than $1,000 plus one coupon payment. A purchaser of an 8% coupon bond 1 day before the bond's maturity would receive $1,040 (par value plus semiannual interest) on the following day and so should be willing to pay a total price of $1,040 for the bond. The bond price is quoted net of accrued interest in the financial pages and thus appears as $1,000.[3]

Corporate Bonds

Like the government, corporations borrow money by issuing bonds. Figure 14.2 is a sample of corporate bond listings for a few actively traded corporate bonds. Although some bonds trade on a formal exchange operated by the New York Stock Exchange, most bonds are traded over-the-counter in a network of bond dealers linked by a computer quotation system. (See Chapter 3 for a comparison of exchange versus OTC trading.) In practice, the bond market can be quite "thin," in that there are few investors interested in trading a particular issue at any particular time.

The bond listings in Figure 14.2 include the coupon, maturity, price, and yield to maturity of each bond. The "rating" column is the estimation of bond safety given by the three major bond-rating agencies—Moody's, Standard & Poor's, and Fitch. Bonds with gradations of A ratings are safer than those with B ratings or below. Notice that as a general rule, safer bonds with higher ratings promise lower yields to maturity than other bonds with similar maturities. We will return to this topic toward the end of the chapter.

Call Provisions on Corporate Bonds Although the Treasury no longer issues callable bonds, some corporate bonds are issued with call provisions allowing the issuer to repurchase the bond at a specified *call price* before the maturity date. For example, if a company issues a bond with a high coupon rate when market interest rates are high, and interest rates later fall, the firm might like to retire the high-coupon debt and issue new bonds at a lower coupon rate to reduce interest payments. This is called *refunding.* Callable bonds typically come with a period of call protection, an initial time during which the bonds are not callable. Such bonds are referred to as *deferred* callable bonds.

ISSUER NAME	SYMBOL	COUPON	MATURITY	RATING MOODY'S/S&P/FITCH	HIGH	LOW	LAST	CHANGE	YIELD %
Gatx	GMT.IK	8.875%	Jun 2009	Baa1/BBB/BBB–	107.545	107.538	107.545	–0.100	5.433
Marshall & Ilsley	MI.YL	3.800%	Feb 2008	Aa3/A+/A+	98.514	98.470	98.514	0.064	5.263
Capital One	COF.HK	7.686%	Aug 2036	Baa2/BBB–/BBB–	113.895	113.390	113.733	0.257	6.621
Entergy Gulf States	ETR.KC	6.180%	Mar 2035	Baa3/BBB+/BBB	99.950	94.616	99.469	0.219	6.220
AOL Time Warner	AOL.HG	6.875%	May 2012	Baa2/BBB+/BBB	107.205	105.402	106.565	0.720	5.427
Household Intl	HI.HJG	8.875%	Feb 2008	Aa3/AA–/AA–	100.504	100.504	100.504	–0.109	5.348
SBC Comm	SBC.IF	5.875%	Feb 2012	A2/A/A	102.116	102.001	102.001	–0.156	5.415
American General Finance	AIG.GOU	5.750%	Sep 2016	A1/A+/A+	101.229	101.135	101.135	–0.530	5.595

FIGURE 14.2 Listing of corporate bonds

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[3]In contrast to bonds, stocks do not trade at flat prices with adjustments for "accrued dividends." Whoever owns the stock when it goes "ex-dividend" receives the entire dividend payment, and the stock price reflects the value of the upcoming dividend. The price therefore typically falls by about the amount of the dividend on the "ex-day." There is no need to differentiate between reported and invoice prices for stocks.

The option to call the bond is valuable to the firm, allowing it to buy back the bonds and refinance at lower interest rates when market rates fall. Of course, the firm's benefit is the bondholder's burden. Holders of called bonds must forfeit their bonds for the call price, thereby giving up the attractive coupon rate on their original investment. To compensate investors for this risk, callable bonds are issued with higher coupons and promised yields to maturity than noncallable bonds.

CONCEPT CHECK 1

Suppose that Verizon issues two bonds with identical coupon rates and maturity dates. One bond is callable, however, whereas the other is not. Which bond will sell at a higher price?

Convertible Bonds **Convertible bonds** give bondholders an option to exchange each bond for a specified number of shares of common stock of the firm. The *conversion ratio* is the number of shares for which each bond may be exchanged. Suppose a convertible bond is issued at par value of $1,000 and is convertible into 40 shares of a firm's stock. The current stock price is $20 per share, so the option to convert is not profitable now. Should the stock price later rise to $30, however, each bond may be converted profitably into $1,200 worth of stock. The *market conversion value* is the current value of the shares for which the bonds may be exchanged. At the $20 stock price, for example, the bond's conversion value is $800. The *conversion premium* is the excess of the bond value over its conversion value. If the bond were selling currently for $950, its premium would be $150.

Convertible bondholders benefit from price appreciation of the company's stock. Again, this benefit comes at a price: Convertible bonds offer lower coupon rates and stated or promised yields to maturity than do nonconvertible bonds. However, the actual return on the convertible bond may exceed the stated yield to maturity if the option to convert becomes profitable.

We discuss convertible and callable bonds further in Chapter 20.

Puttable Bonds While the callable bond gives the issuer the option to extend or retire the bond at the call date, the *extendable* or **put bond** gives this option to the bondholder. If the bond's coupon rate exceeds current market yields, for instance, the bondholder will choose to extend the bond's life. If the bond's coupon rate is too low, it will be optimal not to extend; the bondholder instead reclaims principal, which can be invested at current yields.

Floating-Rate Bonds **Floating-rate bonds** make interest payments that are tied to some measure of current market rates. For example, the rate might be adjusted annually to the current T-bill rate plus 2%. If the 1-year T-bill rate at the adjustment date is 4%, the bond's coupon rate over the next year would then be 6%. This arrangement means that the bond always pays approximately current market rates.

The major risk involved in floaters has to do with changes in the firm's financial strength. The yield spread is fixed over the life of the security, which may be many years. If the financial health of the firm deteriorates, then investors will demand a greater yield premium than is offered by the security. In this case, the price of the bond will fall. Although the coupon rate on floaters adjusts to changes in the general level of market interest rates, it does not adjust to changes in the financial condition of the firm.

Preferred Stock

Although preferred stock strictly speaking is considered to be equity, it often is included in the fixed-income universe. This is because, like bonds, preferred stock promises to pay a specified stream of dividends. However, unlike bonds, the failure to pay the promised dividend does not result in corporate bankruptcy. Instead, the dividends owed simply cumulate, and the common stockholders may not receive any dividends until the preferred stockholders have been paid in full. In the event of bankruptcy, preferred stockholders' claims to the firm's assets have lower priority than those of bondholders, but higher priority than those of common stockholders.

Preferred stock commonly pays a fixed dividend. Therefore, it is in effect a perpetuity, providing a level cash flow indefinitely. In the last two decades, however, adjustable or floating-rate preferred stock has become popular, in some years accounting for about half of new issues. Floating-rate preferred stock is much like floating-rate bonds. The dividend rate is linked to a measure of current market interest rates and is adjusted at regular intervals.

Unlike interest payments on bonds, dividends on preferred stock are not considered tax-deductible expenses to the firm. This reduces their attractiveness as a source of capital to issuing firms. On the other hand, there is an offsetting tax advantage to preferred stock. When one corporation buys the preferred stock of another corporation, it pays taxes on only 30% of the dividends received. For example, if the firm's tax bracket is 35%, and it receives $10,000 in preferred dividend payments, it will pay taxes on only $3,000 of that income: Total taxes owed on the income will be $.35 \times \$3,000 = \$1,050$. The firm's effective tax rate on preferred dividends is therefore only $.30 \times 35\% = 10.5\%$. Given this tax rule, it is not surprising that most preferred stock is held by corporations.

Preferred stock rarely gives its holders full voting privileges in the firm. However, if the preferred dividend is skipped, the preferred stockholders may then be provided some voting power.

Other Issuers

There are, of course, several issuers of bonds in addition to the Treasury and private corporations. For example, state and local governments issue municipal bonds. The outstanding feature of these is that interest payments are tax-free. We examined municipal bonds and the value of the tax exemption in Chapter 2.

Government agencies such as the Federal Home Loan Bank Board, the Farm Credit agencies, and the mortgage pass-through agencies Ginnie Mae, Fannie Mae, and Freddie Mac also issue considerable amounts of bonds. These too were reviewed in Chapter 2.

International Bonds

International bonds are commonly divided into two categories, *foreign bonds* and *Eurobonds.* Foreign bonds are issued by a borrower from a country other than the one in which the bond is sold. The bond is denominated in the currency of the country in which it is marketed. For example, if a German firm sells a dollar-denominated bond in the United States, the bond is considered a foreign bond. These bonds are given colorful names based on the countries in which they are marketed. For example, foreign bonds sold in the United States are called *Yankee bonds.* Like other bonds sold in the United States, they are registered with the Securities and Exchange Commission. Yen-denominated bonds sold in Japan by non-Japanese issuers are called *Samurai bonds.* British pound-denominated foreign bonds sold in the United Kingdom are called *bulldog bonds.*

In contrast to foreign bonds, Eurobonds are bonds issued in the currency of one country but sold in other national markets. For example, the Eurodollar market refers to dollar-denominated bonds sold outside the United States (not just in Europe), although London is the largest market for Eurodollar bonds. Because the Eurodollar market falls outside U.S. jurisdiction, these bonds are not regulated by U.S. federal agencies. Similarly, Euroyen bonds are yen-denominated bonds selling outside Japan, Eurosterling bonds are pound denominated Eurobonds selling outside the United Kingdom, and so on.

Innovation in the Bond Market

Issuers constantly develop innovative bonds with unusual features; these issues illustrate that bond design can be extremely flexible. Here are examples of some novel bonds. They should give you a sense of the potential variety in security design.

Inverse Floaters These are similar to the floating-rate bonds we described earlier, except that the coupon rate on these bonds *falls* when the general level of interest rates rises. Investors in these bonds suffer doubly when rates rise. Not only does the present value of each dollar of cash flow from the bond fall as the discount rate rises, but the level of those cash flows falls as well. Of course, investors in these bonds benefit doubly when rates fall.

Asset-Backed Bonds Walt Disney has issued bonds with coupon rates tied to the financial performance of several of its films. Similarly, "David Bowie bonds" have been issued with payments that will be tied to royalties on some of his albums. These are examples of asset-backed securities. The income from a specified group of assets is used to service the debt. More conventional asset-backed securities are mortgage-backed securities or securities backed by auto or credit card loans, as we discussed in Chapter 2.

Catastrophe Bonds Oriental Land Company, which manages Tokyo Disneyland, issued a bond in 1999 with a final payment that depended on whether there had been an earthquake near the park. The Swiss insurance firm Winterthur has issued a bond whose payments will be cut if a severe hailstorm in Switzerland results in extensive payouts on Winterthur policies. These bonds are a way to transfer "catastrophe risk" from the firm to the capital markets. Investors in these bonds receive compensation for taking on the risk in the form of higher coupon rates. But in the event of a catastrophe, the bondholders will give up all or part of their investments. "Disaster" can be defined by total insured losses or by criteria such as wind speed in a hurricane or Richter level in an earthquake. Issuance of catastrophe bonds has surged in recent years, rising from about $1 billion in 2000 to $9 billion in 2006, as insurers have sought ways to spread their risks across a wider spectrum of the capital market.

Indexed Bonds Indexed bonds make payments that are tied to a general price index or the price of a particular commodity. For example, Mexico has issued 20-year bonds with payments that depend on the price of oil. Some bonds are indexed to the general price level. The United States Treasury started issuing such inflation-indexed bonds in January 1997. They are called Treasury Inflation Protected Securities (TIPS). By tying the par value of the bond to the general level of prices, coupon payments as well as the final repayment of par value on these bonds increase in direct proportion to the Consumer Price Index. Therefore, the interest rate on these bonds is a risk-free real rate.

To illustrate how TIPS work, consider a newly issued bond with a 3-year maturity, par value of $1,000, and a coupon rate of 4%. For simplicity, we will assume the bond makes annual coupon payments. Assume that inflation turns out to be 2%, 3%, and 1% in the next 3 years. Table 14.1 shows how the bond cash flows will be calculated. The first payment comes at the end of the first year, at $t = 1$. Because inflation over the year was 2%, the par value of the bond increases from $1,000 to $1,020; because the coupon rate is 4%, the coupon payment is 4% of this amount, or $40.80. Notice that par value increases by the

TABLE 14.1

Principal and interest payments for a Treasury Inflation Protected Security

Time	Inflation in Year Just Ended	Par Value	Coupon Payment	+	Principal Repayment	=	Total Payment
0		$1,000.00					
1	2%	1,020.00	$40.80		$ 0		$ 40.80
2	3	1,050.60	42.02		0		42.02
3	1	1,061.11	42.44		1,061.11		1,103.55

inflation rate, and because the coupon payments are 4% of par, they too increase in proportion to the general price level. Therefore, the cash flows paid by the bond are fixed in *real* terms. When the bond matures, the investor receives a final coupon payment of $42.44 plus the (price-level-indexed) repayment of principal, $1,061.11.[4]

The *nominal* rate of return on the bond in the first year is

$$\text{Nominal return} = \frac{\text{Interest} + \text{Price Appreciation}}{\text{Initial Price}} = \frac{40.80 + 20}{1000} = 6.08\%$$

The real rate of return is precisely the 4% real yield on the bond:

$$\text{Real return} = \frac{1 + \text{Nominal return}}{1 + \text{Inflation}} - 1 = \frac{1.0608}{1.02} - 1 = .04,\ \text{or } 4\%$$

One can show in a similar manner (see Problem 16 in the end-of-chapter problems) that the rate of return in each of the 3 years is 4% as long as the real yield on the bond remains constant. If real yields do change, then there will be capital gains or losses on the bond. In early 2008, the real yield on long-term TIPS bonds was about 1.75%.

14.2 BOND PRICING

Because a bond's coupon and principal repayments all occur months or years in the future, the price an investor would be willing to pay for a claim to those payments depends on the value of dollars to be received in the future compared to dollars in hand today. This "present value" calculation depends in turn on market interest rates. As we saw in Chapter 5, the nominal risk-free interest rate equals the sum of (1) a real risk-free rate of return and (2) a premium above the real rate to compensate for expected inflation. In addition, because most bonds are not riskless, the discount rate will embody an additional premium that reflects bond-specific characteristics such as default risk, liquidity, tax attributes, call risk, and so on.

We simplify for now by assuming there is one interest rate that is appropriate for discounting cash flows of any maturity, but we can relax this assumption easily. In practice, there may be different discount rates for cash flows accruing in different periods. For the time being, however, we ignore this refinement.

To value a security, we discount its expected cash flows by the appropriate discount rate. The cash flows from a bond consist of coupon payments until the maturity date plus the final payment of par value. Therefore,

$$\text{Bond value} = \text{Present value of coupons} + \text{Present value of par value}$$

[4]By the way, total nominal income (i.e., coupon plus that year's increase in principal) is treated as taxable income in each year.

If we call the maturity date T and call the interest rate r, the bond value can be written as

$$\text{Bond value} = \sum_{t=1}^{T} \frac{\text{Coupon}}{(1+r)^t} + \frac{\text{Par value}}{(1+r)^T} \tag{14.1}$$

The summation sign in Equation 14.1 directs us to add the present value of each coupon payment; each coupon is discounted based on the time until it will be paid. The first term on the right-hand side of Equation 14.1 is the present value of an annuity. The second term is the present value of a single amount, the final payment of the bond's par value.

You may recall from an introductory finance class that the present value of a $1 annuity that lasts for T periods when the interest rate equals r is $\frac{1}{r}\left[1 - \frac{1}{(1+r)^T}\right]$. We call this expression the T-period *annuity factor* for an interest rate of r.[5] Similarly, we call $\frac{1}{(1+r)^T}$ the *PV factor*, that is, the present value of a single payment of $1 to be received in T periods. Therefore, we can write the price of the bond as

$$\begin{aligned} \text{Price} &= \text{Coupon} \times \frac{1}{r}\left[1 - \frac{1}{(1+r)^T}\right] + \text{Par value} \times \frac{1}{(1+r)^T} \\ &= \text{Coupon} \times \text{Annuity factor}(r,\ T) + \text{Par value} \times \text{PV factor}(r,\ T) \end{aligned} \tag{14.2}$$

EXAMPLE 14.2 Bond Pricing

We discussed earlier an 8% coupon, 30-year maturity bond with par value of $1,000 paying 60 semiannual coupon payments of $40 each. Suppose that the interest rate is 8% annually, or $r = 4\%$ per 6-month period. Then the value of the bond can be written as

$$\begin{aligned} \text{Price} &= \sum_{t=1}^{60} \frac{\$40}{(1.04)^t} + \frac{\$1{,}000}{(1.04)^{60}} \\ &= \$40 \times \text{Annuity factor}(4\%,\ 60) + \$1{,}000 \times \text{PV factor}(4\%,\ 60) \end{aligned} \tag{14.3}$$

It is easy to confirm that the present value of the bond's 60 semiannual coupon payments of $40 each is $904.94 and that the $1,000 final payment of par value has a present value of $95.06, for a total bond value of $1,000. You can either calculate the value directly from Equation 14.2, perform these calculations on any financial calculator,[6] use a spreadsheet program (see the Excel Applications box), or use a set of present value tables.

[5] Here is a quick derivation of the formula for the present value of an annuity. An annuity lasting T periods can be viewed as equivalent to a perpetuity whose first payment comes at the end of the current period *less* another perpetuity whose first payment comes at the end of the $(T + 1)$st period. The immediate perpetuity net of the delayed perpetuity provides exactly T payments. We know that the value of a $1 per period perpetuity is $1/$r$. Therefore, the present value of the delayed perpetuity is $1/$r$ discounted for T additional periods, or $\frac{1}{r} \times \frac{1}{(1+r)^T}$. The present value of the annuity is the present value of the first perpetuity minus the present value of the delayed perpetuity, or $\frac{1}{r}\left[1 - \frac{1}{(1+r)^T}\right]$.

[6] On your financial calculator, you would enter the following inputs: n (number of periods) = 60; FV (face or future value) = 1000; PMT (payment each period) = 40; i (per period interest rate) = 4%; then you would compute the price of the bond (COMP PV or CPT PV). You should find that the price is $1,000. Actually, most calculators will display the result as *negative* $1,000. This is because most (but not all) calculators treat the initial purchase price of the bond as a cash *outflow*. We will discuss calculators and spreadsheets more fully in a few pages.

In this example, the coupon rate equals the market interest rate, and the bond price equals par value. If the interest rate were not equal to the bond's coupon rate, the bond would not sell at par value. For example, if the interest rate were to rise to 10% (5% per 6 months), the bond's price would fall by $189.29 to $810.71, as follows:

$$\$40 \times \text{Annuity factor}(5\%,\ 60) + \$1{,}000 \times \text{PV factor}(5\%,\ 60)$$
$$= \$757.17 + \$53.54 = \$810.71$$

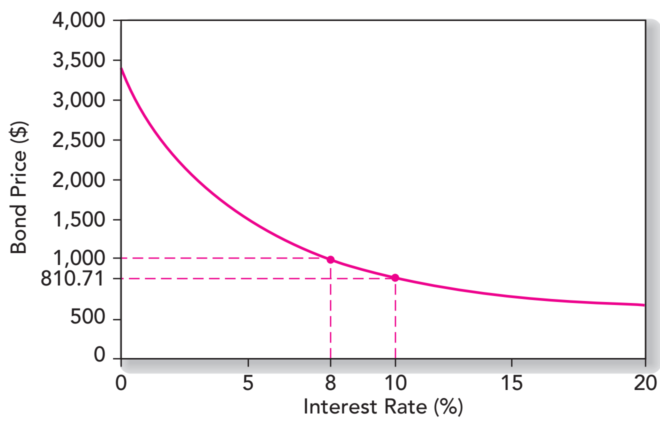

FIGURE 14.3 The inverse relationship between bond prices and yields. Price of an 8% coupon bond with 30-year maturity making semiannual payments

At a higher interest rate, the present value of the payments to be received by the bondholder is lower. Therefore, the bond price will fall as market interest rates rise. This illustrates a crucial general rule in bond valuation. When interest rates rise, bond prices must fall because the present value of the bond's payments is obtained by discounting at a higher interest rate.

Figure 14.3 shows the price of the 30-year, 8% coupon bond for a range of interest rates, including 8%, at which the bond sells at par, and 10%, at which it sells for $810.71. The negative slope illustrates the inverse relationship between prices and yields. Note also from the figure (and from Table 14.2) that the shape of the curve implies that an increase in the interest rate results in a price decline that is smaller than the price gain resulting from a decrease of equal magnitude in the interest rate. This property of bond prices is called *convexity* because of the convex shape of the bond price curve. This curvature reflects the fact that progressive increases in the interest rate result in progressively smaller reductions in the bond price.[7] Therefore, the price curve becomes flatter at higher interest rates. We return to the issue of convexity in Chapter 16.

CONCEPT CHECK 2

Calculate the price of the 30-year, 8% coupon bond for a market interest rate of 3% per half-year. Compare the capital gains for the interest rate decline to the losses incurred when the rate increases to 5%.

Corporate bonds typically are issued at par value. This means that the underwriters of the bond issue (the firms that market the bonds to the public for the issuing corporation) must choose a coupon rate that very closely approximates market yields. In a primary issue of bonds, the underwriters attempt to sell the newly issued bonds directly to their customers. If the coupon rate is inadequate, investors will not pay par value for the bonds.

After the bonds are issued, bondholders may buy or sell bonds in secondary markets, such as the one operated by the New York Stock Exchange or the over-the-counter market,

[7]The progressively smaller impact of interest increases results from the fact that at higher rates the bond is worth less. Therefore, an additional increase in rates operates on a smaller initial base, resulting in a smaller price reduction.

TABLE 14.2

Bond prices at different interest rates (8% coupon bond, coupons paid semiannually)

Time to Maturity	Bond Price at Given Market Interest Rate				
	4%	6%	8%	10%	12%
1 year	1,038.83	1,029.13	1,000.00	981.41	963.33
10 years	1,327.03	1,148.77	1,000.00	875.35	770.60
20 years	1,547.11	1,231.15	1,000.00	828.41	699.07
30 years	1,695.22	1,276.76	1,000.00	810.71	676.77

where most bonds trade. In these secondary markets, bond prices move in accordance with market forces. The bond prices fluctuate inversely with the market interest rate.

The inverse relationship between price and yield is a central feature of fixed-income securities. Interest rate fluctuations represent the main source of risk in the fixed-income market, and we devote considerable attention in Chapter 16 to assessing the sensitivity of bond prices to market yields. For now, however, it is sufficient to highlight one key factor that determines that sensitivity, namely, the maturity of the bond.

A general rule in evaluating bond price risk is that, keeping all other factors the same, the longer the maturity of the bond, the greater the sensitivity of price to fluctuations in the interest rate. For example, consider Table 14.2, which presents the price of an 8% coupon bond at different market yields and times to maturity. For any departure of the interest rate from 8% (the rate at which the bond sells at par value), the change in the bond price is greater for longer times to maturity.

This makes sense. If you buy the bond at par with an 8% coupon rate, and market rates subsequently rise, then you suffer a loss: You have tied up your money earning 8% when alternative investments offer higher returns. This is reflected in a capital loss on the bond—a fall in its market price. The longer the period for which your money is tied up, the greater the loss, and correspondingly the greater the drop in the bond price. In Table 14.2, the row for 1-year maturity bonds shows little price sensitivity—that is, with only 1 year's earnings at stake, changes in interest rates are not too threatening. But for 30-year maturity bonds, interest rate swings have a large impact on bond prices. The force of discounting is greatest for the longest-term bonds.

This is why short-term Treasury securities such as T-bills are considered to be the safest. They are free not only of default risk but also largely of price risk attributable to interest rate volatility.

Bond Pricing between Coupon Dates

Equation 14.2 for bond prices assumes that the next coupon payment is in precisely one payment period, either a year for an annual payment bond or 6 months for a semiannual payment bond. But you probably want to be able to price bonds all 365 days of the year, not just on the one or two dates each year that it makes a coupon payment!

In principle, the fact that the bond is between coupon dates does not affect the pricing problem. The procedure is always the same: compute the present value of each remaining payment and sum up. But if you are between coupon dates, there will be fractional periods remaining until each payment, and this does complicate the arithmetic computations.

Fortunately, bond pricing functions are included in most spreadsheet programs such as Excel. The spreadsheet allows you to enter today's date as well as the maturity date of the bond, and so can provide prices for bonds at any date. The nearby box shows you how.

As we pointed out earlier, bond prices are typically quoted net of accrued interest. These prices, which appear in the financial press, are called *flat prices.* The actual *invoice price* that a buyer pays for the bond includes accrued interest. Thus,

$$\text{Invoice price} = \text{Flat price} + \text{Accrued interest}$$

When a bond pays its coupon, flat price equals invoice price, because at that moment accrued interest reverts to zero. However, this will be the exceptional case, not the rule.

Excel pricing functions provide the flat price of the bond. To find the invoice price, we need to add accrued interest. Fortunately, Excel also provides functions that count the days since the last coupon payment date and thus can be used to compute accrued interest. The nearby box also illustrates how to use these functions. The box provides examples using bonds that have just paid a coupon and so have zero accrued interest, as well as a bond that is between coupon dates.

14.3 BOND YIELDS

We have noted that the current yield of a bond measures only the cash income provided by the bond as a percentage of bond price and ignores any prospective capital gains or losses. We would like a measure of rate of return that accounts for both current income and the price increase or decrease over the bond's life. The yield to maturity is the standard measure of the total rate of return. However, it is far from perfect, and we will explore several variations of this measure.

Yield to Maturity

In practice, an investor considering the purchase of a bond is not quoted a promised rate of return. Instead, the investor must use the bond price, maturity date, and coupon payments to infer the return offered by the bond over its life. The **yield to maturity** (YTM) is defined as the interest rate that makes the present value of a bond's payments equal to its price. This interest rate is often interpreted as a measure of the average rate of return that will be earned on a bond if it is bought now and held until maturity. To calculate the yield to maturity, we solve the bond price equation for the interest rate given the bond's price.

EXAMPLE 14.3 Yield to Maturity

Suppose an 8% coupon, 30-year bond is selling at $1,276.76. What average rate of return would be earned by an investor purchasing the bond at this price? We find the interest rate at which the present value of the remaining 60 semiannual payments equals the bond price. This is the rate consistent with the observed price of the bond. Therefore, we solve for r in the following equation:

$$\$1{,}276.76 = \sum_{t=1}^{60} \frac{\$40}{(1+r)^t} + \frac{\$1{,}000}{(1+r)^{60}}$$

or, equivalently,

$$1{,}276.76 = 40 \times \text{Annuity factor}(r, 60) + 1{,}000 \times \text{PV factor}(r, 60)$$

These equations have only one unknown variable, the interest rate, r. You can use a financial calculator or spreadsheet to confirm that the solution is $r = .03$, or 3%, per half-year.[8] This is considered the bond's yield to maturity.

The financial press reports yields on an annualized basis, and annualizes the bond's semiannual yield using simple interest techniques, resulting in an annual percentage rate, or APR. Yields annualized using simple interest are also called "bond equivalent yields." Therefore, the semiannual yield would be doubled and reported in the newspaper as a bond equivalent yield of 6%. The *effective* annual yield of the bond, however, accounts for compound interest. If one earns 3% interest every 6 months, then after 1 year, each dollar invested grows with interest to $\$1 \times (1.03)^2 = \$1.0609$, and the effective annual interest rate on the bond is 6.09%.

Excel also provides a function for yield to maturity that is especially useful in between coupon dates. It is

= YIELD(settlement date, maturity date, annual coupon rate, bond price, redemption value as percent of par value, number of coupon payments per year)

The bond price used in the function should be the reported flat price, without accrued interest. For example, to find the yield to maturity of the bond in Example 14.3, we would use column B of Spreadsheet 14.1. If the coupons were paid only annually, we would change the entry for payments per year to 1 (see cell D8), and the yield would fall slightly to 5.99%.

The bond's yield to maturity is the internal rate of return on an investment in the bond. The yield to maturity can be interpreted as the compound rate of return over the life of the bond under the assumption that all bond coupons can be reinvested at that yield.[9] Yield to maturity is widely accepted as a proxy for average return.

	A	B	C	D	E
1		Semiannual coupons		Annual coupons	
2					
3	Settlement date	1/1/2000		1/1/2000	
4	Maturity date	1/1/2030		1/1/2030	
5	Annual coupon rate	0.08		0.08	
6	Bond price (flat)	127.676		127.676	
7	Redemption value (% of face value)	100		100	
8	Coupon payments per year	2		1	
9					
10	**Yield to maturity (decimal)**	**0.0600**		**0.0599**	
11					
12	The formula entered here is: =YIELD(B3,B4,B5,B6,B7,B8)				

SPREADSHEET 14.1

Finding yield to maturity in Excel

eXcel
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[8]On your financial calculator, you would enter the following inputs: $n = 60$ periods; PV = −1,276.76; FV = 1000; PMT = 40; then you would compute the interest rate (COMP i or CPT i). Notice that we enter the present value, or PV, of the bond as *minus* \$1,276.76. Again, this is because most calculators treat the initial purchase price of the bond as a cash outflow. Spreadsheet 14.1 shows how to find yield to maturity using Excel. Without a financial calculator or spreadsheet, you still could solve the equation, but you would need to use a trial-and-error approach.

[9]If the reinvestment rate does not equal the bond's yield to maturity, the compound rate of return will differ from YTM. This is demonstrated in Examples 14.5 and 14.6.

Excel and most other spreadsheet programs provide built-in functions to compute bond prices and yields. They typically ask you to input both the date you buy the bond (called the *settlement date*) and the maturity date of the bond. The Excel function for bond price is

= PRICE(settlement date, maturity date, annual coupon rate, yield to maturity, redemption value as percent of par value, number of coupon payments per year)

For the 4.25% coupon January 2011 maturity bond highlighted in Figure 14.1, we would enter the values in the following spreadsheet. Alternatively, we could simply enter the following function in Excel:

= PRICE(DATE(2007,1,15), DATE(2011,1,15), .0425, .0474, 100, 2)

The DATE function in Excel, which we use for both the settlement and maturity date, uses the format DATE(year,month,day). The first date is January 15, 2007, when the bond is purchased, and the second is January 15, 2011, when it matures. Most bonds pay coupons on the 15th of the month.

Notice that the coupon rate and yield to maturity are expressed as decimals, not percentages. In most cases, redemption value is 100 (i.e., 100% of par value), and the resulting price similarly is expressed as a percent of par value. Occasionally, however, you may encounter bonds that pay off at a premium or discount to par value. One example would be callable bonds, discussed shortly.

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The value of the bond returned by the pricing function is 98.234 (cell B12), which matches the price reported in Table 14.1 to the nearest 32nd of a point. This bond has just paid a coupon. In other words, the settlement date is precisely at the beginning of the coupon period, so no adjustment for accrued interest is necessary.

To illustrate the procedure for bonds between coupon payments, consider the 4% coupon November 2012 bond, also appearing in Figure 14.1. Using the entries in column D of the spreadsheet, we find in cell D12 that the (flat) price of the bond is 96.410, which matches the price given in the figure except for a few cents' rounding error.

What about the bond's invoice price? Rows 13 through 16 make the necessary adjustments. The function described in cell C13 counts the days since the last coupon. This day count is based on the bond's settlement date, maturity date, coupon period (1 = annual; 2 = semiannual), and day count convention (choice 1 uses actual days). The function described in cell C14 counts the total days in each coupon payment period. Therefore, the entries for accrued interest in row 15 are the semiannual coupon multiplied by the fraction of a coupon period that has elapsed since the last payment. Finally, the invoice prices in row 16 are the sum of flat price plus accrued interest.

As a final example, suppose you wish to find the price of the bond in Example 14.2. It is a 30-year maturity bond with a coupon rate of 8% (paid semiannually). The market interest rate given in the latter part of the example is 10%. However, you are not given a specific settlement or maturity date. You can still use the PRICE function to value the bond. Simply choose an *arbitrary* settlement date (January 1, 2000, is convenient) and let the maturity date be 30 years hence. The appropriate inputs appear in column F of the spreadsheet, with the resulting price, 81.071% of face value, appearing in cell F16.

	A	B	C	D	E	F	G
1		**4.250% coupon bond,**		**4% coupon bond,**		**8% coupon bond,**	
2		**maturing Jan 2011**	**Formula in column B**	**maturing Nov 2012**		**30-year maturity**	
3							
4	Settlement date	1/15/2007	= DATE (2007, 1, 15)	1/15/2007		1/1/2000	
5	Maturity date	1/15/2011	= DATE (2011, 1, 15)	11/15/2012		1/1/2030	
6	Annual coupon rate	0.0425		0.04		0.08	
7	Yield to maturity	0.0474		0.0471		0.1	
8	Redemption value (% of face value)	100		100		100	
9	Coupon payments per year	2		2		2	
10							
11							
12	**Flat price (% of par)**	**98.234**	**=PRICE(B4,B5,B6,B7,B8,B9)**	**96.410**		**81.071**	
13	Days since last coupon	0	=COUPDAYBS(B4,B5,2,1)	61		0	
14	Days in coupon period	181	=COUPDAYS(B4,B5,2,1)	181		182	
15	Accrued interest	0	=(B13/B14)*B6*100/2	0.0674		0	
16	**Invoice price**	**98.234**	**=B12+B15**	**97.084**		**81.071**	

Yield to maturity differs from the **current yield** of a bond, which is the bond's annual coupon payment divided by the bond price. For example, for the 8%, 30-year bond currently selling at $1,276.76, the current yield would be $80/$1,276.76 = .0627, or 6.27%, per year. In contrast, recall that the effective annual yield to maturity is 6.09%. For this bond, which is selling at a premium over par value ($1,276 rather than $1,000), the coupon rate (8%) exceeds the current yield (6.27%), which exceeds the yield to maturity (6.09%). The coupon rate exceeds current yield because the coupon rate divides the coupon payments by par value ($1,000) rather than by the bond price ($1,276). In turn, the current yield exceeds yield to maturity because the yield to maturity accounts for the built-in capital loss on the bond; the bond bought today for $1,276 will eventually fall in value to $1,000 at maturity.

Example 14.3 illustrates a general rule: For **premium bonds** (bonds selling above par value), coupon rate is greater than current yield, which in turn is greater than yield to maturity. For **discount bonds** (bonds selling below par value), these relationships are reversed (see Concept Check 3).

It is common to hear people talking loosely about the yield on a bond. In these cases, they almost always are referring to the yield to maturity.

CONCEPT CHECK 3

What will be the relationship among coupon rate, current yield, and yield to maturity for bonds selling at discounts from par? Illustrate using the 8% (semiannual payment) coupon bond, assuming it is selling at a yield to maturity of 10%.

Yield to Call

Yield to maturity is calculated on the assumption that the bond will be held until maturity. What if the bond is callable, however, and may be retired prior to the maturity date? How should we measure average rate of return for bonds subject to a call provision?

Figure 14.4 illustrates the risk of call to the bondholder. The colored line is the value at various market interest rates of a "straight" (i.e., noncallable) bond with par value $1,000, an 8% coupon rate, and a 30-year time to maturity. If interest rates fall, the bond price, which equals the present value of the promised payments, can rise substantially.

Now consider a bond that has the same coupon rate and maturity date but is callable at 110% of par value, or $1,100. When interest rates fall, the present value of the bond's *scheduled* payments rises, but the call provision allows the issuer to repurchase the bond at the call price. If the call price is less than the present value of the scheduled payments, the issuer may call the bond back from the bondholder.

The dark line in Figure 14.4 is the value of the callable bond. At high interest rates, the risk of call is negligible because the present value of scheduled payments is less than the call price; therefore the values of the straight and callable bonds converge. At lower rates, however, the values of the bonds begin to

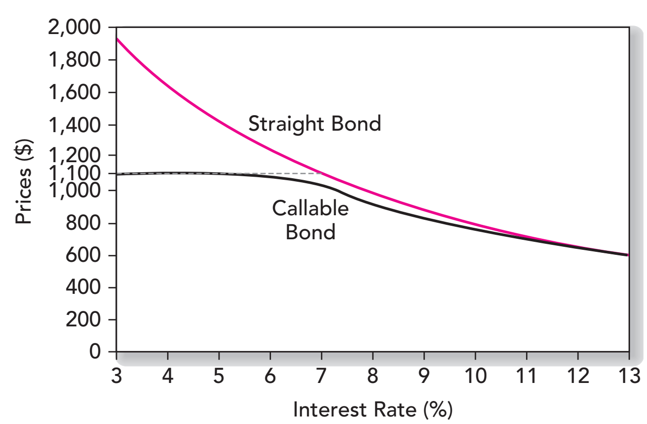

FIGURE 14.4 Bond prices: Callable and straight debt. Coupon = 8%; maturity = 30 years; semiannual payments.

diverge, with the difference reflecting the value of the firm's option to reclaim the callable bond at the call price. At very low rates, the present value of scheduled payments exceeds the call price, so the bond is called. Its value at this point is simply the call price, $1,100.

This analysis suggests that bond market analysts might be more interested in a bond's yield to call rather than yield to maturity especially if the bond is likely to be called. The yield to call is calculated just like the yield to maturity except that the time until call replaces time until maturity, and the call price replaces the par value. This computation is sometimes called "yield to first call," as it assumes the issuer will call the bond as soon as it may do so.

EXAMPLE 14.4 Yield to Call

Suppose the 8% coupon, 30-year maturity bond sells for $1,150 and is callable in 10 years at a call price of $1,100. Its yield to maturity and yield to call would be calculated using the following inputs:

	Yield to Call	Yield to Maturity
Coupon payment	$40	$40
Number of semiannual periods	20 periods	60 periods
Final payment	$1,100	$1,000
Price	$1,150	$1,150

Yield to call is then 6.64%. [To confirm this on a calculator, input n = 20; PV = (−)1,150; FV = 1100; PMT = 40; compute i as 3.32%, or 6.64% bond equivalent yield.] Yield to maturity is 6.82%. [To confirm, input n = 60; PV = (−)1,150; FV = 1000; PMT = 40; compute i as 3.41% or 6.82% bond equivalent yield. In Excel, you can calculate yield to call as = YIELD(DATE(2000,01,01), DATE(2010,01,01), .08, 115, 110, 2). Notice that redemption value is input as 110, i.e., 110% of par value.]

We have noted that most callable bonds are issued with an initial period of call protection. In addition, an implicit form of call protection operates for bonds selling at deep discounts from their call prices. Even if interest rates fall a bit, deep-discount bonds still will sell below the call price and thus will not be subject to a call.

Premium bonds that might be selling near their call prices, however, are especially apt to be called if rates fall further. If interest rates fall, a callable premium bond is likely to provide a lower return than could be earned on a discount bond whose potential price appreciation is not limited by the likelihood of a call. Investors in premium bonds often are more interested in the bond's yield to call rather than yield to maturity as a consequence, because it may appear to them that the bond will be retired at the call date.

CONCEPT CHECK 4 and 5

4. The yield to maturity on two 10-year maturity bonds currently is 7%. Each bond has a call price of $1,100. One bond has a coupon rate of 6%, the other 8%. Assume for simplicity that bonds are called as soon as the present value of their remaining payments exceeds their call price. What will be the capital gain on each bond if the market interest rate suddenly falls to 6%?
5. A 20-year maturity 9% coupon bond paying coupons semiannually is callable in 5 years at a call price of $1,050. The bond currently sells at a yield to maturity of 8%. What is the yield to call?

Realized Compound Return versus Yield to Maturity

We have noted that yield to maturity will equal the rate of return realized over the life of the bond if all coupons are reinvested at an interest rate equal to the bond's yield to maturity. Consider, for example, a 2-year bond selling at par value paying a 10% coupon once a year. The yield to maturity is 10%. If the \$100 coupon payment is reinvested at an interest rate of 10%, the \$1,000 investment in the bond will grow after 2 years to \$1,210, as illustrated in Figure 14.5, panel A. The coupon paid in the first year is reinvested and grows with interest to a second-year value of \$110, which together with the second coupon payment and payment of par value in the second year results in a total value of \$1,210.

To summarize, the initial value of the investment is $V_0 = \$1{,}000$. The final value in 2 years is $V_2 = \$1{,}210$. The compound rate of return, therefore, is calculated as follows:

$$\begin{aligned} V_0(1+r)^2 &= V_2 \\ \$1{,}000(1+r)^2 &= \$1{,}210 \\ r &= .10 = 10\% \end{aligned}$$

With a reinvestment rate equal to the 10% yield to maturity, the **realized compound return** equals yield to maturity.

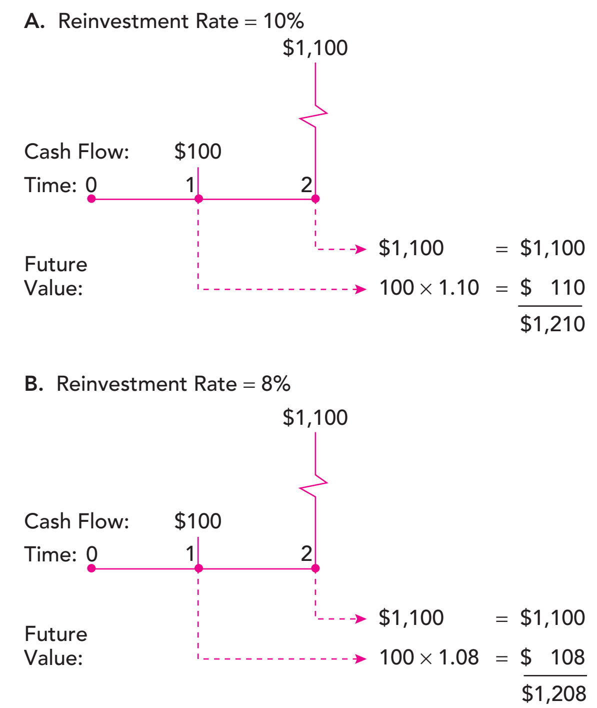

FIGURE 14.5 Growth of invested funds

But what if the reinvestment rate is not 10%? If the coupon can be invested at more than 10%, funds will grow to more than \$1,210, and the realized compound return will exceed 10%. If the reinvestment rate is less than 10%, so will be the realized compound return. Consider the following example.

EXAMPLE 14.5 Realized Compound Return

If the interest rate earned on the first coupon is less than 10%, the final value of the investment will be less than \$1,210, and the realized compound return will be less than 10%. To illustrate, suppose the interest rate at which the coupon can be invested equals 8%. The following calculations are illustrated in Figure 14.5, panel B.

Future value of first coupon payment with interest earnings	$= \$100 \times 1.08 =$	\$ 108
+ Cash payment in second year (final coupon plus par value)		\$1,100
= Total value of investment with reinvested coupons		\$1,208

The realized compound return is the compound rate of growth of invested funds, assuming that all coupon payments are reinvested. The investor purchased the bond for par at \$1,000, and this investment grew to \$1,208.

$$\begin{aligned} V_0(1+r)^2 &= V_2 \\ \$1{,}000(1+r)^2 &= \$1{,}208 \\ r &= .0991 = 9.91\% \end{aligned}$$

Example 14.5 highlights the problem with conventional yield to maturity when reinvestment rates can change over time. Conventional yield to maturity will not equal realized compound return. However, in an economy with future interest rate uncertainty, the rates at which interim coupons will be reinvested are not yet known. Therefore, although realized compound return can be computed *after* the investment period ends, it cannot be computed in advance without a forecast of future reinvestment rates. This reduces much of the attraction of the realized return measure.

Forecasting the realized compound yield over various holding periods or investment horizons is called **horizon analysis.** The forecast of total return depends on your forecasts of *both* the price of the bond when you sell it at the end of your horizon *and* the rate at which you are able to reinvest coupon income. The sales price depends in turn on the yield to maturity at the horizon date. With a longer investment horizon, however, reinvested coupons will be a larger component of your final proceeds.

EXAMPLE 14.6 Horizon Analysis

Suppose you buy a 30-year, 7.5% (annual payment) coupon bond for $980 (when its yield to maturity is 7.67%) and plan to hold it for 20 years. Your forecast is that the bond's yield to maturity will be 8% when it is sold and that the reinvestment rate on the coupons will be 6%. At the end of your investment horizon, the bond will have 10 years remaining until expiration, so the forecast sales price (using a yield to maturity of 8%) will be $966.45. The 20 coupon payments will grow with compound interest to $2,758.92. (This is the future value of a 20-year $75 annuity with an interest rate of 6%.)

Based on these forecasts, your $980 investment will grow in 20 years to $966.45 + $2,758.92 = $3,725.37. This corresponds to an annualized compound return of 6.90%:

$$
\begin{aligned}
V_0(1+r)^{20} &= V_{20} \\
\$980(1+r)^{20} &= \$3,725.37 \\
r &= .0690 = 6.90\%
\end{aligned}
$$

Examples 14.5 and 14.6 demonstrate that as interest rates change, bond investors are actually subject to two sources of offsetting risk. On the one hand, when rates rise, bond prices fall, which reduces the value of the portfolio. On the other hand, reinvested coupon income will compound more rapidly at those higher rates. This **reinvestment rate risk** will offset the impact of price risk. In Chapter 16, we will explore this trade-off in more detail and will discover that by carefully tailoring their bond portfolios, investors can precisely balance these two effects for any given investment horizon.

14.4 BOND PRICES OVER TIME

As we noted earlier, a bond will sell at par value when its coupon rate equals the market interest rate. In these circumstances, the investor receives fair compensation for the time value of money in the form of the recurring coupon payments. No further capital gain is necessary to provide fair compensation.

When the coupon rate is lower than the market interest rate, the coupon payments alone will not provide investors as high a return as they could earn elsewhere in the market. To receive a fair return on such an investment, investors also need to earn price appreciation on their bonds. The bonds, therefore, would have to sell below par value to provide a "built-in" capital gain on the investment.

EXAMPLE 14.7 Fair Holding-Period Return

To illustrate built-in capital gains or losses, suppose a bond was issued several years ago when the interest rate was 7%. The bond's annual coupon rate was thus set at 7%. (We will suppose for simplicity that the bond pays its coupon annually.) Now, with 3 years left in the bond's life, the interest rate is 8% per year. The bond's market price is the present value of the remaining annual coupons plus payment of par value. That present value is[10]

$$\$70 \times \text{Annuity factor}(8\%,\ 3) + \$1{,}000 \times \text{PV factor}(8\%,\ 3) = \$974.23$$

which is less than par value.

In another year, after the next coupon is paid, the bond would sell at

$$\$70 \times \text{Annuity factor}(8\%,\ 2) + \$1{,}000 \times \text{PV factor}(8\%,\ 2) = \$982.17$$

thereby yielding a capital gain over the year of $7.94. If an investor had purchased the bond at $974.23, the total return over the year would equal the coupon payment plus capital gain, or $70 + $7.94 = $77.94. This represents a rate of return of $77.94/$974.23, or 8%, exactly the current rate of return available elsewhere in the market.

CONCEPT CHECK 6

At what price will the bond in Example 14.7 sell in yet another year, when only 1 year remains until maturity? What is the rate of return to an investor who purchases the bond at $982.17 and sells it 1 year hence?

When bond prices are set according to the present value formula, any discount from par value provides an anticipated capital gain that will augment a below-market coupon rate just sufficiently to provide a fair total rate of return. Conversely, if the coupon rate exceeds the market interest rate, the interest income by itself is greater than that available elsewhere in the market. Investors will bid up the price of these bonds above their par values. As the bonds approach maturity, they will fall in value because fewer of these above-market coupon payments remain. The resulting capital losses offset the large coupon payments so that the bondholder again receives only a fair rate of return.

Problem 12 at the end of the chapter asks you to work through the case of the high-coupon bond. Figure 14.6 traces out the price paths of high- and low-coupon bonds (net of accrued interest) as time to maturity approaches, at least for the case in which the market interest rate is constant. The low-coupon bond enjoys capital gains, whereas the high-coupon bond suffers capital losses.[11]

We use these examples to show that each bond offers investors the same total rate of return. Although the capital gain versus income components differ, the price of each bond is set to provide competitive rates, as we should expect in well-functioning capital markets. Security returns all should be comparable on an after-tax risk-adjusted basis. It they are not, investors will try to sell low-return securities, thereby driving down their prices until the total return at the now-lower price is competitive with other securities. Prices should

[10]Using a calculator, enter $n = 3$, $i = 8$, PMT = 70, FV = 1,000, and compute PV.

[11]If interest rates are volatile, the price path will be "jumpy," vibrating around the price path in Figure 14.6 and reflecting capital gains or losses as interest rates fluctuate. Ultimately, however, the price must reach par value at the maturity date, so the price of the premium bond will fall over time while that of the discount bond will rise.

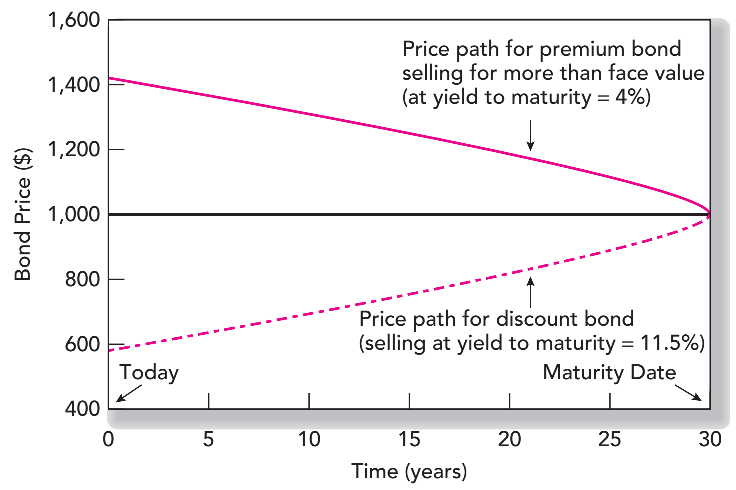

FIGURE 14.6 Prices over time of 30-year maturity, 6.5% coupon bonds. Bond price approaches par value as maturity approaches.

continue to adjust until all securities are fairly priced in that expected returns are comparable, given appropriate risk and tax adjustments.

Yield to Maturity versus Holding-Period Return

In Example 14.7, the holding-period return and the yield to maturity were equal. The bond yield started and ended the year at 8%, and the bond's holding-period return also equaled 8%. This turns out to be a general result. When the yield to maturity is unchanged over the period, the rate of return on the bond will equal that yield. As we noted, this should not be surprising: The bond must offer a rate of return competitive with those available on other securities.

However, when yields fluctuate, so will a bond's rate of return. Unanticipated changes in market rates will result in unanticipated changes in bond returns and, after the fact, a bond's holding-period return can be better or worse than the yield at which it initially sells. An increase in the bond's yield acts to reduce its price, which means that the holding-period return will be less than the initial yield. Conversely, a decline in yield will result in a holding-period return greater than the initial yield.

EXAMPLE 14.8 Yield to Maturity versus Holding-Period Return

Consider a 30-year bond paying an annual coupon of \$80 and selling at par value of \$1,000. The bond's initial yield to maturity is 8%. If the yield remains at 8% over the year, the bond price will remain at par, so the holding-period return also will be 8%. But if the yield falls below 8%, the bond price will increase. Suppose the yield falls and the price increases to \$1,050. Then the holding-period return is greater than 8%:

$$\text{Holding-period return} = \frac{\$80 + (\$1{,}050 - \$1{,}000)}{\$1{,}000} = .13, \text{ or } 13\%$$

CONCEPT CHECK 7

Show that if yield to maturity increases, then holding-period return is *less* than initial yield. For example, suppose in Example 14.8 that by the end of the first year, the bond's yield to maturity is 8.5%. Find the 1-year holding-period return and compare it to the bond's initial 8% yield to maturity.

Here is another way to think about the difference between yield to maturity and holding-period return. Yield to maturity depends only on the bond's coupon, *current* price, and par value at maturity. All of these values are observable today, so yield to maturity can be easily calculated. Yield to maturity can be interpreted as a measure of the *average* rate of return if the investment in the bond is held until the bond matures. In contrast, holding-period return is the rate of return over a particular investment period and depends on the market price of the bond at the end of that holding period; of course this price is *not* known today. Because bond prices over the holding period will respond to unanticipated changes in interest rates, holding-period return can at most be forecast.

Zero-Coupon Bonds and Treasury Strips

Original-issue discount bonds are less common than coupon bonds issued at par. These are bonds that are issued intentionally with low coupon rates that cause the bond to sell at a discount from par value. An extreme example of this type of bond is the *zero-coupon bond*, which carries no coupons and provides all its return in the form of price appreciation. Zeros provide only one cash flow to their owners, on the maturity date of the bond.

U.S. Treasury bills are examples of short-term zero-coupon instruments. If the bill has face value of $10,000, the Treasury issues or sells it for some amount less than $10,000, agreeing to repay $10,000 at maturity. All of the investor's return comes in the form of price appreciation.

Longer-term zero-coupon bonds are commonly created from coupon-bearing notes and bonds with the help of the U.S. Treasury. A bond dealer who purchases a Treasury coupon bond may ask the Treasury to break down the cash flows to be paid by the bond into a series of independent securities, where each security is a claim to one of the payments of the original bond. For example, a 10-year coupon bond would be "stripped" of its 20 semiannual coupons, and each coupon payment would be treated as a stand-alone zero-coupon bond. The maturities of these bonds would thus range from 6 months to 10 years. The final payment of principal would be treated as another stand-alone zero-coupon security. Each of the payments is now treated as an independent security and is assigned its own CUSIP number (by the Committee on Uniform Securities Identification Procedures), the security identifier that allows for electronic trading over the Fedwire system, a network that connects all Federal Reserve banks and their branches. The payments are still considered obligations of the U.S. Treasury. The Treasury program under which coupon stripping is performed is called STRIPS (Separate Trading of Registered Interest and Principal of Securities), and these zero-coupon securities are called Treasury *strips*.

What should happen to prices of zeros as time passes? On their maturity dates, zeros must sell for par value. Before maturity, however, they should sell at discounts from par, because of the time value of money. As time passes, price should approach par value. In fact, if the interest rate is constant, a zero's price will increase at exactly the rate of interest.

To illustrate this property, consider a zero with 30 years until maturity, and suppose the market interest rate is 10% per year. The price of the bond today will be $\$1{,}000/(1.10)^{30} = \$57.31$. Next year, with only 29 years until maturity, if the yield is still 10%, the price will be $\$1{,}000/(1.10)^{29} = \$63.04$, a 10% increase over its previous-year value. Because the par value of the bond is now discounted for 1 year fewer, its price has increased by the 1-year discount factor.

Figure 14.7 presents the price path of a 30-year zero-coupon bond until its maturity date for an

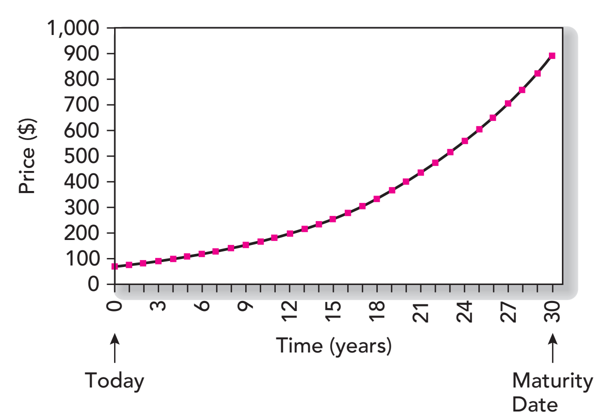

FIGURE 14.7 The price of a 30-year zero-coupon bond over time at a yield to maturity of 10%. Price equals $1{,}000/(1.10)^T$, where T is time until maturity.

annual market interest rate of 10%. The bond prices rise exponentially, not linearly, until its maturity.

After-Tax Returns

The tax authorities recognize that the "built-in" price appreciation on original-issue discount (OID) bonds such as zero-coupon bonds represents an implicit interest payment to the holder of the security. The IRS, therefore, calculates a price appreciation schedule to impute taxable interest income for the built-in appreciation during a tax year, even if the asset is not sold or does not mature until a future year. Any additional gains or losses that arise from changes in market interest rates are treated as capital gains or losses if the OID bond is sold during the tax year.

EXAMPLE 14.9 Taxation of Original-Issue Discount Bonds

If the interest rate originally is 10%, the 30-year zero would be issued at a price of $\$1{,}000/(1.10)^{30} = \$57.31$. The following year, the IRS calculates what the bond price would be if the yield were still 10%. This is $\$1{,}000/(1.10)^{29} = \$63.04$. Therefore, the IRS imputes interest income of $\$63.04 - \$57.31 = \$5.73$. This amount is subject to tax. Notice that the *imputed* interest income is based on a "constant yield method" that ignores any changes in market interest rates.

If interest rates actually fall, let's say to 9.9%, the bond price will be $\$1{,}000/(1.099)^{29} = \$64.72$. If the bond is sold, then the difference between \$64.72 and \$63.04 is treated as capital gains income and taxed at the capital gains tax rate. If the bond is not sold, then the price difference is an unrealized capital gain and does not result in taxes in that year. In either case, the investor must pay taxes on the \$5.73 of imputed interest at the rate on ordinary income.

The procedure illustrated in Example 14.9 applies as well to the taxation of other original-issue discount bonds, even if they are not zero-coupon bonds. Consider, as an example, a 30-year maturity bond that is issued with a coupon rate of 4% and a yield to maturity of 8%. For simplicity, we will assume that the bond pays coupons once annually. Because of the low coupon rate, the bond will be issued at a price far below par value, specifically at a price of \$549.69. If the bond's yield to maturity is still 8%, then its price in 1 year will rise to \$553.66. (Confirm this for yourself.) This would provide a pretax holding-period return (HPR) of exactly 8%:

$$\text{HPR} = \frac{\$40 + (\$553.66 - \$549.69)}{\$549.69} = .08$$

The increase in the bond price based on a constant yield, however, is treated as interest income, so the investor is required to pay taxes on both the explicit coupon income, \$40, as well as the imputed interest income of $\$553.66 - \$549.69 = \$3.97$. If the bond's yield actually changes during the year, the difference between the bond's price and the constant-yield value of \$553.66 would be treated as capital gains income if the bond is sold.

CONCEPT CHECK 8

Suppose that the yield to maturity of the 4% coupon, 30-year maturity bond falls to 7% by the end of the first year and that the investor sells the bond after the first year. If the investor's federal plus state tax rate on interest income is 38% and the combined tax rate on capital gains is 20%, what is the investor's after-tax rate of return?

14.5 DEFAULT RISK AND BOND PRICING

Although bonds generally *promise* a fixed flow of income, that income stream is not riskless unless the investor can be sure the issuer will not default on the obligation. While U.S. government bonds may be treated as free of default risk, this is not true of corporate bonds. Therefore, the actual payments on these bonds are uncertain, for they depend to some degree on the ultimate financial status of the firm.

Bond default risk, usually called **credit risk,** is measured by Moody's Investor Services, Standard & Poor's Corporation, and Fitch Investors Service, all of which provide financial information on firms as well as quality ratings of large corporate and municipal bond issues. International sovereign bonds, which also entail default risk, especially in emerging markets, also are commonly rated for default risk. Each rating firm assigns letter grades to the bonds of corporations and municipalities to reflect their assessment of the safety of the bond issue. The top rating is AAA or Aaa, a designation awarded to only about a dozen firms. Moody's modifies each rating class with a 1, 2, or 3 suffix (e.g., Aaa1, Aaa2, Aaa3) to provide a finer gradation of ratings. The other agencies use a + or − modification.

Those rated BBB or above (S&P, Fitch) or Baa and above (Moody's) are considered **investment-grade bonds,** whereas lower-rated bonds are classified as **speculative-grade** or **junk bonds.** Defaults on low-grade issues are not uncommon. For example, almost half of the bonds that were rated CCC by Standard & Poor's at issue have defaulted within 10 years. Highly rated bonds rarely default, but even these bonds are not free of credit risk. For example, in May 2001 WorldCom sold $11.8 billion of bonds with an investment-grade rating. Only a year later, the firm filed for bankruptcy and its bondholders lost more than 80% of their investment. Certain regulated institutional investors such as insurance companies have not always been allowed to invest in speculative-grade bonds.

Figure 14.8 provides the definitions of each bond rating classification.

Junk Bonds

Junk bonds, also known as *high-yield bonds,* are nothing more than speculative-grade (low-rated or unrated) bonds. Before 1977, almost all junk bonds were "fallen angels," that is, bonds issued by firms that originally had investment-grade ratings but that had since been downgraded. In 1977, however, firms began to issue "original-issue junk."

Much of the credit for this innovation is given to Drexel Burnham Lambert, and especially its trader Michael Milken. Drexel had long enjoyed a niche as a junk bond trader and had established a network of potential investors in junk bonds. Firms not able to muster an investment-grade rating were happy to have Drexel (and other investment bankers) market their bonds directly to the public, as this opened up a new source of financing. Junk issues were a lower-cost financing alternative than borrowing from banks.

High-yield bonds gained considerable notoriety in the 1980s when they were used as financing vehicles in leveraged buyouts and hostile takeover attempts. Shortly thereafter, however, the junk bond market suffered. The legal difficulties of Drexel and Michael Milken in connection with Wall Street's insider trading scandals of the late 1980s tainted the junk bond market.

At the height of Drexel's difficulties, the high-yield bond market nearly dried up. Since then, the market has rebounded dramatically. However, it is worth noting that the average credit quality of high-yield debt issued today is higher than the average quality in the boom years of the 1980s.

Bond Ratings								
	Very High Quality		High Quality		Speculative		Very Poor	
Standard & Poor's	AAA	AA	A	BBB	BB	B	CCC	D
Moody's	Aaa	Aa	A	Baa	Ba	B	Caa	C

At times both Moody's and Standard & Poor's have used adjustments to these ratings: S&P uses plus and minus signs: A + is the strongest A rating and A − the weakest. Moody's uses a 1, 2, or 3 designation, with 1 indicating the strongest.

Moody's	S&P	
Aaa	AAA	Debt rated Aaa and AAA has the highest rating. Capacity to pay interest and principal is extremely strong.
Aa	AA	Debt rated Aa and AA has a very strong capacity to pay interest and repay principal. Together with the highest rating, this group comprises the high-grade bond class.
A	A	Debt rated A has a strong capacity to pay interest and repay principal, although it is somewhat more susceptible to the adverse effects of changes in circumstances and economic conditions than debt in higher-rated categories.
Baa	BBB	Debt rated Baa and BBB is regarded as having an adequate capacity to pay interest and repay principal. Whereas it normally exhibits adequate protection parameters, adverse economic conditions or changing circumstances are more likely to lead to a weakened capacity to pay interest and repay principal for debt in this category than in higher-rated categories. These bonds are medium-grade obligations.
Ba B Caa Ca	BB B CCC CC	Debt rated in these categories is regarded, on balance, as predominantly speculative with respect to capacity to pay interest and repay principal in accordance with the terms of the obligation. BB and Ba indicate the lowest degree of speculation, and CC and Ca the highest degree of speculation. Although such debt will likely have some quality and protective characteristics, these are outweighed by large uncertainties or major risk exposures to adverse conditions. Some issues may be in default.
C	C	This rating is reserved for income bonds on which no interest is being paid.
D	D	Debt rated D is in default, and payment of interest and/or repayment of principal is in arrears.

FIGURE 14.8 Definitions of each bond rating class

Source: Stephen A. Ross and Randolph W. Westerfield, *Corporate Finance,* Copyright 1988 (St. Louis: Times Mirror/Mosby College Publishing, reproduced with permission from the McGarw-Hill Companies, Inc.). Data from various editions of *Standard & Poor's Bond Guide* and *Moody's Bond Guide.*

Determinants of Bond Safety

Bond rating agencies base their quality ratings largely on an analysis of the level and trend of some of the issuer's financial ratios. The key ratios used to evaluate safety are

1. *Coverage ratios*—Ratios of company earnings to fixed costs. For example, the *times-interest-earned ratio* is the ratio of earnings before interest payments and taxes to interest obligations. The *fixed-charge coverage ratio* includes lease payments and

sinking fund payments with interest obligations to arrive at the ratio of earnings to all fixed cash obligations (sinking funds are described below). Low or falling coverage ratios signal possible cash flow difficulties.

2. *Leverage ratio—Debt-to-equity ratio.* A too-high leverage ratio indicates excessive indebtedness, signaling the possibility the firm will be unable to earn enough to satisfy the obligations on its bonds.
3. *Liquidity ratios*—The two most common liquidity ratios are the *current ratio* (current assets/current liabilities) and the *quick ratio* (current assets excluding inventories/current liabilities). These ratios measure the firm's ability to pay bills coming due with its most liquid assets.
4. *Profitability ratios*—Measures of rates of return on assets or equity. Profitability ratios are indicators of a firm's overall financial health. The *return on assets* (earnings before interest and taxes divided by total assets) or *return on equity* (net income/equity) are the most popular of these measures. Firms with higher returns on assets or equity should be better able to raise money in security markets because they offer prospects for better returns on the firm's investments.
5. *Cash flow-to-debt ratio*—This is the ratio of total cash flow to outstanding debt.

Standard & Poor's periodically computes median values of selected ratios for firms in several rating classes, which we present in Table 14.3. Of course, ratios must be evaluated in the context of industry standards, and analysts differ in the weights they place on particular ratios. Nevertheless, Table 14.3 demonstrates the tendency of ratios to improve along with the firm's rating class. And default rates vary dramatically with bond rating. Historically, only about 1% of bonds originally rated AA or better at issuance had defaulted after 15 years. That ratio is around 7.5% for BBB-rated bonds, and 40% for B-rated bonds. Credit risk clearly varies dramatically across rating classes.

Many studies have tested whether financial ratios can in fact be used to predict default risk. One of the best-known series of tests was conducted by Edward Altman, who used discriminant analysis to predict bankruptcy. With this technique a firm is assigned a score based on its financial characteristics. If its score exceeds a cut-off value, the firm is deemed

	3-year (2002 to 2004) medians						
	AAA	AA	A	BBB	BB	B	CCC
EBIT interest coverage multiple	23.8	19.5	8.0	4.7	2.5	1.2	0.4
EBITDA interest coverage multiple	25.5	24.6	10.2	6.5	3.5	1.9	0.9
Funds from operations/total debt (%)	203.3	79.9	48.0	35.9	22.4	11.5	5.0
Free operating cash flow/total debt (%)	127.6	44.5	25.0	17.3	8.3	2.8	(2.1)
Total debt/EBITDA multiple	0.4	0.9	1.6	2.2	3.5	5.3	7.9
Return on capital (%)	27.6	27.0	17.5	13.4	11.3	8.7	3.2
Total debt/total debt + equity (%)	12.4	28.3	37.5	42.5	53.7	75.9	113.5

TABLE 14.3

Financial ratios by rating class, long-term debt

Note: EBITDA is earnings before interest, taxes, depreciation, and amortization

Source: *Corporate Rating Criteria*, Standard & Poor's, 2006.

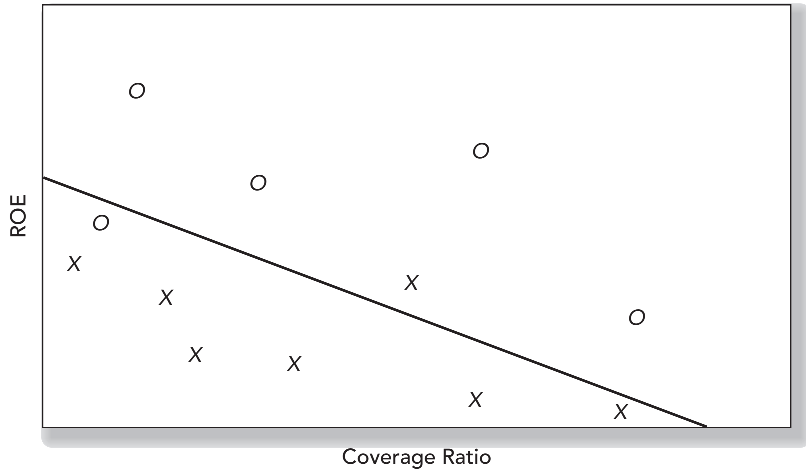

FIGURE 14.9 Discriminant analysis

creditworthy. A score below the cut-off value indicates significant bankruptcy risk in the near future.

To illustrate the technique, suppose that we were to collect data on the return on equity (ROE) and coverage ratios of a sample of firms, and then keep records of any corporate bankruptcies. In Figure 14.9 we plot the ROE and coverage ratios for each firm using *X* for firms that eventually went bankrupt and *O* for those that remained solvent. Clearly, the *X* and *O* firms show different patterns of data, with the solvent firms typically showing higher values for the two ratios.

The discriminant analysis determines the equation of the line that best separates the *X* and *O* observations. Suppose that the equation of the line is $.75 = .9 \times \text{ROE} + .4 \times \text{Coverage}$. Then, based on its own financial ratios, each firm is assigned a "*Z*-score" equal to $.9 \times \text{ROE} + .4 \times \text{Coverage}$. If its *Z*-score exceeds .75, the firm plots above the line and is considered a safe bet; *Z*-scores below .75 foretell financial difficulty.

Altman[12] found the following equation to best separate failing and nonfailing firms:

$$Z = 3.3\frac{\text{EBIT}}{\text{Total assets}} + 99.9\frac{\text{Sales}}{\text{Assets}} + .6\frac{\text{Market value of equity}}{\text{Book value of debt}}$$
$$+ 1.4\frac{\text{Retained earnings}}{\text{Total assets}} + 1.2\frac{\text{Working capital}}{\text{Total assets}}$$

where EBIT = earnings before interest and taxes.

CONCEPT CHECK 9

Suppose we add a new variable equal to current liabilities/current assets to Altman's equation. Would you expect this variable to receive a positive or negative coefficient?

Bond Indentures

A bond is issued with an *indenture*, which is the contract between the issuer and the bondholder. Part of the indenture is a set of restrictions that protect the rights of the bondholders. Such restrictions include provisions relating to collateral, sinking funds, dividend policy, and further borrowing. The issuing firm agrees to these *protective covenants* in order to market its bonds to investors concerned about the safety of the bond issue.

Sinking Funds Bonds call for the payment of par value at the end of the bond's life. This payment constitutes a large cash commitment for the issuer. To help ensure the commitment

[12]Edward I. Altman, "Financial Ratios, Discriminant Analysis, and the Prediction of Corporate Bankruptcy," *Journal of Finance* 23 (September 1968).

does not create a cash flow crisis, the firm agrees to establish a **sinking fund** to spread the payment burden over several years. The fund may operate in one of two ways:

1. The firm may repurchase a fraction of the outstanding bonds in the open market each year.
2. The firm may purchase a fraction of the outstanding bonds at a special call price associated with the sinking fund provision. The firm has an option to purchase the bonds at either the market price or the sinking fund price, whichever is lower. To allocate the burden of the sinking fund call fairly among bondholders, the bonds chosen for the call are selected at random based on serial number.[13]

The sinking fund call differs from a conventional bond call in two important ways. First, the firm can repurchase only a limited fraction of the bond issue at the sinking fund call price. At best, some indentures allow firms to use a *doubling option,* which allows repurchase of double the required number of bonds at the sinking fund call price. Second, while callable bonds generally have call prices above par value, the sinking fund call price usually is set at the bond's par value.

Although sinking funds ostensibly protect bondholders by making principal repayment more likely, they can hurt the investor. The firm will choose to buy back discount bonds (selling below par) at market price, while exercising its option to buy back premium bonds (selling above par) at par. Therefore, if interest rates fall and bond prices rise, firms will benefit from the sinking fund provision that enables them to repurchase their bonds at below-market prices. In these circumstances, the firm's gain is the bondholder's loss.

One bond issue that does not require a sinking fund is a *serial bond* issue. In a serial bond issue, the firm sells bonds with staggered maturity dates. As bonds mature sequentially, the principal repayment burden for the firm is spread over time, just as it is with a sinking fund. One advantage of serial bonds over sinking fund issues is that there is no uncertainty introduced by the possibility that a particular bond will be called for the sinking fund. The disadvantage of serial bonds, however, is that bonds of different maturity dates are not interchangeable, which reduces the liquidity of the issue.

Subordination of Further Debt One of the factors determining bond safety is total outstanding debt of the issuer. If you bought a bond today, you would be understandably distressed to see the firm tripling its outstanding debt tomorrow. Your bond would be of lower credit quality than it appeared when you bought it. To prevent firms from harming bondholders in this manner, **subordination clauses** restrict the amount of additional borrowing. Additional debt might be required to be subordinated in priority to existing debt; that is, in the event of bankruptcy, *subordinated* or *junior* debtholders will not be paid unless and until the prior senior debt is fully paid off. For this reason, subordination is sometimes called a "me-first rule," meaning the senior (earlier) bondholders are to be paid first in the event of bankruptcy.

Dividend Restrictions Covenants also limit the dividends firms may pay. These limitations protect the bondholders because they force the firm to retain assets rather than paying them out to stockholders. A typical restriction disallows payments of dividends

[13]Although it is less common, the sinking fund provision also may call for periodic payments to a trustee, with the payments invested so that the accumulated sum can be used for retirement of the entire issue at maturity.

& Mobil Corp. debenture 8s, due 2032:
Rating — Aa2

AUTH—$250,000,000.
OUTSTG—Dec. 31, 1993, $250,000,000.
DATED—Oct. 30, 1991.
INTEREST—F&A 12.
TRUSTEE—Chemical Bank.
DENOMINATION—Fully registered, $1,000 and integral multiples thereof. Transferable and exchangeable without service charge.
CALLABLE—As a whole or in part, at any time, on or after Aug. 12, 2002, at the option of Co. on at least 30 but not more than the 60 days' notice to each Aug. 11 as follows:

2003..........105.007	2004..........104.756	2005..........104.506
2006..........104.256	2007..........104.005	2008..........103.755
2009..........103.505	2010..........103.254	2011..........103.004
2012..........102.754	2013..........102.503	2014..........102.253
2015..........102.003	2016..........101.752	2017..........101.502
2018..........101.252	2019..........101.001	2020..........100.751
2021..........100.501	2022..........100.250	

and thereafter at 100 plus accrued interest.
SECURITY—Not secured. Ranks equally with all other unsecured and unsubordinated indebtedness of Co. Co. nor any Affiliate will not incur any indebtedness; provided that Co. will not create as security for any indebtedness for borrowed money, any mortgage, pledge, security interest or lien on any stock or indebtedness is directly owned by Co. without effectively providing that the debt securities shall be secured equally and ratably with such indebtedness. so long as such indebtedness shall be so secured.
INDENTURE MODIFICATION—Indenture may be modified, except as provided with, consent of 66 2/3% of debs. outstg.
RIGHTS ON DEFAULT—Trustee, or 25% of debs. outstg., may declare principal due and payable (30 days' grace for payment of interest).

LISTED—On New York Stock Exchange.
PURPOSE—Proceeds used for general corporate purposes.
OFFERED—($250,000,000) at 99.51 plus accrued interest (proceeds to Co., 99.11) on Aug. 5, 1992 thru Merrill Lynch & Co., Donaldson, Lufkin & Jenerette Securities Corp., PaineWebber Inc., Prudential Securities Inc., Smith Barney, Harris Upham & Co. Inc. and associates.

FIGURE 14.10 Callable bond issued by Mobil

Source: *Mergent's Industrial Manual,* Mergent's Investor Services, 1994. Reprinted with permission. All rights reserved.

if cumulative dividends paid since the firm's inception exceed cumulative retained earnings plus proceeds from sales of stock.

Collateral Some bonds are issued with specific collateral behind them. **Collateral** can take several forms, but it represents a particular asset of the firm that the bondholders receive if the firm defaults on the bond. If the collateral is property, the bond is called a *mortgage bond.* If the collateral takes the form of other securities held by the firm, the bond is a *collateral trust bond.* In the case of equipment, the bond is known as an *equipment obligation bond.* This last form of collateral is used most commonly by firms such as railroads, where the equipment is fairly standard and can be easily sold to another firm should the firm default and the bondholders acquire the collateral.

Because of the specific collateral that backs them, collateralized bonds generally are considered the safest variety of corporate bonds. General **debenture** bonds by contrast do not provide for specific collateral; they are *unsecured* bonds. The bondholder relies solely on the general earning power of the firm for the bond's safety. If the firm defaults, debenture owners become general creditors of the firm. Because they are safer, collateralized bonds generally offer lower yields than general debentures.

Figure 14.10 shows the terms of a bond issued by Mobil as described in *Moody's Industrial Manual.* The bond is registered and listed on the NYSE. It was issued in 1991 but was not callable until 2002. Although the call price started at 105.007% of par value, it falls gradually until it reaches par after 2020. Most of the terms of the bond are typical and illustrate many of the indenture provisions we have mentioned. However, in recent years there has been a marked trend away from the use of call provisions.

Yield to Maturity and Default Risk

Because corporate bonds are subject to default risk, we must distinguish between the bond's promised yield to maturity and its expected yield. The promised or stated yield will be realized only if the firm meets the obligations of the bond issue. Therefore, the stated yield is the *maximum possible* yield to maturity of the bond. The expected yield to maturity must take into account the possibility of a default.

For example, in November 2001, as Enron approached bankruptcy, its 6.4% coupon bonds due in 2006 were selling at about 20% of par value, resulting in a yield to maturity of about 57%. Investors did not really expect these bonds to provide a 57% rate of return. They recognized that bondholders were very unlikely to receive all the payments promised in the bond contract and that the yield based on *expected* cash flows was far less than the yield based on *promised* cash flows.

EXAMPLE 14.10 Expected vs. Promised Yield to Maturity

Suppose a firm issued a 9% coupon bond 20 years ago. The bond now has 10 years left until its maturity date but the firm is having financial difficulties. Investors believe that the firm will be able to make good on the remaining interest payments, but that at the maturity date, the firm will be forced into bankruptcy, and bondholders will receive only 70% of par value. The bond is selling at $750.

Yield to maturity (YTM) would then be calculated using the following inputs:

	Expected YTM	Stated YTM
Coupon payment	$45	$45
Number of semiannual periods	20 periods	20 periods
Final payment	$700	$1,000
Price	$750	$750

The yield to maturity based on promised payments is 13.7%. Based on the expected payment of $700 at maturity, however, the yield to maturity would be only 11.6%. The stated yield to maturity is greater than the yield investors actually expect to receive.

Example 14.10 suggests that when a bond becomes more subject to default risk, its price will fall, and therefore its promised yield to maturity will rise. Similarly, the default premium, the spread between the stated yield to maturity and that on otherwise-comparable Treasury bonds, will rise. However, its expected yield to maturity, which ultimately is tied to the systematic risk of the bond, will be far less affected. Let's continue Example 14.10.

EXAMPLE 14.11 Default Risk and the Default Premium

Suppose that the condition of the firm in Example 14.10 deteriorates further, and investors now believe that the bond will pay off only 55% of face value at maturity. Investors now demand an expected yield to maturity of 12% (i.e., 6% semiannually), which is 0.4% higher than in Example 14.10. But the price of the bond will fall from $750 to $688 [$n$ = 20; i = 6; FV = 550; PMT = $45]. At this price, the stated yield to maturity based on promised cash flows is 15.2%. While the expected yield to maturity has increased by 0.4%, the drop in price has caused the promised yield to maturity to rise by 1.5%.

CONCEPT CHECK 10

What is the expected yield to maturity if the firm is in even worse condition? Investors expect a final payment of only $500, and the bond price has fallen to $650.

To compensate for the possibility of default, corporate bonds must offer a **default premium.** The default premium is the difference between the promised yield on a corporate bond and the yield of an otherwise-identical government bond that is riskless in terms of default. If the firm remains solvent and actually pays the investor all of the promised cash flows, the investor will realize a

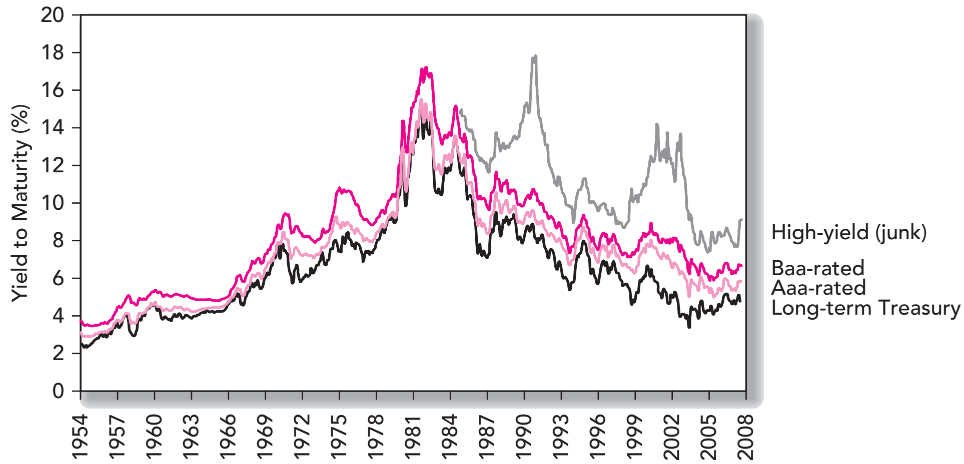

FIGURE 14.11 Yields on long-term bonds, 1954–2006

higher yield to maturity than would be realized from the government bond. If, however, the firm goes bankrupt, the corporate bond is likely to provide a lower return than the government bond. The corporate bond has the potential for both better and worse performance than the default-free Treasury bond. In other words, it is riskier.

The pattern of default premiums offered on risky bonds is sometimes called the *risk structure of interest rates.* The greater the default risk, the higher the default premium. Figure 14.11 shows yield to maturity of bonds of different risk classes since 1954 and yields on junk bonds since 1986. You can see here clear evidence of credit-risk premiums on promised yields.

One particular manner in which yield spreads seem to vary over time is related to the business cycle. Yield spreads tend to be wider when the economy is in a recession. Apparently, investors perceive a higher probability of bankruptcy when the economy is faltering, even holding bond rating constant. They require a commensurately higher default premium. This is sometimes termed a *flight to quality,* meaning that investors move their funds into safer bonds unless they can obtain larger premiums on lower-rated securities.

Credit Risk and Collateralized Debt Obligations

Collateralized debt obligations, or **CDO**s, emerged in the last decade as a major mechanism to reallocate credit risk in the fixed-income markets. To create a CDO, a financial institution, commonly a bank, first establishes a legally distinct entity to buy and later resell a portfolio of bonds or other loans. A common vehicle for this purpose is the so-called Structured Investment Vehicle (SIV).[14] The SIV raises funds, often by issuing short-term commercial paper, and uses the proceeds to buy corporate bonds or other

[14]The legal separation of the bank from the SIV allows the ownership of the loans to be conducted off the bank's balance sheet, and thus avoids capital requirements the bank would otherwise encounter.

		Senior-Subordinated Tranche Structure	Typical Terms
		Senior tranche	70–90% of notional principal, coupon similar to Aa-Aaa rated bonds
		Mezzanine 1	5–15% of principal, investment-grade rating
Bank	Structured Investment Vehicle, SIV		
		Mezzanine 2	5–15% of principal, higher-quality junk rating
		Equity/first loss/ residual tranche	<2%, unrated, coupon rate with 20% credit spread

FIGURE 14.12 Collateralized debt obligations

forms of debt such as mortgage loans or credit card debt. These loans are first pooled together and then split into a series of classes known as *tranches*. (*Tranche* is the French word for "slice.")

Each tranche is given a different level of seniority in terms of its claims on the underlying loan pool, and each can be sold as a stand-alone security. As the loans in the underlying pool make their interest payments, the proceeds are distributed to pay interest to each tranche in order of seniority. This priority structure implies that each tranche has a different exposure to credit risk.

Figure 14.12 illustrates a typical setup. The senior tranche is on top. Its investors may account for perhaps 80% of the principal of the entire pool. But it has first claim on *all* the debt service, and therefore bears little credit exposure. For example, using our numbers, even if 20% of the debt pool defaults, the senior tranche can be paid in full. Once the highest seniority tranche is paid off, the next-lower class (e.g., the mezzanine 1 tranche in Figure 14.12) receives the proceeds from the pool of loans until its claims also are satisfied.

Of course, shielding senior tranches from default risk means that the risk is concentrated on the lower tranches. The bottom tranche—called alternatively the equity, first-loss, or residual tranche—has last call on payments from the pool of loans, or, put differently, is at the head of the line in terms of absorbing default or delinquency risk. Using junior tranches to insulate senior tranches from credit risk in this manner, one can create Aaa-rated bonds even from a junk-bond portfolio. And, in fact, while Aaa-rated bonds are extremely few and far between, Aaa-rated CDO tranches are common.

Not surprisingly, investors in tranches with the greatest exposure to credit risk demand the highest coupon rates. Therefore, while the lower mezzanine and equity tranches bear the most risk, they will provide the highest returns if credit experience turns out favorably. Ideally, investors with greater expertise in evaluating credit risk are the natural investors in these securities. Often, the originating bank holds the residual tranche. This arrangement makes sense, because it provides incentives to the originator to perform careful credit analysis of the bonds included in the structure. The bank

CREDIT AND BLAME

The rating agencies Standard & Poor's (S&P), Moody's, and Fitch have earned huge sums in the past ten years offering opinions on the creditworthiness of an alphabet soup of mortgage-related securities created by over-eager banks. But did the fat fees lead to a drop in standards?

The agencies feel aggrieved at the criticism. So far, defaults have hit only three of the mortgage tranches it has rated. Of more complex products, collateralized-debt obligations (CDOs) downgrades have affected just 1% of securities by value.

The agencies are neither the only, nor indeed the main, culprits for the subprime crisis. The American mortgage industry was rotten from top to bottom, from buyers lying about their incomes to qualify for loans, through brokers accepting buyers with poor credit histories, to investors who bought bonds in the secondary market without conducting enough research.

Nevertheless, the agencies' business is built upon a rather shaky foundation. Rules devised by regulators, such as America's Securities Exchange Commission (SEC) and bank watchdogs, have made ratings a formal part of the financial system. The agencies have thus been handed a lucrative oligopoly. Moreover, they have a conflict of interest, since they are paid by the issuers whose securities they rate.

It is very hard to see how this combination can be justified. If the agencies' views are given a regulatory imprimatur, they should be subject to legal challenge. Alternatively, if they are simply independent expressions of opinion, then either investors, not issuers, should pay them, or they should be divorced from the regulatory system.

Joshua Rosner of Graham Fisher, an investment firm, thinks that the agencies should both be more transparent and improve their monitoring. Following bonds once they trade in the secondary market is much less lucrative for the agencies, he argues, and they devote far fewer resources to it. Although the agencies' models make it clear what rating they will give a bond on issue, it is less clear what will cause them to downgrade it later on.

Another response would be to make the agencies legally liable for their views. But the potential damage claim for making a duff rating would be so large that agencies might either be driven out of business or made excessively cautious by the threat of legal action.

The agencies could be asked to earn their fees from someone other than the issuers. But who? It is hard to believe that investors would pay: By hook or by crook, ratings would become public knowledge. The problem of free-riders means that there would not be enough research.

Source: *The Economist*, September 6, 2007.

therefore retains significant interest in the management of the relationship with the borrowers.

Mortgage-backed CDOs were an investment disaster in 2007. These were CDOs formed by pooling not corporate debt, but subprime mortgage loans made to individuals whose credit standing did not allow them to qualify for conventional mortgages. When home prices stalled in 2007 and interest rates on these typically adjustable-rate loans reset to market levels, mortgage delinquencies and home foreclosures soared, and investors in these securities lost billions of dollars. Even some highly rated tranches suffered extreme losses as default rates turned out to be far higher than anticipated. The SIVs, which had financed their purchase of these loans by issuing short-term asset-backed commercial paper, came under extreme pressure as investors were unwilling to roll over the paper into new issues once they reassessed the credit risk of the loan pools backing their investments.

Not surprisingly, the rating agencies that had certified these tranches as investment-grade came under considerable fire. Questions were raised concerning conflicts of interest: Because the rating agencies are paid by bond issuers, the agencies were accused of responding to pressure to ease their standards. See the above box for more on the ensuing controversy.

SUMMARY

1. Fixed-income securities are distinguished by their promise to pay a fixed or specified stream of income to their holders. The coupon bond is a typical fixed-income security.
2. Treasury notes and bonds have original maturities greater than 1 year. They are issued at or near par value, with their prices quoted net of accrued interest.
3. Callable bonds should offer higher promised yields to maturity to compensate investors for the fact that they will not realize full capital gains should the interest rate fall and the bonds be called away from them at the stipulated call price. Bonds often are issued with a period of call protection. In addition, discount bonds selling significantly below their call price offer implicit call protection.
4. Put bonds give the bondholder rather than the issuer the option to terminate or extend the life of the bond.
5. Convertible bonds may be exchanged, at the bondholder's discretion, for a specified number of shares of stock. Convertible bondholders "pay" for this option by accepting a lower coupon rate on the security.
6. Floating-rate bonds pay a coupon rate at a fixed premium over a reference short-term interest rate. Risk is limited because the rate is tied to current market conditions.
7. The yield to maturity is the single interest rate that equates the present value of a security's cash flows to its price. Bond prices and yields are inversely related. For premium bonds, the coupon rate is greater than the current yield, which is greater than the yield to maturity. The order of these inequalities is reversed for discount bonds.
8. The yield to maturity is often interpreted as an estimate of the average rate of return to an investor who purchases a bond and holds it until maturity. This interpretation is subject to error, however. Related measures are yield to call, realized compound yield, and expected (versus promised) yield to maturity.
9. Prices of zero-coupon bonds rise exponentially over time, providing a rate of appreciation equal to the interest rate. The IRS treats this built-in price appreciation as imputed taxable interest income to the investor.
10. When bonds are subject to potential default, the stated yield to maturity is the maximum possible yield to maturity that can be realized by the bondholder. In the event of default, however, that promised yield will not be realized. To compensate bond investors for default risk, bonds must offer default premiums, that is, promised yields in excess of those offered by default-free government securities. If the firm remains healthy, its bonds will provide higher returns than government bonds. Otherwise the returns may be lower.
11. Bond safety is often measured using financial ratio analysis. Bond indentures are another safeguard to protect the claims of bondholders. Common indentures specify sinking fund requirements, collateralization of the loan, dividend restrictions, and subordination of future debt.
12. Collateralized debt obligations are used to reallocate the credit risk of a pool of loans. The pool is sliced into tranches, with each tranche assigned a different level of seniority in terms of its claims on the cash flows from the underlying loans. High seniority tranches are usually quite safe, with credit risk concentrated on the lower level tranches. Each tranche can be sold as a stand-alone security.

Related web sites for this chapter are available at **www.mhhe.com/bkm**

KEY TERMS

debt securities
bond
par value
face value
coupon rate
bond indenture
zero-coupon bonds
convertible bonds
put bond
floating-rate bonds
yield to maturity
current yield
premium bonds
discount bonds
realized compound return
horizon analysis
reinvestment rate risk
credit risk
investment-grade bonds
speculative-grade or junk bonds
sinking fund
subordination clauses
collateral
debenture
default premium
collateralized debt obligations

PROBLEM SETS

Quiz

1. Two bonds have identical times to maturity and coupon rates. One is callable at 105, the other at 110. Which should have the higher yield to maturity? Why?
2. The stated yield to maturity and realized compound yield to maturity of a (default-free) zero-coupon bond will always be equal. Why?
3. Why do bond prices go down when interest rates go up? Don't lenders like high interest rates?

Problems

4. Which security has a higher *effective* annual interest rate?
   a. A 3-month T-bill selling at $97,645 with par value $100,000.
   b. A coupon bond selling at par and paying a 10% coupon semiannually.
5. Treasury bonds paying an 8% coupon rate with *semiannual* payments currently sell at par value. What coupon rate would they have to pay in order to sell at par if they paid their coupons *annually?* (Hint: what is the effective annual yield on the bond?)
6. Consider a bond with a 10% coupon and with yield to maturity = 8%. If the bond's yield to maturity remains constant, then in 1 year, will the bond price be higher, lower, or unchanged? Why?
7. Consider an 8% coupon bond selling for $953.10 with 3 years until maturity making *annual* coupon payments. The interest rates in the next 3 years will be, with certainty, $r_1 = 8\%$, $r_2 = 10\%$, and $r_3 = 12\%$. Calculate the yield to maturity and realized compound yield of the bond.
8. Assume you have a 1-year investment horizon and are trying to choose among three bonds. All have the same degree of default risk and mature in 10 years. The first is a zero-coupon bond that pays $1,000 at maturity. The second has an 8% coupon rate and pays the $80 coupon once per year. The third has a 10% coupon rate and pays the $100 coupon once per year.
   a. If all three bonds are now priced to yield 8% to maturity, what are their prices?
   b. If you expect their yields to maturity to be 8% at the beginning of next year, what will their prices be then? What is your before-tax holding-period return on each bond? If your tax bracket is 30% on ordinary income and 20% on capital gains income, what will your after-tax rate of return be on each?
   c. Recalculate your answer to (*b*) under the assumption that you expect the yields to maturity on each bond to be 7% at the beginning of next year.
9. A 20-year maturity bond with par value of $1,000 makes semiannual coupon payments at a coupon rate of 8%. Find the bond equivalent and effective annual yield to maturity of the bond if the bond price is:
   a. $950.
   b. $1,000.
   c. $1,050.
10. Repeat Problem 9 using the same data, but assuming that the bond makes its coupon payments annually. Why are the yields you compute lower in this case?
11. Fill in the table below for the following zero-coupon bonds, all of which have par values of $1,000.

Price	Maturity (years)	Bond-Equivalent Yield to Maturity
$400	20	—
$500	20	—
$500	10	—
—	10	10%
—	10	8%
$400	—	8%

12. Consider a bond paying a coupon rate of 10% per year semiannually when the market interest rate is only 4% per half-year. The bond has 3 years until maturity.
    a. Find the bond's price today and 6 months from now after the next coupon is paid.
    b. What is the total (6-month) rate of return on the bond?
13. A bond with a coupon rate of 7% makes semiannual coupon payments on January 15 and July 15 of each year. *The Wall Street Journal* reports the asked price for the bond on January 30 at 100:02. What is the invoice price of the bond? The coupon period has 182 days.
14. A bond has a current yield of 9% and a yield to maturity of 10%. Is the bond selling above or below par value? Explain.
15. Is the coupon rate of the bond in Problem 14 more or less than 9%?
16. Return to Table 14.1 and calculate both the real and nominal rates of return on the TIPS bond in the second and third years.
17. A newly issued 20-year maturity, zero-coupon bond is issued with a yield to maturity of 8% and face value $1,000. Find the imputed interest income in the first, second, and last year of the bond's life.
18. A newly issued 10-year maturity, 4% coupon bond making *annual* coupon payments is sold to the public at a price of $800. What will be an investor's taxable income from the bond over the coming year? The bond will not be sold at the end of the year. The bond is treated as an original-issue discount bond.
19. A 30-year maturity, 8% coupon bond paying coupons semiannually is callable in 5 years at a call price of $1,100. The bond currently sells at a yield to maturity of 7% (3.5% per half-year).
    a. What is the yield to call?
    b. What is the yield to call if the call price is only $1,050?
    c. What is the yield to call if the call price is $1,100, but the bond can be called in 2 years instead of 5 years?
20. A 10-year bond of a firm in severe financial distress has a coupon rate of 14% and sells for $900. The firm is currently renegotiating the debt, and it appears that the lenders will allow the firm to reduce coupon payments on the bond to one-half the originally contracted amount. The firm can handle these lower payments. What is the stated and expected yield to maturity of the bonds? The bond makes its coupon payments annually.
21. A 2-year bond with par value $1,000 making annual coupon payments of $100 is priced at $1,000. What is the yield to maturity of the bond? What will be the realized compound yield to maturity if the 1-year interest rate next year turns out to be (*a*) 8%, (*b*) 10%, (*c*) 12%?
22. Suppose that today's date is April 15. A bond with a 10% coupon paid semiannually every January 15 and July 15 is listed in *The Wall Street Journal* as selling at an asked price of 101:04. If you buy the bond from a dealer today, what price will you pay for it?
23. Assume that two firms issue bonds with the following characteristics. Both bonds are issued at par.

	ABC Bonds	XYZ Bonds
Issue size	$1.2 billion	$150 million
Maturity	10 years*	20 years
Coupon	9%	10%
Collateral	First mortgage	General debenture
Callable	Not callable	In 10 years
Call price	None	110
Sinking fund	None	Starting in 5 years

*Bond is extendible at the discretion of the bondholder for an additional 10 years.

Ignoring credit quality, identify four features of these issues that might account for the lower coupon on the ABC debt. Explain.

24. A large corporation issued both fixed and floating-rate notes 5 years ago, with terms given in the following table:

	9% Coupon Notes	Floating-Rate Note
Issue size	$250 million	$280 million
Original maturity	20 years	10 years
Current price (% of par)	93	98
Current coupon	9%	8%
Coupon adjusts	Fixed coupon	Every year
Coupon reset rule	—	1-year T-bill rate + 2%
Callable	10 years after issue	10 years after issue
Call price	106	102.50
Sinking fund	None	None
Yield to maturity	9.9%	—
Price range since issued	$85–$112	$97–$102

a. Why is the price range greater for the 9% coupon bond than the floating-rate note?
b. What factors could explain why the floating-rate note is not always sold at par value?
c. Why is the call price for the floating-rate note not of great importance to investors?
d. Is the probability of call for the fixed-rate note high or low?
e. If the firm were to issue a fixed-rate note with a 15-year maturity, what coupon rate would it need to offer to issue the bond at par value?
f. Why is an entry for yield to maturity for the floating-rate note not appropriate?

25. Masters Corp. issues two bonds with 20-year maturities. Both bonds are callable at $1,050. The first bond is issued at a deep discount with a coupon rate of 4% and a price of $580 to yield 8.4%. The second bond is issued at par value with a coupon rate of 8¾%.

a. What is the yield to maturity of the par bond? Why is it higher than the yield of the discount bond?
b. If you expect rates to fall substantially in the next 2 years, which bond would you prefer to hold?
c. In what sense does the discount bond offer "implicit call protection"?

Challenge Problem

26. A newly issued bond pays its coupons once annually. Its coupon rate is 5%, its maturity is 20 years, and its yield to maturity is 8%.

a. Find the holding-period return for a 1-year investment period if the bond is selling at a yield to maturity of 7% by the end of the year.
b. If you sell the bond after 1 year, what taxes will you owe if the tax rate on interest income is 40% and the tax rate on capital gains income is 30%? The bond is subject to original-issue discount tax treatment.
c. What is the after-tax holding-period return on the bond?
d. Find the realized compound yield *before taxes* for a 2-year holding period, assuming that (1) you sell the bond after 2 years, (2) the bond yield is 7% at the end of the second year, and (3) the coupon can be reinvested for 1 year at a 3% interest rate.
e. Use the tax rates in (*b*) above to compute the *after-tax* 2-year realized compound yield. Remember to take account of OID tax rules.

CFA® PROBLEMS

1. Leaf Products may issue a 10-year maturity fixed-income security, which might include a sinking fund provision and either refunding or call protection.

a. Describe a sinking fund provision.
b. Explain the impact of a sinking fund provision on:

i. The expected average life of the proposed security.
ii. Total principal and interest payments over the life of the proposed security.

c. From the investor's point of view, explain the rationale for demanding a sinking fund provision.

2. Bonds of Zello Corporation with a par value of $1,000 sell for $960, mature in 5 years, and have a 7% annual coupon rate paid semiannually.
   a. Calculate the:
      i. Current yield.
      ii. Yield to maturity (to the nearest whole percent, i.e., 3%, 4%, 5%, etc.).
      iii. Realized compound yield for an investor with a 3-year holding period and a reinvestment rate of 6% over the period. At the end of 3 years the 7% coupon bonds with 2 years remaining will sell to yield 7%.
   b. Cite one major shortcoming for each of the following fixed-income yield measures:
      i. Current yield.
      ii. Yield to maturity.
      iii. Realized compound yield.
3. On May 30, 2008, Janice Kerr is considering one of the newly issued 10-year AAA corporate bonds shown in the following exhibit.

Description	Coupon	Price	Callable	Call Price
Sentinal, due May 30, 2018	6.00%	100	Noncallable	NA
Colina, due May 30, 2018	6.20%	100	Currently callable	102

   a. Suppose that market interest rates decline by 100 basis points (i.e., 1%). Contrast the effect of this decline on the price of each bond.
   b. Should Kerr prefer the Colina over the Sentinal bond when rates are expected to rise or to fall?
   c. What would be the effect, if any, of an increase in the *volatility* of interest rates on the prices of each bond?
4. A convertible bond has the following features:

Coupon	5.25%
Maturity	June 15, 2027
Market price of bond	$77.50
Market price of underlying common stock	$28.00
Annual dividend	$1.20
Conversion ratio	20.83 shares

   Calculate the conversion premium for this bond.
5. *a.* Explain the impact on the offering yield of adding a call feature to a proposed bond issue.
   b. Explain the impact on the bond's expected life of adding a call feature to a proposed bond issue.
   c. Describe one advantage and one disadvantage of including callable bonds in a portfolio.
6. *a.* An investment in a coupon bond will provide the investor with a return equal to the bond's yield to maturity at the time of purchase if:
      i. The bond is not called for redemption at a price that exceeds its par value.
      ii. All sinking fund payments are made in a prompt and timely fashion over the life of the issue.
      iii. The reinvestment rate is the same as the bond's yield to maturity and the bond is held until maturity.
      iv. All of the above.
   b. A bond with a call feature:
      i. Is attractive because the immediate receipt of principal plus premium produces a high return.
      ii. Is more apt to be called when interest rates are high because the interest savings will be greater.

iii. Will usually have a higher yield to maturity than a similar noncallable bond.
iv. None of the above.

c. In which *one* of the following cases is the bond selling at a discount?

i. Coupon rate is greater than current yield, which is greater than yield to maturity.
ii. Coupon rate, current yield, and yield to maturity are all the same.
iii. Coupon rate is less than current yield, which is less than yield to maturity.
iv. Coupon rate is less than current yield, which is greater than yield to maturity.

d. Consider a 5-year bond with a 10% coupon that has a present yield to maturity of 8%. If interest rates remain constant, 1 year from now the price of this bond will be:

i. Higher
ii. Lower
iii. The same
iv. Par

STANDARD & POOR'S

Use the *Financial Highlights* section of Market Insight (**www.mhhe.com/edumarketinsight**) to obtain Standard & Poor's Issuer Credit Ratings of at least ten firms in the database. Try to choose a sample with a wide range of ratings. Next use Market Insight's Annual Ratio Report (in the *Excel Analytics* section) to obtain for each firm the financial ratios shown in Table 14.3. What is the relationship between the firms' credit ratings and their ratios? Can you tell from your sample firms which of these ratios are the more important determinants of credit rating?

E-Investments

Credit Spreads

At **www.bondsonline.com** review the *Industrial Spreads* for various ratings (click the links on the left-side menus to follow the links to *Today's Markets, Corporate Bond Spreads*). These are spreads above U.S. Treasuries of comparable maturities. What factors tend to explain the yield differences? How might these yield spreads differ during an economic boom versus a recession?

From the home page, select *Today's Markets* from the left-side menu and then select the link for *Composite Bond Yields*. How do the Yield Curves for Treasury, Agency, Corporate, and Municipal bonds compare to each other?

SOLUTIONS TO CONCEPT CHECKS

1. The callable bond will sell at the *lower* price. Investors will not be willing to pay as much if they know that the firm retains a valuable option to reclaim the bond for the call price if interest rates fall.

2. At a semiannual interest rate of 3%, the bond is worth $40 × Annuity factor (3%, 60) + $1,000 × PV factor(3%, 60) = $1,276.76, which results in a capital gain of $276.76. This exceeds the capital loss of $189.29 ($1,000 − $810.71) when the semiannual interest rate increased to 5%.

3. Yield to maturity exceeds current yield, which exceeds coupon rate. Take as an example the 8% coupon bond with a yield to maturity of 10% per year (5% per half year). Its price is $810.71, and therefore its current yield is 80/810.71 = .0987, or 9.87%, which is higher than the coupon rate but lower than the yield to maturity.

4. The bond with the 6% coupon rate currently sells for 30 × Annuity factor (3.5%, 20) + 1,000 × PV factor(3.5%, 20) = $928.94. If the interest rate immediately drops to 6% (3% per half-year), the bond price will rise to $1,000, for a capital gain of $71.06, or 7.65%. The 8% coupon bond currently sells for $1,071.06. If the interest rate falls to 6%, the present value of the *scheduled* payments increases to $1,148.77. However, the bond will be called at $1,100, for a capital gain of only $28.94, or 2.70%.

5. The current price of the bond can be derived from its yield to maturity. Using your calculator, set: $n = 40$ (semiannual periods); payment = $45 per period; future value = $1,000; interest rate = 4% per semiannual period. Calculate present value as $1,098.96. Now we can calculate yield to call. The time to call is 5 years, or 10 semiannual periods. The price at which the bond will be called is $1,050. To find yield to call, we set: $n = 10$ (semiannual periods); payment = $45 per period; future value = $1,050; present value = $1,098.96. Calculate yield to call as 3.72%.

6. Price = $70 × Annuity factor(8%, 1) + $1,000 × PV factor(8%, 1) = $990.74

$$\text{Rate of return to investor} = \frac{\$70 + (\$990.74 - \$982.17)}{\$982.17} = .080 = 8\%$$

7. By year-end, remaining maturity is 29 years. If the yield to maturity were still 8%, the bond would still sell at par and the holding-period return would be 8%. At a higher yield, price and return will be lower. Suppose, for example, that the yield to maturity rises to 8.5%. With annual payments of $80 and a face value of $1,000, the price of the bond will be $946.70 [$n = 29$; $i = 8.5\%$; PMT = $80; FV = $1,000]. The bond initially sold at $1,000 when issued at the start of the year. The holding-period return is

$$\text{HPR} = \frac{80 + (946.70 - 1{,}000)}{1{,}000} = .0267 = 2.67\%$$

which is less than the initial yield to maturity of 8%.

8. At the lower yield, the bond price will be $631.67 [$n = 29$, $i = 7\%$, FV = $1,000, PMT = $40]. Therefore, total after-tax income is

Coupon	$40 × (1 − .38) =	$24.80
Imputed interest	($553.66 − $549.69) × (1 − .38) =	2.46
Capital gains	($631.67 − $553.66) × (1 − .20) =	62.41
Total income after taxes		$89.67

Rate of return = 89.67/549.69 = .163 = 16.3%.

9. It should receive a negative coefficient. A high ratio of liabilities to assets is a poor omen for a firm that should lower its credit rating.

10. The coupon payment is $45. There are 20 semiannual periods. The final payment is assumed to be $500. The present value of expected cash flows is $650. The expected yield to maturity is 6.317% semiannual or annualized, 12.63%, bond equivalent yield.

PART V

CHAPTER EIGHTEEN 18

EQUITY VALUATION MODELS

AS OUR DISCUSSION of market efficiency indicated, finding undervalued securities is hardly easy. At the same time, there are enough chinks in the armor of the efficient market hypothesis that the search for such securities should not be dismissed out of hand. Moreover, it is the ongoing search for mispriced securities that maintains a nearly efficient market. Even infrequent discoveries of minor mispricing would justify the salary of a stock market analyst.

This chapter describes the valuation models that stock market analysts use to uncover mispriced securities. The models presented are those used by *fundamental analysts,* those analysts who use information concerning the current and prospective profitability of a company to assess its fair market value. We start with a discussion of alternative measures of the value of a company. From there, we progress to quantitative tools called *dividend discount models,* which security analysts commonly use to measure the value of a firm as an ongoing concern. Next we turn to price–earnings, or P/E, ratios, explaining why they are of such interest to analysts but also highlighting some of their shortcomings. We explain how P/E ratios are tied to dividend valuation models and, more generally, to the growth prospects of the firm.

We close the chapter with a discussion and extended example of free cash flow models used by analysts to value firms based on forecasts of the cash flows that will be generated from the firms' business endeavors. Finally, we apply the several valuation tools covered in the chapter to a real firm and find that there is some disparity in their conclusions—a conundrum that will confront any security analyst—and consider reasons for these discrepancies.

18.1 VALUATION BY COMPARABLES

The purpose of fundamental analysis is to identify stocks that are mispriced relative to some measure of "true" value that can be derived from observable financial data. There are many convenient sources of such data. For U.S. companies, the Securities and Exchange

Commission provides information at its EDGAR Web site, **www.sec.gov/edgar.shtml.** The SEC requires all public companies (except foreign companies and companies with less than $10 million in assets and 500 shareholders) to file registration statements, periodic reports, and other forms electronically through EDGAR. Anyone can access and download this information.

Many Web sites provide analysis of the data. An example is Standard & Poor's Market Insight service, which includes COMPUSTAT.[1] Table 18.1 shows COMPUSTAT's selection of financial highlights for Microsoft Corporation on October 25, 2007.

The price of a share of Microsoft common stock on that day is shown as $31.25, and the total market value of all 9,380 million shares outstanding was $293,125 million. Under the heading Valuation, Table 18.1 reports the ratios of Microsoft's stock price to four different items taken from its latest financial statements (each divided by the number of outstanding shares): operating earnings, book value, sales revenue, and cash flow. Microsoft's price-to-earnings (P/E) ratio is 21.6, the price-to-book value is 9.4, and price-to-sales is 5.7. Such comparative valuation ratios are used to assess the valuation of one firm versus others in the same industry. In the column to the right in Table 18.1 are comparable ratios for the average firm in the PC software industry.

For example, an analyst might compare the price/cash flow ratio for Microsoft—18.9, to the industry average ratio of 19.3. By comparison with this standard, Microsoft appears

TABLE 18.1

Financial highlights for Microsoft Corporation, October 25, 2007

Current Qtr Ended:	Jun. 2007	Current Year Ended:	Jun. 2007
Miscellaneous			
Current price	31.250000	Comn sharehldrs (actual)	148344
Comn shares outstdg (mil)	9380.000	Employees (actual)	79000
Market capitalization (mil)	293125.000	S&P issuer credit rating	
Latest 12 Months	**Company**		**1 Yr Chng (%)**
Sales (mil)	51122.000		15.4
EBITDA (mil)	19964.000		8.0
Net income (mil)	14065.000		11.6
EPS from Ops	1.45		12.4
Dividends/Share	0.390000		14.7
Valuation	**Company**		**Industry Avg**
Price/EPS from Ops	21.6		22.4
Price/Book	9.4		6.3
Price/Sales	5.7		5.2
Price/Cash flow	18.9		19.3
Profitability (%)			
Return on equity	45.2		27.4
Return on assets	22.3		13.8
Oper profit margin	36.2		31.0
Net profit margin	27.5		22.5
Financial Risk			
Debt/Equity	0.0		18.4

Source: COMPUSTAT Company Profiles, October 25, 2007.

[1]A subscription to S&P Market Insight's educational version comes with this textbook.

to be slightly underpriced. The price-to-sales ratio is useful for firms and industries that are in a start-up phase. Earnings figures for start-up firms are often negative and not reported, so analysts shift their focus from earnings per share to sales revenue per share.

The market price of a share of Microsoft stock was 9.4 times its book value. **Book value** is the net worth of a company as reported on its balance sheet. For the average firm in the PC software industry it was 6.3. By comparison with this standard, Microsoft seems a bit overvalued.

Limitations of Book Value

Shareholders in a firm are sometimes called "residual claimants," which means that the value of their stake is what is left over when the liabilities of the firm are subtracted from its assets. Shareholders' equity is this net worth. However, the values of both assets and liabilities recognized in financial statements are based on historical—not current—values. For example, the book value of an asset equals the *original* cost of acquisition less some adjustment for depreciation, even if the market price of that asset has changed over time. Moreover, depreciation allowances are used to allocate the original cost of the asset over several years, but do not reflect loss of actual value.

Whereas book values are based on original cost, market values measure *current* values of assets and liabilities. The market value of the shareholders' equity investment equals the difference between the current values of all assets and liabilities. (The stock price is just the market value of shareholders' equity divided by the number of outstanding shares.) We've emphasized that current values generally will not match historical ones. Equally or even more important, many assets, for example, the value of a good brand name or specialized expertise developed over many years, may not even be included on the financial statements. Market prices therefore reflect the value of the firm as a going concern. It would be unusual if the market price of a stock were exactly equal to its book value.

Can book value represent a "floor" for the stock's price, below which level the market price can never fall? Although Microsoft's book value per share in 2007 was less than its market price, other evidence disproves this notion. While it is not common, there are always some firms selling at a market price below book value. In early 2008, for example, such troubled firms included Northwest Airlines and Countrywide Financial Corp.

A better measure of a floor for the stock price is the firm's **liquidation value** per share. This represents the amount of money that could be realized by breaking up the firm, selling its assets, repaying its debt, and distributing the remainder to the shareholders. The reasoning behind this concept is that if the market price of equity drops below liquidation value, the firm becomes attractive as a takeover target. A corporate raider would find it profitable to buy enough shares to gain control and then actually to liquidate.

Another approach to valuing a firm is the **replacement cost** of its assets less its liabilities. Some analysts believe the market value of the firm cannot remain for long too far above its replacement cost because if it did, competitors would try to replicate the firm. The competitive pressure of other similar firms entering the same industry would drive down the market value of all firms until they came into equality with replacement cost.

This idea is popular among economists, and the ratio of market price to replacement cost is known as **Tobin's *q*,** after the Nobel Prize–winning economist James Tobin. In the long run, according to this view, the ratio of market price to replacement cost will tend toward 1, but the evidence is that this ratio can differ significantly from 1 for very long periods.

Although focusing on the balance sheet can give some useful information about a firm's liquidation value or its replacement cost, the analyst must usually turn to expected future cash flows for a better estimate of the firm's value as a going concern. We now examine the quantitative models that analysts use to value common stock in terms of the future earnings and dividends the firm will yield.

18.2 INTRINSIC VALUE VERSUS MARKET PRICE

The most popular model for assessing the value of a firm as a going concern starts from the observation that an investor in stock expects a return consisting of cash dividends and capital gains or losses. We begin by assuming a 1-year holding period and supposing that ABC stock has an expected dividend per share, $E(D_1)$, of \$4, the current price of a share, P_0, is \$48, and the expected price at the end of a year, $E(P_1)$, is \$52. For now, don't worry about how you derive your forecast of next year's price. At this point we ask only whether the stock seems attractively priced *today* given your forecast of *next year's* price.

The *expected* holding-period return is $E(D_1)$ plus the expected price appreciation, $E(P_1) - P_0$, all divided by the current price, P_0:

$$\text{Expected HPR} = E(r) = \frac{E(D_1) + [E(P_1) - P_0]}{P_0}$$

$$= \frac{4 + (52 - 48)}{48} = .167, \text{ or } 16.7\%$$

Thus, the stock's expected holding-period return is the sum of the expected dividend yield, $E(D_1)/P_0$, and the expected rate of price appreciation, the capital gains yield, $[E(P_1) - P_0]/P_0$.

But what is the required rate of return for ABC stock? The CAPM states that when stock market prices are at equilibrium levels, the rate of return that investors can expect to earn on a security is $r_f + \beta[E(r_M) - r_f]$. Thus, the CAPM may be viewed as providing the rate of return an investor can expect to earn on a security given its risk as measured by beta. This is the return that investors will require of any other investment with equivalent risk. We will denote this required rate of return as k. If a stock is priced "correctly," it will offer investors a "fair" return, that is, its *expected* return will equal its *required* return. Of course, the goal of a security analyst is to find stocks that are mispriced. For example, an underpriced stock will provide an expected return greater than the required return.

Suppose that $r_f = 6\%$, $E(r_M) - r_f = 5\%$, and the beta of ABC is 1.2. Then the value of k is

$$k = 6\% + 1.2 \times 5\% = 12\%$$

The expected holding period return, 16.7%, therefore exceeds the required rate of return based on ABC's risk by a margin of 4.7%. Naturally, the investor will want to include more of ABC stock in the portfolio than a passive strategy would indicate.

Another way to see this is to compare the intrinsic value of a share of stock to its market price. The **intrinsic value,** denoted V_0, is defined as the present value of all cash payments to the investor in the stock, including dividends as well as the proceeds from the ultimate sale of the stock, discounted at the appropriate risk-adjusted interest rate, k. If the intrinsic value, or the investor's own estimate of what the stock is really worth, exceeds the market price, the stock is considered undervalued and a good investment. In the case of ABC, using a 1-year investment horizon and a forecast that the stock can be sold at the end of the year at price $P_1 = \$52$, the intrinsic value is

$$V_0 = \frac{E(D_1) + E(P_1)}{1 + k} = \frac{\$4 + \$52}{1.12} = \$50$$

Equivalently, at a price of \$50, the investor would derive a 12% rate of return—just equal to the required rate of return—on an investment in the stock. However, at the current price of \$48, the stock is underpriced compared to intrinsic value. At this price, it provides better than a fair rate of return relative to its risk. In other words, using the terminology of the CAPM, it is a positive-alpha stock, and investors will want to buy more of it than they would following a passive strategy.

If the intrinsic value turns out to be lower than the current market price, investors should buy less of it than under the passive strategy. It might even pay to go short on ABC stock, as we discussed in Chapter 3.

In market equilibrium, the current market price will reflect the intrinsic value estimates of all market participants. This means the individual investor whose V_0 estimate differs from the market price, P_0, in effect must disagree with some or all of the market consensus estimates of $E(D_1)$, $E(P_1)$, or k. A common term for the market consensus value of the required rate of return, k, is the **market capitalization rate,** which we use often throughout this chapter.

CONCEPT CHECK 1

You expect the price of IBX stock to be \$59.77 per share a year from now. Its current market price is \$50, and you expect it to pay a dividend 1 year from now of \$2.15 per share.

a. What is the stock's expected dividend yield, rate of price appreciation, and holding-period return?

b. If the stock has a beta of 1.15, the risk-free rate is 6% per year, and the expected rate of return on the market portfolio is 14% per year, what is the required rate of return on IBX stock?

c. What is the intrinsic value of IBX stock, and how does it compare to the current market price?

18.3 DIVIDEND DISCOUNT MODELS

Consider an investor who buys a share of Steady State Electronics stock, planning to hold it for 1 year. The intrinsic value of the share is the present value of the dividend to be received at the end of the first year, D_1, and the expected sales price, P_1. We will henceforth use the simpler notation P_1 instead of $E(P_1)$ to avoid clutter. Keep in mind, though, that future prices and dividends are unknown, and we are dealing with expected values, not certain values. We've already established

$$V_0 = \frac{D_1 + P_1}{1 + k} \quad \textbf{(18.1)}$$

Although this year's dividends are fairly predictable given a company's history, you might ask how we can estimate P_1, the year-end price. According to Equation 18.1, V_1 (the year-end intrinsic value) will be

$$V_1 = \frac{D_2 + P_2}{1 + k}$$

If we assume the stock will be selling for its intrinsic value next year, then $V_1 = P_1$, and we can substitute this value for P_1 into Equation 18.1 to find

$$V_0 = \frac{D_1}{1 + k} + \frac{D_2 + P_2}{(1 + k)^2}$$

This equation may be interpreted as the present value of dividends plus sales price for a 2-year holding period. Of course, now we need to come up with a forecast of P_2. Continuing in the same way, we can replace P_2 by $(D_3 + P_3)/(1 + k)$, which relates P_0 to the value of dividends plus the expected sales price for a 3-year holding period.

More generally, for a holding period of H years, we can write the stock value as the present value of dividends over the H years, plus the ultimate sale price, P_H:

$$V_0 = \frac{D_1}{1+k} + \frac{D_2}{(1+k)^2} + \cdots + \frac{D_H + P_H}{(1+k)^H} \tag{18.2}$$

Note the similarity between this formula and the bond valuation formula developed in Chapter 14. Each relates price to the present value of a stream of payments (coupons in the case of bonds, dividends in the case of stocks) and a final payment (the face value of the bond, or the sales price of the stock). The key differences in the case of stocks are the uncertainty of dividends, the lack of a fixed maturity date, and the unknown sales price at the horizon date. Indeed, one can continue to substitute for price indefinitely, to conclude

$$V_0 = \frac{D_1}{1+k} + \frac{D_2}{(1+k)^2} + \frac{D_3}{(1+k)^3} + \cdots \tag{18.3}$$

Equation 18.3 states that the stock price should equal the present value of all expected future dividends into perpetuity. This formula is called the **dividend discount model (DDM)** of stock prices.

It is tempting, but incorrect, to conclude from Equation 18.3 that the DDM focuses exclusively on dividends and ignores capital gains as a motive for investing in stock. Indeed, we assume explicitly in Equation 18.1 that capital gains (as reflected in the expected sales price, P_1) are part of the stock's value. Our point is that the price at which you can sell a stock in the future depends on dividend forecasts at that time.

The reason only dividends appear in Equation 18.3 is not that investors ignore capital gains. It is instead that those capital gains will be determined by dividend forecasts at the time the stock is sold. That is why in Equation 18.2 we can write the stock price as the present value of dividends plus sales price for *any* horizon date. P_H is the present value at time H of all dividends expected to be paid after the horizon date. That value is then discounted back to today, time 0. The DDM asserts that stock prices are determined ultimately by the cash flows accruing to stockholders, and those are dividends.[2]

The Constant-Growth DDM

Equation 18.3 as it stands is still not very useful in valuing a stock because it requires dividend forecasts for every year into the indefinite future. To make the DDM practical, we need to introduce some simplifying assumptions. A useful and common first pass at the problem is to assume that dividends are trending upward at a stable growth rate that we will call g. Then if $g = .05$, and the most recently paid dividend was $D_0 = 3.81$, expected future dividends are

$$\begin{aligned} D_1 &= D_0(1+g) &&= 3.81 \times 1.05 &&= 4.00 \\ D_2 &= D_0(1+g)^2 &&= 3.81 \times (1.05)^2 &&= 4.20 \\ D_3 &= D_0(1+g)^3 &&= 3.81 \times (1.05)^3 &&= 4.41 \end{aligned}$$

and so on. Using these dividend forecasts in Equation 18.3, we solve for intrinsic value as

$$V_0 = \frac{D_0(1+g)}{1+k} + \frac{D_0(1+g)^2}{(1+k)^2} + \frac{D_0(1+g)^3}{(1+k)^3} + \cdots$$

[2]If investors never expected a dividend to be paid, then this model implies that the stock would have no value. To reconcile the DDM with the fact that non-dividend-paying stocks do have a market value, one must assume that investors expect that some day it may pay out some cash, even if only a liquidating dividend.

This equation can be simplified to[3]

$$V_0 = \frac{D_0(1+g)}{k-g} = \frac{D_1}{k-g} \qquad \textbf{(18.4)}$$

Note in Equation 18.4 that we divide D_1 (not D_0) by $k - g$ to calculate intrinsic value. If the market capitalization rate for Steady State is 12%, now we can use Equation 18.4 to show that the intrinsic value of a share of Steady State stock is

$$\frac{\$4.00}{.12 - .05} = \$57.14$$

Equation 18.4 is called the **constant-growth DDM,** or the Gordon model, after Myron J. Gordon, who popularized the model. It should remind you of the formula for the present value of a perpetuity. If dividends were expected not to grow, then the dividend stream would be a simple perpetuity, and the valuation formula would be[4] $V_0 = D_1/k$. Equation 18.4 is a generalization of the perpetuity formula to cover the case of a *growing* perpetuity. As g increases (for a given value of D_1), the stock price also rises.

EXAMPLE 18.1 Preferred Stock and the DDM

Preferred stock that pays a fixed dividend can be valued using the constant-growth dividend discount model. The constant-growth rate of dividends is simply zero. For example, to value a preferred stock paying a fixed dividend of \$2 per share when the discount rate is 8%, we compute

$$V_0 = \frac{\$2}{.08 - 0} = \$25$$

[3]We prove that the intrinsic value, V_0, of a stream of cash dividends growing at a constant rate g is equal to $\frac{D_1}{k-g}$ as follows. By definition,

$$V_0 = \frac{D_1}{1+k} + \frac{D_1(1+g)}{(1+k)^2} + \frac{D_1(1+g)^2}{(1+k)^3} + \cdots \qquad \text{(a)}$$

Multiplying through by $(1 + k)/(1 + g)$, we obtain

$$\frac{(1+k)}{(1+g)}V_0 = \frac{D_1}{(1+g)} + \frac{D_1}{(1+k)} + \frac{D_1(1+g)}{(1+k)^2} + \cdots \qquad \text{(b)}$$

Subtracting equation (a) from equation (b), we find that

$$\frac{1+k}{1+g}V_0 - V_0 = \frac{D_1}{(1+g)}$$

which implies

$$\frac{(k-g)V_0}{(1+g)} = \frac{D_1}{(1+g)}$$

$$V_0 = \frac{D_1}{k-g}$$

[4]Recall from introductory finance that the present value of a \$1 per year perpetuity is $1/k$. For example, if $k = 10\%$, the value of the perpetuity is \$1/.10 = \$10. Notice that if $g = 0$ in Equation 18.4, the constant-growth DDM formula is the same as the perpetuity formula.

EXAMPLE 18.2 The Constant-Growth DDM

High Flyer Industries has just paid its annual dividend of \$3 per share. The dividend is expected to grow at a constant rate of 8% indefinitely. The beta of High Flyer stock is 1.0, the risk-free rate is 6%, and the market risk premium is 8%. What is the intrinsic value of the stock? What would be your estimate of intrinsic value if you believed that the stock was riskier, with a beta of 1.25?

Because a \$3 dividend has just been paid and the growth rate of dividends is 8%, the forecast for the year-end dividend is $\$3 \times 1.08 = \$3.24$. The market capitalization rate is $6\% + 1.0 \times 8\% = 14\%$. Therefore, the value of the stock is

$$V_0 = \frac{D_1}{k - g} = \frac{\$3.24}{.14 - .08} = \$54$$

If the stock is perceived to be riskier, its value must be lower. At the higher beta, the market capitalization rate is $6\% \times 1.25 \times 8\% = 16\%$, and the stock is worth only

$$\frac{\$3.24}{.16 - .08} = \$40.50$$

The constant-growth DDM is valid only when g is less than k. If dividends were expected to grow forever at a rate faster than k, the value of the stock would be infinite. If an analyst derives an estimate of g that is greater than k, that growth rate must be unsustainable in the long run. The appropriate valuation model to use in this case is a multistage DDM such as those discussed below.

The constant-growth DDM is so widely used by stock market analysts that it is worth exploring some of its implications and limitations. The constant-growth rate DDM implies that a stock's value will be greater:

1. The larger its expected dividend per share.
2. The lower the market capitalization rate, k.
3. The higher the expected growth rate of dividends.

Another implication of the constant-growth model is that the stock price is expected to grow at the same rate as dividends. To see this, suppose Steady State stock is selling at its intrinsic value of \$57.14, so that $V_0 = P_0$. Then

$$P_0 = \frac{D_1}{k - g}$$

Note that price is proportional to dividends. Therefore, next year, when the dividends paid to Steady State stockholders are expected to be higher by $g = 5\%$, price also should increase by 5%. To confirm this, note

$$D_2 = \$4(1.05) = \$4.20$$

$$P_1 = \frac{D_2}{k - g} = \frac{\$4.20}{.12 - .05} = \$60.00$$

which is 5% higher than the current price of \$57.14. To generalize,

$$P_1 = \frac{D_2}{k-g} = \frac{D_1(1+g)}{k-g} = \frac{D_1}{k-g}(1+g)$$
$$= P_0(1+g)$$

Therefore, the DDM implies that in the case of constant growth of dividends, the rate of price appreciation in any year will equal that constant-growth rate, g. Note that for a stock whose market price equals its intrinsic value ($V_0 = P_0$), the expected holding-period return will be

$$E(r) = \text{Dividend yield} + \text{Capital gains yield}$$
$$= \frac{D_1}{P_0} + \frac{P_1 - P_0}{P_0} = \frac{D_1}{P_0} + g \quad \textbf{(18.5)}$$

This formula offers a means to infer the market capitalization rate of a stock, for if the stock is selling at its intrinsic value, then $E(r) = k$, implying that $k = D_1/P_0 + g$. By observing the dividend yield, D_1/P_0, and estimating the growth rate of dividends, we can compute k. This equation is also known as the *discounted cash flow (DCF) formula*.

This is an approach often used in rate hearings for regulated public utilities. The regulatory agency responsible for approving utility pricing decisions is mandated to allow the firms to charge just enough to cover costs plus a "fair" profit, that is, one that allows a competitive return on the investment the firm has made in its productive capacity. In turn, that return is taken to be the expected return investors require on the stock of the firm. The $D_1/P_0 + g$ formula provides a means to infer that required return.

EXAMPLE 18.3 The Constant-Growth Model

Suppose that Steady State Electronics wins a major contract for its new computer chip. The very profitable contract will enable it to increase the growth rate of dividends from 5% to 6% without reducing the current dividend from the projected value of \$4.00 per share. What will happen to the stock price? What will happen to future expected rates of return on the stock?

The stock price ought to increase in response to the good news about the contract, and indeed it does. The stock price jumps from its original value of \$57.14 to a postannouncement price of

$$\frac{D_1}{k-g} = \frac{\$4.00}{.12 - .06} = \$66.67$$

Investors who are holding the stock when the good news about the contract is announced will receive a substantial windfall.

On the other hand, at the new price the expected rate of return on the stock is 12%, just as it was before the new contract was announced.

$$E(r) = \frac{D_1}{P_0} + g = \frac{\$4.00}{\$66.67} + 0.06 = 0.12, \text{ or } 12\%$$

This result makes sense. Once the news about the contract is reflected in the stock price, the expected rate of return will be consistent with the risk of the stock. Because the risk of the stock has not changed, neither should the expected rate of return.

CONCEPT CHECK 2

a. IBX's stock dividend at the end of this year is expected to be \$2.15, and it is expected to grow at 11.2% per year forever. If the required rate of return on IBX stock is 15.2% per year, what is its intrinsic value?

b. If IBX's current market price is equal to this intrinsic value, what is next year's expected price?

c. If an investor were to buy IBX stock now and sell it after receiving the \$2.15 dividend a year from now, what is the expected capital gain (i.e., price appreciation) in percentage terms? What is the dividend yield, and what would be the holding-period return?

Convergence of Price to Intrinsic Value

Now suppose that the current market price of ABC stock is only \$48 per share and, therefore, that the stock now is undervalued by \$2 per share. In this case the expected rate of price appreciation depends on an additional assumption about whether the discrepancy between the intrinsic value and the market price will disappear, and if so, when.

One fairly common assumption is that the discrepancy will never disappear and that the market price will trend upward at rate g forever. This implies that the discrepancy between intrinsic value and market price also will grow at the same rate. In our example:

Now	Next Year
$V_0 = \$50$	$V_1 = \$50 \times 1.04 = \$52$
$P_0 = \$48$	$P_1 = \$48 \times 1.04 = \$49.92$
$V_0 - P_0 = \$2$	$V_1 - P_1 = \$2 \times 1.04 = \$2.08$

Under this assumption the expected HPR will exceed the required rate, because the dividend yield is higher than it would be if P_0 were equal to V_0. In our example the dividend yield would be 8.33% instead of 8%, so that the expected HPR would be 12.33% rather than 12%:

$$E(r) = \frac{D_1}{P_0} + g = \frac{\$4}{\$48} + .04 = .0833 + .04 = .1233$$

An investor who identifies this undervalued stock can get an expected dividend that exceeds the required yield by 33 basis points. This excess return is earned *each year,* and the market price never catches up to intrinsic value.

An alternative assumption is that the gap between market price and intrinsic value will disappear by the end of the year. In that case we would have $P_1 = V_1 = \$52$, and

$$E(r) = \frac{D_1}{P_0} + \frac{P_1 - P_0}{P_0} = \frac{4}{48} + \frac{52 - 48}{48} = .0833 + .0833 = .1667$$

The assumption of complete catch-up to intrinsic value produces a much larger 1-year HPR. In future years, however, the stock is expected to generate only fair rates of return.

Many stock analysts assume that a stock's price will approach its intrinsic value gradually over time—for example, over a 5-year period. This puts their expected 1-year HPR somewhere between the bounds of 12.33% and 16.67%.

Stock Prices and Investment Opportunities

Consider two companies, Cash Cow, Inc., and Growth Prospects, each with expected earnings in the coming year of $5 per share. Both companies could in principle pay out all of these earnings as dividends, maintaining a perpetual dividend flow of $5 per share. If the market capitalization rate were k = 12.5%, both companies would then be valued at D_1/k = $5/.125 = $40 per share. Neither firm would grow in value, because with all earnings paid out as dividends, and no earnings reinvested in the firm, both companies' capital stock and earnings capacity would remain unchanged over time; earnings[5] and dividends would not grow.

Now suppose one of the firms, Growth Prospects, engages in projects that generate a return on investment of 15%, which is greater than the required rate of return, k = 12.5%. It would be foolish for such a company to pay out all of its earnings as dividends. If Growth Prospects retains or plows back some of its earnings into its profitable projects, it can earn a 15% rate of return for its shareholders, whereas if it pays out all earnings as dividends, it forgoes the projects, leaving shareholders to invest the dividends in other opportunities at a fair market rate of only 12.5%. Suppose, therefore, that Growth Prospects chooses a lower **dividend payout ratio** (the fraction of earnings paid out as dividends), reducing payout from 100% to 40%, maintaining a **plowback ratio** (the fraction of earnings reinvested in the firm) at 60%. The plowback ratio is also referred to as the **earnings retention ratio.**

The dividend of the company, therefore, will be $2 (40% of $5 earnings) instead of $5. Will share price fall? No—it will rise! Although dividends initially fall under the earnings reinvestment policy, subsequent growth in the assets of the firm because of reinvested profits will generate growth in future dividends, which will be reflected in today's share price.

Figure 18.1 illustrates the dividend streams generated by Growth Prospects under two dividend policies. A low-reinvestment-rate plan allows the firm to pay higher initial dividends, but results in a lower dividend growth rate. Eventually, a high-reinvestment-rate plan will provide higher dividends. If the dividend growth generated by the reinvested earnings is high enough, the stock will be worth more under the high-investment strategy.

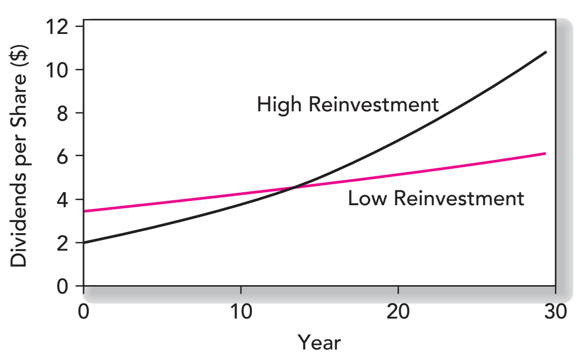

FIGURE 18.1 Dividend growth for two earnings reinvestment policies

How much growth will be generated? Suppose Growth Prospects starts with plant and equipment of $100 million and is all equity financed. With a return on investment or equity (ROE) of 15%, total earnings are ROE × $100 million = .15 × $100 million = $15 million. There are 3 million shares of stock outstanding, so earnings per share are $5, as posited above. If 60% of the $15 million in this year's earnings is reinvested, then the value of the firm's assets will increase by .60 × $15 million = $9 million, or by 9%. The percentage increase in assets is the rate at which income was generated (ROE) times the plowback ratio (the fraction of earnings reinvested in the firm), which we will denote as b.

[5]Actually, we are referring here to earnings net of the funds necessary to maintain the productivity of the firm's capital, that is, earnings net of "economic depreciation." In other words, the earnings figure should be interpreted as the maximum amount of money the firm could pay out each year in perpetuity without depleting its productive capacity. For this reason, the net earnings number may be quite different from the accounting earnings figure that the firm reports in its financial statements. We explore this further in the next chapter.

Now endowed with 9% more assets, the company earns 9% more income, and pays out 9% higher dividends. The growth rate of the dividends, therefore, is[6]

$$g = \text{ROE} \times b = .15 \times .60 = .09$$

If the stock price equals its intrinsic value, it should sell at

$$P_0 = \frac{D_1}{k - g} = \frac{\$2}{.125 - .09} = \$57.14$$

When Growth Prospects pursued a no-growth policy and paid out all earnings as dividends, the stock price was only $40. Therefore, you can think of $40 as the value per share of the assets the company already has in place.

When Growth Prospects decided to reduce current dividends and reinvest some of its earnings in new investments, its stock price increased. The increase in the stock price reflects the fact that the planned investments provide an expected rate of return greater than the required rate. In other words, the investment opportunities have positive net present value. The value of the firm rises by the NPV of these investment opportunities. This net present value is also called the **present value of growth opportunities,** or **PVGO.**

Therefore, we can think of the value of the firm as the sum of the value of assets already in place, or the no-growth value of the firm, plus the net present value of the future investments the firm will make, which is the PVGO. For Growth Prospects, PVGO = $17.14 per share:

$$\begin{aligned} \text{Price} &= \text{No-growth value per share} + \text{PVGO} \\ P_0 &= \frac{E_1}{k} + \text{PVGO} \qquad \textbf{(18.6)} \\ 57.14 &= 40 + 17.14 \end{aligned}$$

We know that in reality, dividend cuts almost always are accompanied by steep drops in stock prices. Does this contradict our analysis? Not necessarily: Dividend cuts are usually taken as bad news about the future prospects of the firm, and it is the *new information* about the firm—not the reduced dividend yield per se—that is responsible for the stock price decline.

In one well-known case, Florida Power & Light announced a cut in its dividend, not because of financial distress but because it wanted to better position itself for a period of deregulation. At first, the stock market did not believe this rationale—the stock price dropped 14% on the day of the announcement. But within a month, the market became convinced that the firm had in fact made a strategic decision that would improve growth prospects, and the share price actually rose *above* its preannouncement value. Even including the initial price drop, the share price outperformed both the S&P 500 and the S&P utility index in the year following the dividend cut.

It is important to recognize that growth per se is not what investors desire. Growth enhances company value only if it is achieved by investment in projects with attractive profit opportunities (i.e., with ROE > k). To see why, let's now consider Growth Prospects's unfortunate sister company, Cash Cow, Inc. Cash Cow's ROE is only 12.5%, just equal to the required rate of return, k. The net present value of its investment opportunities

[6]We can derive this relationship more generally by noting that with a fixed ROE, earnings (which equal ROE × book value) will grow at the same rate as the book value of the firm. Abstracting from issuance of new shares of stock, we find the growth rate of book value equals reinvested earnings/book value. Therefore,

$$g = \frac{\text{Reinvested earnings}}{\text{Book value}} = \frac{\text{Reinvested earnings}}{\text{Total earnings}} \times \frac{\text{Total earnings}}{\text{Book value}} = b \times \text{ROE}$$

is zero. We've seen that following a zero-growth strategy with $b = 0$ and $g = 0$, the value of Cash Cow will be $E_1/k = \$5/.125 = \$40$ per share. Now suppose Cash Cow chooses a plowback ratio of $b = .60$, the same as Growth Prospects's plowback. Then g would increase to

$$g = \text{ROE} \times b = .125 \times .60 = .075$$

but the stock price is still

$$P_0 = \frac{D_1}{k - g} = \frac{\$2}{.125 - .075} = \$40$$

—no different from the no-growth strategy.

In the case of Cash Cow, the dividend reduction used to free funds for reinvestment in the firm generates only enough growth to maintain the stock price at the current level. This is as it should be: If the firm's projects yield only what investors can earn on their own, shareholders cannot be made better off by a high-reinvestment-rate policy. This demonstrates that "growth" is not the same as growth opportunities. To justify reinvestment, the firm must engage in projects with better prospective returns than those shareholders can find elsewhere. Notice also that the PVGO of Cash Cow is zero: $\text{PVGO} = P_0 - E_1/k = 40 - 40 = 0$. With $\text{ROE} = k$, there is no advantage to plowing funds back into the firm; this shows up as PVGO of zero. In fact, this is why firms with considerable cash flow but limited investment prospects are called "cash cows." The cash these firms generate is best taken out of, or "milked from," the firm.

EXAMPLE 18.4 Growth Opportunities

Takeover Target is run by entrenched management that insists on reinvesting 60% of its earnings in projects that provide an ROE of 10%, despite the fact that the firm's capitalization rate is $k = 15\%$. The firm's year-end dividend will be \$2 per share, paid out of earnings of \$5 per share. At what price will the stock sell? What is the present value of growth opportunities? Why would such a firm be a takeover target for another firm?

Given current management's investment policy, the dividend growth rate will be

$$g = \text{ROE} \times b = 10\% \times .60 = 6\%$$

and the stock price should be

$$P_0 = \frac{\$2}{.15 - .06} = \$22.22$$

The present value of growth opportunities is

$$\begin{aligned} \text{PVGO} &= \text{Price per share} - \text{No-growth value per share} \\ &= \$22.22 - E_1/k = \$22.22 - \$5/.15 = -\$11.11 \end{aligned}$$

PVGO is *negative.* This is because the net present value of the firm's projects is negative: The rate of return on those assets is less than the opportunity cost of capital.

Such a firm would be subject to takeover, because another firm could buy the firm for the market price of \$22.22 per share and increase the value of the firm by changing its investment policy. For example, if the new management simply paid out all earnings as dividends, the value of the firm would increase to its no-growth value, $E_1/k = \$5/.15 = \$33.33$.

CONCEPT CHECK 3

a. Calculate the price of a firm with a plowback ratio of .60 if its ROE is 20%. Current earnings, E_1, will be $5 per share, and $k = 12.5\%$.

b. What if ROE is 10%, which is less than the market capitalization rate? Compare the firm's price in this instance to that of a firm with the same ROE and E_1, but a plowback ratio of $b = 0$.

Life Cycles and Multistage Growth Models

As useful as the constant-growth DDM formula is, you need to remember that it is based on a simplifying assumption, namely, that the dividend growth rate will be constant forever. In fact, firms typically pass through life cycles with very different dividend profiles in different phases. In early years, there are ample opportunities for profitable reinvestment in the company. Payout ratios are low, and growth is correspondingly rapid. In later years, the firm matures, production capacity is sufficient to meet market demand, competitors enter the market, and attractive opportunities for reinvestment may become harder to find. In this mature phase, the firm may choose to increase the dividend payout ratio, rather than retain earnings. The dividend level increases, but thereafter it grows at a slower rate because the company has fewer growth opportunities.

Table 18.2 illustrates this pattern. It gives Value Line's forecasts of return on assets, dividend payout ratio, and 3-year growth rate in earnings per share for a sample of the

TABLE 18.2 Financial ratios in two industries

	Return on Assets (%)	Payout Ratio (%)	Growth Rate 2008–2011
Computer Software			
Adobe Systems	13.5%	0.0%	14.0%
Cognizant	18.0	0.0	20.8
Compuware	14.0	0.0	11.5
Intuit	15.0	0.0	9.5
Microsoft	40.0	30.0	15.9
Oracle	30.0	0.0	17.2
Red Hat	9.5	0.0	17.8
Parametric Tech	15.0	0.0	13.7
SAP	24.0	27.0	12.7
Median	15.0%	0.0%	14.0%
Electric Utilities			
Central Hudson G&E	5.5%	72.0%	4.5%
Central Vermont	5.5	57.0	2.1
Consolidated Edison	6.0	70.0	1.0
Energy East	6.0	76.0	7.7
Northeast Utilities	5.5	49.0	6.8
Nstar	9.5	60.0	10.1
Pennsylvania Power	13.0	50.0	18.6
Public Services Enter.	9.5	43.0	2.2
United Illuminating	6.5	80.0	3.3
Median	6.0%	60.0%	4.5%

Source: *Value Line Investment Survey,* 2007. Reprinted with permission of Value Line Investment Survey.

firms included in the computer software industry versus those of East Coast electric utilities. (We compare return on assets rather than return on equity because the latter is affected by leverage, which tends to be far greater in the electric utility industry than in the software industry. Return on assets measures operating income per dollar of total assets, regardless of whether the source of the capital supplied is debt or equity. We will return to this issue in the next chapter.)

By and large, the software firms have attractive investment opportunities. The median return on assets of these firms is forecast to be 15%, and the firms have responded with high plowback ratios. Most of these firms pay no dividends at all. The high return on assets and high plowback result in rapid growth. The median growth rate of earnings per share in this group is projected at 14%.

In contrast, the electric utilities are more representative of mature firms. Their median return on assets is lower, 6%; dividend payout is higher, 60%; and median growth is lower, 4.5%.

We conclude that the higher payouts of the electric utilities reflect their more limited opportunities to reinvest earnings at attractive rates of return. Consistent with this view, Microsoft's announcement in 2004 that it would sharply increase its dividend and initiate multi-billion-dollar stock buybacks was widely seen as an indication that the firm was maturing into a lower-growth stage. It was generating far more cash than it had the opportunity to invest attractively, and so was paying out that cash to its shareholders.

To value companies with temporarily high growth, analysts use a multistage version of the dividend discount model. Dividends in the early high-growth period are forecast and their combined present value is calculated. Then, once the firm is projected to settle down to a steady-growth phase, the constant-growth DDM is applied to value the remaining stream of dividends.

We can illustrate this with a real-life example. Figure 18.2 is a Value Line Investment Survey report on Honda Motor Co. Some of the relevant information for 2007 is highlighted.

Honda's beta appears at the circled A, its recent stock price at the B, the per-share dividend payments at the C, the ROE (referred to as "return on shareholder equity") at the D, and the dividend payout ratio (referred to as "all dividends to net profits") at the E.[7] The rows ending at C, D, and E are historical time series. The boldfaced, italicized entries under 2008 are estimates for that year. Similarly, the entries in the far right column (labeled 10–12) are forecasts for some time between 2010 and 2012, which we will take to be 2011.

Value Line projects rapid growth in the near term, with dividends rising from $.77 in 2008 to $1.10 in 2011. This rapid growth rate cannot be sustained indefinitely. We can obtain dividend inputs for this initial period by using the explicit forecasts for 2008 and 2011 and linear interpolation for the years between:

2008	$.77	2010	$ .99
2009	$.88	2011	$1.10

Now let us assume the dividend growth rate levels off in 2011. What is a good guess for that steady-state growth rate? Value Line forecasts a dividend payout ratio of .26 and an ROE of 12.5%, implying long-term growth will be

$$g = \text{ROE} \times b = 12.5\% \times (1 - .26) = 9.25\%$$

[7]Because Honda is a Japanese firm, Americans would hold its shares via ADRs, or American depository receipts. ADRs are not shares of the firm, but are *claims* to shares of the underlying foreign stock that are then traded in U.S. security markets. Value Line notes that each Honda ADR is a claim on one common share, but in other cases, each ADR may represent a claim to multiple shares or even fractional shares.

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HONDA MOTOR (ADR) NYSE-HMC | RECENT PRICE 32.10 | P/E RATIO 10.9 (Trailing: 11.4 Median: 13.0) | RELATIVE P/E RATIO 0.62 | DIV'D YLD 2.2% | VALUE LINE

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TIMELINESS 4 Lowered 4/6/07
SAFETY 2 Raised 3/2/07
TECHNICAL 4 Lowered 8/10/07
BETA .90 (1.00 = Market)

2010-12 PROJECTIONS	Price	Gain	Ann'l Total Return
High	70	(+120%)	23%
Low	55	(+70%)	16%

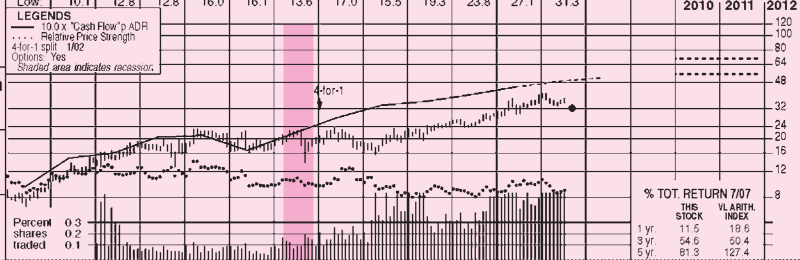
Insider Decisions

NOT REPORTED

U.S. Institutional Decisions

	3Q2006	4Q2006	1Q2007
to Buy	105	124	119
to Sell	66	68	89
Hld's(000)	89617	97208	96461

	1996	1997	1998	1999	2000	2001	2002	2003	2004	2005	2006	2007
High:	15.3	19.1	20.0	23.5	22.7	23.1	23.8	23.6	26.1	29.7	39.9	40.8
Low:	10.1	12.8	12.8	16.0	16.1	13.6	17.0	15.5	19.3	23.8	27.1	31.3



% TOT. RETURN 7/07

	THIS STOCK	VL ARITH. INDEX
1 yr.	11.5	18.6
3 yr.	54.6	50.4
5 yr.	81.3	127.4

1991	1992	1993	1994	1995	1996	1997	1998	1999	2000	2001	2002	2003	2004	2005	2006	2007	2008	VALUE LINE, INC.	10-12
133	116	103	89.00	107	124	133	121	106	124	133	120	106	107	118	118	***117***	***110***	Translation Rate (Yen/$)[A]	***106***
16.97	18.29	19.26	22.87	20.39	21.89	23.15	26.52	29.52	26.77	28.41	34.09	40.49	43.55	45.82	51.53	***57.75***	***59.60***	Sales per ADR [B]	***66.05***
.99	.94	.83	1.00	.94	1.48	1.60	2.05	2.10	1.67	2.15	2.77	3.36	3.58	3.97	4.48	***4.90***	***5.20***	"Cash Flow" per ADR	***6.35***
.25	.17	.12	.28	.34	.90	1.01	1.30	1.27	.96	1.40	1.83	2.30	2.42	2.76	2.75	***2.95***	***3.20***	Earnings per ADR [BC]	***4.20***
.05	.06	.07	.08	.07	.06	.07	.09	.10	.09	.10	.13	.15	.23	.33	.52	***.70***	***.77***	Gross Div'ds Decl'd/ADR [D]	***1.10***
.92	.74	.61	.74	.72	.90	1.19	1.01	1.08	1.18	1.17	1.36	1.43	1.88	2.13	2.78	***3.55***	***3.65***	Cap'l Spending per ADR	***4.05***
4.26	4.59	4.82	5.87	5.49	5.74	6.20	7.51	9.34	9.24	9.93	11.25	14.26	16.56	19.08	20.83	***22.95***	***25.35***	Book Value per ADR	***34.10***
1945.6	1947.3	1947.4	1948.4	1948.6	1948.7	1948.8	1948.8	1948.8	1948.8	1948.8	1948.8	1907.3	1849.7	1840.8	1824.7	***1810.0***	***1800.0***	Equiv ADRs Outst'g [E]	***1775.0***
22.0	32.1	58.7	30.3	26.5	14.3	16.5	13.6	16.0	19.3	14.3	10.8	8.8	10.0	9.8	12.7	***Bold figures are***		Avg Ann'l P/E Ratio	***15.0***
1.41	1.95	3.47	1.99	1.77	.90	.95	.71	.91	1.25	.73	.59	.50	.53	.52	.69	***Value Line***		Relative P/E Ratio	***1.00***
1.0%	1.1%	1.0%	.9%	.7%	.4%	.4%	.5%	.5%	.5%	.5%	.7%	.8%	1.0%	1.2%	1.5%	***estimates***		Avg Ann'l Div'd Yield	***1.8%***
						45111	51688	57536	52170	55357	66429	77232	80549	84345	94018	***104550***	***107265***	Sales ($mill) [B]	***117235***
						10.3%	11.7%	9.8%	8.9%	11.3%	11.4%	10.0%	9.9%	10.0%	11.0%	***10.5%***	***11.0%***	Operating Margin	***12.0%***
						1152.9	1473.8	1624.0	1375.0	1466.0	1840.6	2019.5	2102.2	2232.0	3150.0	***3500***	***3575***	Depreciation ($mill)	***3800***
						1959.6	2530.4	2475.6	1874.4	2727.0	3555.5	4393.4	4527.4	5082.0	5023.0	***5355***	***5775***	Net Profit ($mill)	***7485***
						45.4%	44.1%	41.0%	46.3%	41.9%	40.2%	39.4%	40.6%	38.9%	35.8%	***38.5%***	***38.5%***	Income Tax Rate	***38.5%***
						4.3%	4.9%	4.3%	3.6%	4.9%	5.4%	5.7%	5.6%	6.0%	5.3%	***5.1%***	***5.4%***	Net Profit Margin	***6.4%***
						636.1	868.2	2390.6	d1310	d162.0	1416.4	3013.0	2422.7	5418.0	8182.0	***8740***	***8755***	Working Cap'l ($mill)	***8800***
						5095.9	5583.4	5420.4	2971.5	5388.0	9501.5	13195	14522	15996	16160	***16000***	***15000***	Long-Term Debt ($mill)	***10000***
						12090	14632	18211	18001	19353	21914	27197	30629	35122	38012	***41585***	***45625***	Shr. Equity ($mill)	***60545***
						11.8%	12.9%	10.8%	9.1%	11.1%	11.4%	10.9%	10.1%	10.0%	9.8%	***9.5%***	***9.5%***	Return on Total Capital	***10.5%***
						16.2%	17.3%	13.6%	10.4%	14.1%	16.2%	16.2%	14.8%	14.5%	13.2%	***13.0%***	***12.5%***	Return on Shr. Equity	***12.5%***
						15.2%	16.1%	12.5%	9.4%	13.1%	15.1%	15.0%	13.3%	12.7%	10.7%	***10.0%***	***9.5%***	Retained to Com Eq	***9.0%***
						6%	7%	8%	10%	7%	7%	7%	10%	12%	19%	***24%***	***24%***	All Div'ds to Net Prof	***26%***

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CAPITAL STRUCTURE as of 6/30/07
Total Debt $34746 mill. Due in 5 Yrs $24580 mill.
LT Debt $16620 mill. LT Interest $1330 mill.
(30% of Cap'l)

Pension Assets-3/07 $10.5 bill. Oblig. $13.9 bill.
Pfd Stock None

Common Stock 1821.6 mill. shares
(equivalent to 1821.6 ADRs)

MARKET CAP: $58.5 billion (Large Cap)

CURRENT POSITION ($MILL.)	2005	2006	6/30/07
Cash Assets	6362	8018	6664
Receivables	8199	8950	7905
Inventory (FIFO)	8822	10033	10271
Other	15996	17539	17466
Current Assets	39379	44540	42306
Accts Payable	9363	9892	8730
Debt Due	11772	17310	18126
Other	12826	9156	8677
Current Liab.	33961	36358	35533

ANNUAL RATES of change (per ADR)	Past 10 Yrs.	Past 5 Yrs.	Est'd '04-'06 to '10-'12
Sales	8.0%	10.5%	*6.0%*
"Cash Flow"	13.5%	15.0%	*8.0%*
Earnings	18.0%	17.0%	*8.0%*
Dividends	18.5%	30.0%	*20.5%*
Book Value	12.5%	14.5%	*10.5%*

Fiscal Year Begins	QUARTERLY SALES ($ mill.)[B] Jun.30	Sep.30	Dec.31	Mar.31	Full Fiscal Year
2004	19120	18875	20680	21874	80549
2005	19278	19900	21044	24123	84345
2006	22045	22310	23478	26185	94018
2007	23780	***25250***	***26205***	***29315***	***104550***
2008	***24895***	***25565***	***26885***	***29920***	***107265***

Fiscal Year Begins	EARNINGS PER ADR[BC] Jun.30	Sep.30	Dec.31	Mar.31	Full Fiscal Year
2004	.57	.62	.76	.47	2.42
2005	.51	.62	.61	1.02	2.76
2006	.67	.59	.67	.82	2.75
2007	.74	***.65***	***.71***	***.85***	***2.95***
2008	***.79***	***.70***	***.78***	***.93***	***3.20***

Cal-endar	GROSS QUARTERLY DIV'DS PAID[D] Mar.31	Jun.30	Sep.30	Dec.31	Full Year
2003	--	--	.068	.086	.15
2004	--	--	.099	.126	.23
2005	--	--	.163	.167	.33
2006	--	--	.258	.258	.52
2007	--	.162			

BUSINESS: Honda Motor Company, Ltd. is Japan's third-largest auto maker & the world's biggest motorcycle producer. Builds cars in Japan, United States, Canada, United Kingdom; cycles in 32 countries. Fiscal 2006 unit sales: autos, 3.7 million; cycles, 10.4 mill.; power products, 6.4 mill. Sales by region: United States & Canada, 43%; Japan, 33%; Europe, 9%; other, 15%. R&D, 5.0% of sales. 2006 depreciation rate, 6.2%. Has about 144,785 employees. Top 10 Japanese financial institutions control 33.3% of common (6/07 proxy). President & CEO: Takeo Fukui. Inc.: Japan. Address: 1-1, 2-chome, Minami-Aoyama, Minato-ku. Tokyo 107, Japan. U.S. address: 540 Madison Ave. 32nd Floor, New York, NY 10022. Telephone: 212-355-9191. Internet: www.honda.com.

Honda Motor continues to persevere amid a difficult automotive market. North American unit sales remain weak industrywide, with elevated gasoline prices keeping many drivers off the lots. Although HMC's motorcycle and power product volumes disappointed in the first quarter, overall auto sales held up relatively well, until July that is, as soft demand for *Acura* models led to the U.S. division's 5% year-over-year sales decline. Still, a fuel-efficient reputation should keep sales headed in the right direction, with earnings growth likely to return this year. Any depreciation of the yen, which we anticipate in the second half of the year, should also help inflate results.

This momentum should continue into next year. The company has aggressively increased its production expansion targets. It now aims for 1.62 million units to be manufactured in North America by the fall of 2008, up from its current 1.4 million. The latest plant additions will likely take on manufacturing of the popular new *Accord* and *CR-V* crossover models.

The short-term appeal here is limited . . . It will be tough for auto makers to thrive, given the sluggish state of the North American market. Honda appears poised for growth this year, but investors seem somewhat wary these days against this dismal backdrop. Indeed, the untimely issue was on an ascent until a weak July scared investors away. In the end, the stock's price is a touch above where it was at our June report.

. . . but these shares may interest long-term investors. Despite the difficult U.S. market, HMC continues to gain ground on its peers. In turn, double-digit annual earnings advances are likely out to 2010-2012. The stock offers substantial appreciation potential over that span at its recent price. Its total return prospects should also improve as the company remains focused on increasing its dividend payout to a level closer to that of its auto brethren.

Jason A. Smith *August 31, 2007*

Sales (and Operating Margins*) by Business Line

	2005	2006	2007	2008
Automobiles	68143(7.9%)	75378(6.7%)	***83010(6.5%)***	***86030(8.0%)***
Motorcycles	10435(9.3%)	11623(7.3%)	***13135(8.0%)***	***13460(8.5%)***
Finl Svcs & Other	5767(18.2%)	7016(17.8%)	***8405(15.5%)***	***7775(15.5%)***
Company Total	84345(8.8%)	94017(7.7%)	***104550(8.0%)***	***107265(9.5%)***

*After depreciation and corporate expenses

(A) At fiscal yearend. (B) FY ends Mar. 31 of following year. Most recent qtl'y figs. based on estimate avg. exch. rate for the full year. (C) Dil. egs., based on U.S. accounting. Primary through 1996. Excl. nonrecur. gains: '94, 29¢; '96, 8¢. Next egs. rpt. due late Oct. May not sum due to rounding. (D) Before 15% Japanese withholding tax. Dividends historically paid in early July and December. (E) In mill. of American Depositary Receipts, adj. for split. Each common share represents 1 ADR.

Company's Financial Strength	B++
Stock's Price Stability	85
Price Growth Persistence	85
Earnings Predictability	80

FIGURE 18.2 Value Line Investment Survey report on Honda Motor Co.

Source: Jason A. Smith, Value Line Investment Survey, August 31, 2007. Reprinted with permission of Value Line Investment Survey © 2007 Value Line Publishing, Inc. All rights reserved.

Our estimate of Honda's intrinsic value using an investment horizon of 2011 is therefore obtained from Equation 18.2, which we restate here:

$$V_{2007} = \frac{D_{2008}}{1+k} + \frac{D_{2009}}{(1+k)^2} + \frac{D_{2010}}{(1+k)^3} + \frac{D_{2011} + P_{2011}}{(1+k)^4}$$
$$= \frac{.77}{1+k} + \frac{.88}{(1+k)^2} + \frac{.99}{(1+k)^3} + \frac{1.10 + P_{2011}}{(1+k)^4}$$

Here, P_{2011} represents the forecast price at which we can sell our shares at the end of 2011, when dividends are assumed to enter their constant-growth phase. That price, according to the constant-growth DDM, should be

$$P_{2011} = \frac{D_{2012}}{k-g} = \frac{D_{2011}(1+g)}{k-g} = \frac{1.10 \times 1.0925}{k - .0925}$$

The only variable remaining to be determined to calculate intrinsic value is the market capitalization rate, k.

One way to obtain k is from the CAPM. Observe from the Value Line report that Honda's beta is .90. The risk-free rate on Treasury bonds at the end of 2007 was about 4.5%. Suppose that the market risk premium were forecast at 8%, roughly in line with its historical average. This would imply that the forecast for the market return was

$$\text{Risk-free rate} + \text{Market risk premium} = 4.5\% + 8\% = 12.5\%$$

Therefore, we can solve for the market capitalization rate as

$$k = r_f + \beta[E(r_M) - r_f] = 4.5\% + .9(12.5 - 4.5) = 11.7\%$$

Our forecast for the stock price in 2011 is thus

$$P_{2011} = \frac{\$1.10 \times 1.0925}{.117 - .0925} = \$49.05$$

And today's estimate of intrinsic value is

$$V_{2007} = \frac{.77}{1.117} + \frac{.88}{(1.117)^2} + \frac{.99}{(1.117)^3} + \frac{1.10 + 49.05}{(1.117)^4} = \$34.32$$

We know from the Value Line report that Honda's actual price was \$32.10 (at the circled B). Our intrinsic value analysis indicates that the stock was a bit underpriced. Should we increase our holdings?

Perhaps. But before betting the farm, stop to consider how firm our estimate is. We've had to guess at dividends in the near future, the ultimate growth rate of those dividends, and the appropriate discount rate. Moreover, we've assumed Honda will follow a relatively simple two-stage growth process. In practice, the growth of dividends can follow more complicated patterns. Even small errors in these approximations could upset a conclusion.

For example, suppose that we have overestimated Honda's growth prospects and that the actual ROE in the post-2011 period will be 12% rather than 12.5%, a seemingly minor change. Using the lower return on equity in the dividend discount model would result in an intrinsic value in 2007 of \$30.09, which is considerably *less* than the stock price. Our conclusion regarding intrinsic value versus price is reversed.

The exercise also highlights the importance of performing sensitivity analysis when you attempt to value stocks. Your estimates of stock values are no better than your assumptions. Sensitivity analysis will highlight the inputs that need to be most carefully examined. For example, even small changes in the estimated ROE for the post-2011 period can result in big changes in intrinsic value. Similarly, small changes in the assumed capitalization rate would change intrinsic value substantially. On the other hand, reasonable changes in the dividends forecast between 2008 and 2011 would have a small impact on intrinsic value.

CONCEPT CHECK 4

Confirm that the intrinsic value of Honda using ROE = 12% is $30.09. (*Hint:* First calculate the stock price in 2011. Then calculate the present value of all interim dividends plus the present value of the 2011 sales price.)

Multistage Growth Models

The two-stage growth model that we just considered for Honda is a good start toward realism, but clearly we could do even better if our valuation model allowed for more flexible patterns of growth. Multistage growth models allow dividends per share to grow at several different rates as the firm matures. Many analysts use three-stage growth models. They may assume an initial period of high dividend growth (or instead make year-by-year forecasts of dividends for the short term), a final period of sustainable growth, and a transition period between, during which dividend growth rates taper off from the initial rapid rate to the ultimate sustainable rate. These models are conceptually no harder to work with than a two-stage model, but they require many more calculations and can be tedious to do by hand. It is easy, however, to build an Excel spreadsheet for such a model.

Spreadsheet 18.1 is an example of such a model. Column B contains the inputs we have used so far for Honda. Column E contains dividend forecasts. In cells E2 through E5 we present the Value Line estimates for the next 4 years. Dividend growth in this period is rapid, about 12.62% annually. Rather than assume a sudden transition to constant dividend growth starting in 2011, we assume instead that the dividend growth rate in 2011 will be 12.62%

	A	B	C	D	E	F	G	H	I
1	**Inputs**			Year	Dividend	Div growth	Term value	Investor CF	
2	beta	0.9		2008	0.77			0.77	
3	mkt_prem	0.08		2009	0.88			0.88	
4	rf	0.045		2010	0.99			0.99	
5	k_equity	0.117		2011	1.10			1.10	
6	plowback	0.74		2012	1.24	0.1262		1.24	
7	roe	0.125		2013	1.39	0.1229		1.39	
8	term_gwth	0.0925		2014	1.56	0.1195		1.56	
9				2015	1.74	0.1161		1.74	
10				2016	1.93	0.1127		1.93	
11				2017	2.15	0.1094		2.15	
12	Value line			2018	2.37	0.1060		2.37	
13	forecasts of			2019	2.62	0.1026		2.62	
14	annual dividends			2020	2.88	0.0992		2.88	
15				2021	3.15	0.0959		3.15	
16				2022	3.44	0.0925		3.44	
17	Transitional period			2023	3.76	0.0925	167.77	171.53	
18	with slowing dividend								
19	growth							39.71	= PV of CF
20		Beginning of constant-			E17*(1+F17)/(B5–F17)				
21		growth period						NPV(B5,H2:H17)	

SPREADSHEET 18.1

A three-stage growth model for Honda Motor Co.

eXcel
Please visit us at
www.mhhe.com/bkm

and that it will decline steadily through 2023, finally reaching the constant terminal growth rate of 9.25% (see column F). Each dividend in the transition period is the previous year's dividend times that year's growth rate. Terminal value once the firm enters a constant-growth stage (cell G17) is computed from the constant-growth DDM. Finally, investor cash flow in each period (column H) equals dividends in each year plus the terminal value in 2023. The present value of these cash flows is computed in cell H19 as $39.71, about 16% more than the value we found from the two-stage model. We obtain a greater intrinsic value in this case because we assume that dividend growth only gradually declines to its steady-state value.

18.4 PRICE–EARNINGS RATIO

The Price–Earnings Ratio and Growth Opportunities

Much of the real-world discussion of stock market valuation concentrates on the firm's **price–earnings multiple,** the ratio of price per share to earnings per share, commonly called the P/E ratio. Our discussion of growth opportunities shows why stock market analysts focus on the P/E ratio. Both companies considered, Cash Cow and Growth Prospects, had earnings per share (EPS) of $5, but Growth Prospects reinvested 60% of earnings in prospects with an ROE of 15%, whereas Cash Cow paid out all earnings as dividends. Cash Cow had a price of $40, giving it a P/E multiple of 40/5 = 8.0, whereas Growth Prospects sold for $57.14, giving it a multiple of 57.14/5 = 11.4. This observation suggests the P/E ratio might serve as a useful indicator of expectations of growth opportunities.

We can see how growth opportunities are reflected in P/E ratios by rearranging Equation 18.6 to

$$\frac{P_0}{E_1} = \frac{1}{k}\left(1 + \frac{\text{PVGO}}{E/k}\right) \tag{18.7}$$

When PVGO = 0, Equation 18.7 shows that $P_0 = E_1/k$. The stock is valued like a nongrowing perpetuity of E_1, and the P/E ratio is just $1/k$. However, as PVGO becomes an increasingly dominant contributor to price, the P/E ratio can rise dramatically.

The ratio of PVGO to E/k has a simple interpretation. It is the ratio of the component of firm value due to growth opportunities to the component of value due to assets already in place (i.e., the no-growth value of the firm, E/k). When future growth opportunities dominate the estimate of total value, the firm will command a high price relative to current earnings. Thus a high P/E multiple indicates that a firm enjoys ample growth opportunities.

Let's see if P/E multiples do vary with growth prospects. Between 1988 and 2007, for example, Limited Brands' P/E ratio averaged about 18.3 while Consolidated Edison's (an electric utility) average P/E was only about two-thirds of that. These numbers do not necessarily imply that Limited was overpriced compared to Con Ed. If investors believed Limited would grow faster than Con Ed, the higher price per dollar of earnings would be justified. That is, an investor might well pay a higher price per dollar of *current* earnings if he or she expects that earnings stream to grow more rapidly. In fact, Limited's growth rate has been consistent with its higher P/E multiple. Over this period, its earnings per share grew at 8.5% per year while Con Ed's earnings growth rate was only 1.4%. Figure 18.4 (page 609) shows the EPS history of the two companies.

Clearly, differences in expected growth opportunities are responsible for differences in P/E ratios across firms. The P/E ratio actually is a reflection of the market's optimism concerning a firm's growth prospects. In their use of a P/E ratio, analysts must decide whether they are more or less optimistic than the market. If they are more optimistic, they will recommend buying the stock.

There is a way to make these insights more precise. Look again at the constant-growth DDM formula, $P_0 = D_1/(k - g)$. Now recall that dividends equal the earnings that are *not* reinvested in the firm: $D_1 = E_1(1 - b)$. Recall also that $g = \text{ROE} \times b$. Hence, substituting for D_1 and g, we find that

$$P_0 = \frac{E_1(1 - b)}{k - \text{ROE} \times b}$$

implying the P/E ratio is

$$\frac{P_0}{E_1} = \frac{1 - b}{k - \text{ROE} \times b} \quad \textbf{(18.8)}$$

It is easy to verify that the P/E ratio increases with ROE. This makes sense, because high-ROE projects give the firm good opportunities for growth.[8] We also can verify that the P/E ratio increases for higher plowback, b, as long as ROE exceeds k. This too makes sense. When a firm has good investment opportunities, the market will reward it with a higher P/E multiple if it exploits those opportunities more aggressively by plowing back more earnings into those opportunities.

Remember, however, that growth is not desirable for its own sake. Examine Table 18.3 where we use Equation 18.8 to compute both growth rates and P/E ratios for different combinations of ROE and b. Although growth always increases with the plowback rate (move across the rows in Table 18.3A), the P/E ratio does not (move across the rows in panel B). In the top row of Table 18.3B, the P/E falls as the plowback rate increases. In the middle row, it is unaffected by plowback. In the third row, it increases.

This pattern has a simple interpretation. When the expected ROE is less than the required return, k, investors prefer that the firm pay out earnings as dividends rather than reinvest earnings in the firm at an inadequate rate of return. That is, for ROE lower than k, the value of the firm falls as plowback increases. Conversely, when ROE exceeds k, the firm offers attractive investment opportunities, so the value of the firm is enhanced as those opportunities are more fully exploited by increasing the plowback rate.

Finally, where ROE just equals k, the firm offers "break-even" investment opportunities with a fair rate of return. In this case, investors are indifferent between reinvestment of earnings in the firm or elsewhere at the market capitalization rate, because the rate of return in either case is 12%. Therefore, the stock price is unaffected by the plowback rate.

TABLE 18.3

Effect of ROE and plowback on growth and the P/E ratio

	Plowback Rate (*b*)			
	0	**.25**	**.50**	**.75**
ROE	**A. Growth rate, g**			
10%	0	2.5%	5.0%	7.5%
12	0	3.0	6.0	9.0
14	0	3.5	7.0	10.5
	B. P/E ratio			
10%	8.33	7.89	7.14	5.56
12	8.33	8.33	8.33	8.33
14	8.33	8.82	10.00	16.67

Assumption: $k = 12\%$ per year.

[8]Note that Equation 18.8 is a simple rearrangement of the DDM formula, with $\text{ROE} \times b = g$. Because that formula requires that $g < k$, Equation 18.8 is valid only when $\text{ROE} \times b < k$.

One way to summarize these relationships is to say the higher the plowback rate, the higher the growth rate, but a higher plowback rate does not necessarily mean a higher P/E ratio. A higher plowback rate increases P/E only if investments undertaken by the firm offer an expected rate of return higher than the market capitalization rate. Otherwise, higher plowback hurts investors because it means more money is sunk into projects with inadequate rates of return.

Notwithstanding these fine points, P/E ratios commonly are taken as proxies for the expected growth in dividends or earnings. In fact, a common Wall Street rule of thumb is that the growth rate ought to be roughly equal to the P/E ratio. In other words, the ratio of P/E to g, often called the *PEG ratio,* should be about 1.0. Peter Lynch, the famous portfolio manager, puts it this way in his book *One Up on Wall Street:*

> The P/E ratio of any company that's fairly priced will equal its growth rate. I'm talking here about growth rate of earnings here. . . . If the P/E ratio of Coca Cola is 15, you'd expect the company to be growing at about 15% per year, etc. But if the P/E ratio is less than the growth rate, you may have found yourself a bargain.

EXAMPLE 18.5 P/E Ratio versus Growth Rate

Let's try Lynch's rule of thumb. Assume that

$$r_f = 8\% \quad \text{(roughly the value when Peter Lynch was writing)}$$

$$r_M - r_f = 8\% \quad \text{(about the historical average market risk premium)}$$

$$b = .4 \quad \text{(a typical value for the plowback ratio in the United States)}$$

Therefore, $r_M = r_f +$ market risk premium $= 8\% + 8\% = 16\%$, and $k = 16\%$ for an average ($\beta = 1$) company. If we also accept as reasonable that ROE $= 16\%$ (the same value as the expected return on the stock), we conclude that

$$g = \text{ROE} \times b = 16\% \times .4 = 6.4\%$$

and

$$\frac{P}{E} = \frac{1 - .4}{.16 - .064} = 6.26$$

Thus, the P/E ratio and g are about equal using these assumptions, consistent with the rule of thumb.

However, note that this rule of thumb, like almost all others, will not work in all circumstances. For example, the value of r_f today is more like 5%, so a comparable forecast of r_M today would be

$$r_f + \text{Market risk premium} = 5\% + 8\% = 13\%$$

If we continue to focus on a firm with $\beta = 1$, and if ROE still is about the same as k, then

$$g = 13\% \times .4 = 5.2\%$$

while

$$\frac{P}{E} = \frac{1 - .4}{.13 - .052} = 7.69$$

The P/E ratio and g now diverge and the PEG ratio is now 1.5. Nevertheless, lower-than-average PEG ratios are still widely seen as signaling potential underpricing.

The importance of growth opportunities is nowhere more evident than in the valuation of Internet firms. Many companies that had yet to turn a profit were valued by the market in the late 1990s at billions of dollars. The perceived value of these companies was *exclusively* as growth opportunities. For example, the online auction firm eBay had 1998 profits of $2.4 million, far less than the $45 million profit earned by the traditional auctioneer Sotheby's; yet eBay's market value was more than 10 times greater: $22 billion versus $1.9 billion. (As it turns out, the market was quite right to value eBay so much more aggressively than Sotheby's. By 2006, eBay's net income was over $1 billion, more than 10 times that of Sotheby's, and still growing.

Of course, when company valuation is determined primarily by growth opportunities, those values can be very sensitive to reassessments of such prospects. When the market became more skeptical of the business prospects of most Internet retailers at the close of the 1990s, that is, as it revised the estimates of growth opportunities downward, their stock prices plummeted.

As perceptions of future prospects wax and wane, share price can swing wildly. Growth prospects are intrinsically difficult to tie down; ultimately, however, those prospects drive the value of the most dynamic firms in the economy.

CONCEPT CHECK 5

ABC stock has an expected ROE of 12% per year, expected earnings per share of $2, and expected dividends of $1.50 per share. Its market capitalization rate is 10% per year.

a. What are its expected growth rate, its price, and its P/E ratio?

b. If the plowback ratio were .4, what would be the expected dividend per share, the growth rate, price, and the P/E ratio?

P/E Ratios and Stock Risk

One important implication of any stock-valuation model is that (holding all else equal) riskier stocks will have lower P/E multiples. We can see this quite easily in the context of the constant-growth model by examining the formula for the P/E ratio (Equation 18.8):

$$\frac{P}{E} = \frac{1 - b}{k - g}$$

Riskier firms will have higher required rates of return, that is, higher values of k. Therefore, the P/E multiple will be lower. This is true even outside the context of the constant-growth model. For *any* expected earnings and dividend stream, the present value of those cash flows will be lower when the stream is perceived to be riskier. Hence the stock price and the ratio of price to earnings will be lower.

Of course, you can find many small, risky, start-up companies with very high P/E multiples. This does not contradict our claim that P/E multiples should fall with risk; instead it is evidence of the market's expectations of high growth rates for those companies. This is why we said that high-risk firms will have lower P/E ratios *holding all else equal.* Given a growth projection, the P/E multiple will be lower when risk is perceived to be higher.

Pitfalls in P/E Analysis

No description of P/E analysis is complete without mentioning some of its pitfalls. First, consider that the denominator in the P/E ratio is accounting earnings, which are influenced by somewhat arbitrary accounting rules such as the use of historical cost in depreciation and inventory valuation. In times of high inflation, historic cost depreciation and inventory

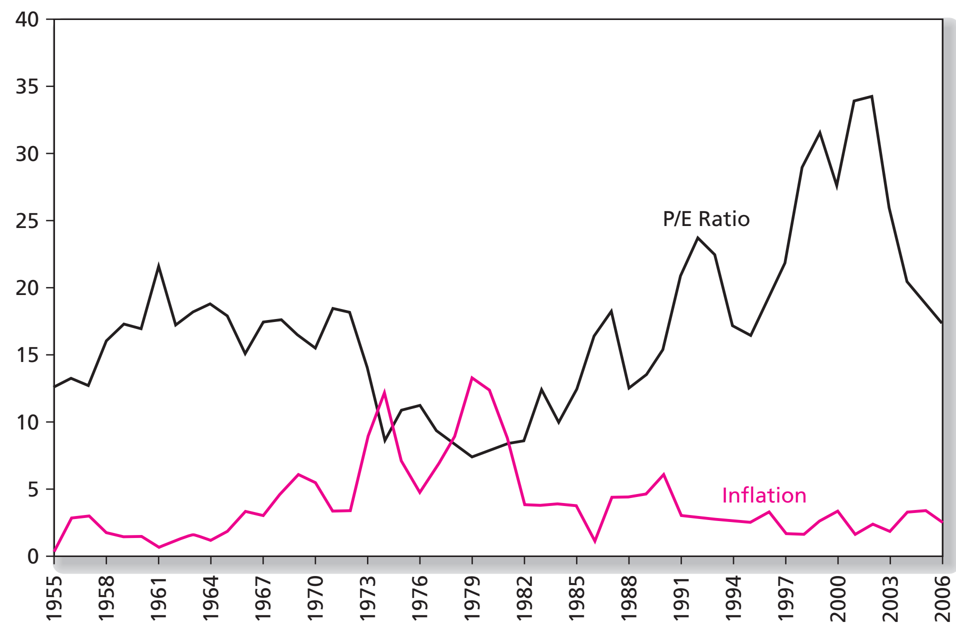

FIGURE 18.3 P/E ratios of the S&P 500 Index and inflation

costs will tend to underrepresent true economic values, because the replacement cost of both goods and capital equipment will rise with the general level of prices. As Figure 18.3 demonstrates, P/E ratios have tended to be lower when inflation has been higher. This reflects the market's assessment that earnings in these periods are of "lower quality," artificially distorted by inflation, and warranting lower P/E ratios.

Earnings management is the practice of using flexibility in accounting rules to improve the apparent profitability of the firm. We will have much to say on this topic in the next chapter on interpreting financial statements. A version of earnings management that became common in the 1990s was the reporting of "pro forma earnings" measures.

Pro forma earnings are calculated ignoring certain expenses, for example, restructuring charges, stock-option expenses, or write-downs of assets from continuing operations. Firms argue that ignoring these expenses gives a clearer picture of the underlying profitability of the firm. Comparisons with earlier periods probably would make more sense if those costs were excluded.

But when there is too much leeway for choosing what to exclude, it becomes hard for investors or analysts to interpret the numbers or to compare them across firms. The lack of standards gives firms considerable leeway to manage earnings.

Even GAAP allows firms considerable discretion to manage earnings. For example, in the late 1990s, Kellogg took restructuring charges, which are supposed to be one-time events, nine quarters in a row. Were these really one-time events, or were they more appropriately treated as ordinary expenses? Given the available leeway in managing earnings, the justified P/E multiple becomes difficult to gauge.

Another confounding factor in the use of P/E ratios is related to the business cycle. We were careful in deriving the DDM to define earnings as being net of *economic* depreciation, that is, the maximum flow of income that the firm could pay out without depleting its productive capacity. But reported earnings are computed in accordance with generally accepted accounting principles and need not correspond to economic earnings. Beyond this, however, notions of a normal or justified P/E ratio, as in Equations 18.7 or 18.8, assume implicitly that earnings rise at a constant rate, or, put another way, on a smooth trend line. In contrast, reported earnings can fluctuate dramatically around a trend line over the course of the business cycle.

Another way to make this point is to note that the "normal" P/E ratio predicted by Equation 18.8 is the ratio of today's price to the *trend value* of future earnings, E_1. The P/E ratio reported in the financial pages of the newspaper, by contrast, is the ratio of price to the most recent *past* accounting earnings. Current accounting earnings can differ considerably from future economic earnings. Because ownership of stock conveys the right to future as well as current earnings, the ratio of price to most recent earnings can vary substantially over the business cycle, as accounting earnings and the trend value of economic earnings diverge by greater and lesser amounts.

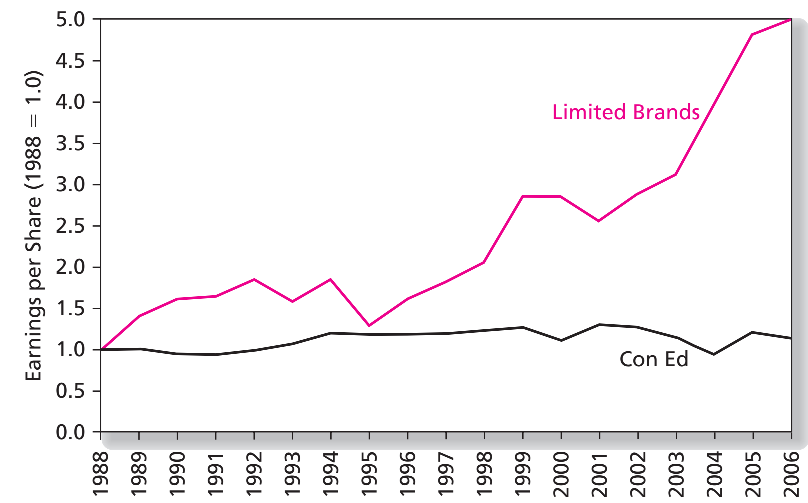

FIGURE 18.4 Earnings growth for two companies

As an example, Figure 18.4 graphs the earnings per share of Limited Brands and Con Ed since 1988. Note that Limited's EPS fluctuate considerably. This reflects the company's relatively high degree of sensitivity to the business cycle. Value Line estimates its beta at 1.15. Con Ed, by contrast, shows much less variation in earnings per share around a smoother and flatter trend line. Its beta was only .70.

Because the market values the entire stream of future dividends generated by the company, when earnings are temporarily depressed, the P/E ratio should tend to be high—that is, the denominator of the ratio responds more sensitively to the business cycle than the numerator. This pattern is borne out well.

Figure 18.5 graphs the P/E ratios of the two firms. Limited, with the more volatile earnings profile, also has a more volatile

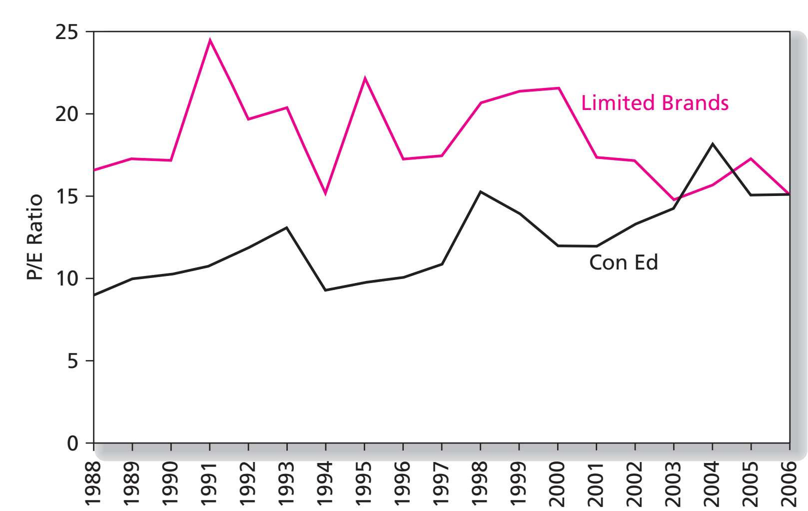

FIGURE 18.5 Price–earnings ratios

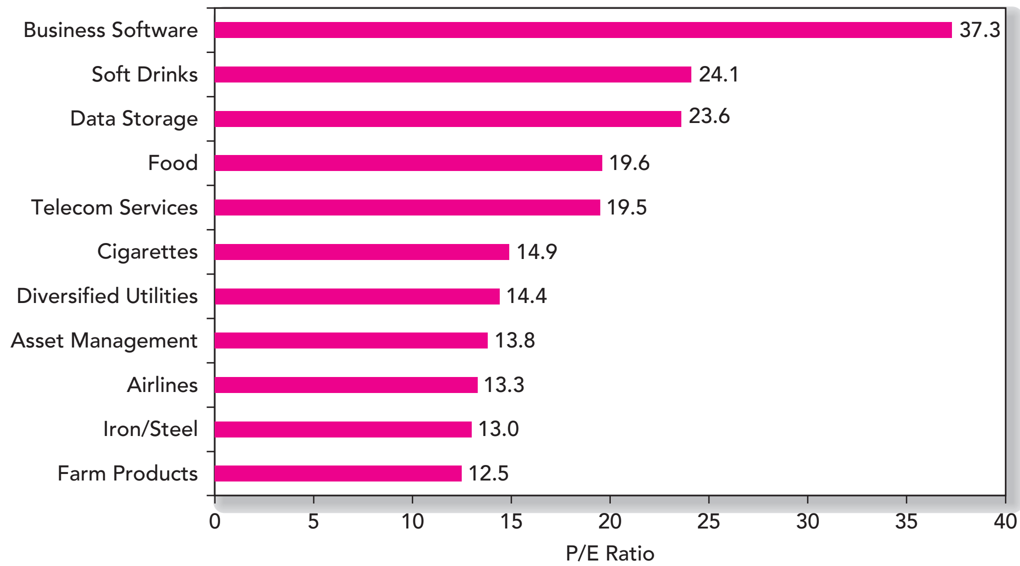

FIGURE 18.6 P/E ratios for different industries, 2007

Source: Data collected from Yahoo! Finance, November 5, 2007.

P/E profile. For example, in 1995, when earnings fell below the trend line (Figure 18.4), the P/E ratio correspondingly jumped (Figure 18.5). The market clearly recognized that earnings were depressed only temporarily. Similarly, the only year in which Con Ed's P/E ratio exceeded Limited's was in 2004, one of the rare years in which Con Ed's earnings fell below its trend line to a meaningful degree.

This example shows why analysts must be careful in using P/E ratios. There is no way to say P/E ratio is overly high or low without referring to the company's long-run growth prospects, as well as to current earnings per share relative to the long-run trend line.

Nevertheless, Figures 18.4 and 18.5 demonstrate a clear relationship between P/E ratios and growth. Despite considerable short-run fluctuations, Limited's EPS clearly trended upward over the period. Con Ed's earnings were essentially flat. Limited's growth prospects are reflected in its consistently higher P/E multiple.

This analysis suggests that P/E ratios should vary across industries, and in fact they do. Figure 18.6 shows P/E ratios in 2007 for a sample of industries. Notice that the industries with the highest multiples—such as business software or data storage—have attractive investment opportunities and relatively high growth rates, whereas the industries with the lowest ratios—farm products or iron/steel manufacturers—are in more mature industries with limited growth prospects. The relationship between P/E and growth is not perfect, which is not surprising in light of the pitfalls discussed in this section, but it appears that as a general rule, the P/E multiple does track growth opportunities.

Combining P/E Analysis and the DDM

Some analysts use P/E ratios in conjunction with earnings forecasts to estimate the price of a stock at an investor's horizon date. The Honda analysis in Figure 18.2 shows that Value Line forecast a P/E ratio for 2011 of 15. EPS for 2011 were forecast at $4.20, implying a price in 2011 of 15 × \$4.20 = \$63. Given an estimate of \$63 for the 2011 sales price, we would compute intrinsic value in 2007 as

$$V_{2007} = \frac{.77}{1.117} + \frac{.88}{(1.117)^2} + \frac{.99}{(1.117)^3} + \frac{1.10 + 63}{(1.117)^4} = \$43.28$$

Other Comparative Valuation Ratios

The price–earnings ratio is an example of a comparative valuation ratio. Such ratios are used to assess the valuation of one firm versus another based on a fundamental indicator such as earnings. For example, an analyst might compare the P/E ratios of two firms in the same industry to test whether the market is valuing one firm "more aggressively" than the other. Other such comparative ratios are commonly used:

Price-to-Book Ratio This is the ratio of price per share divided by book value per share. As we noted earlier in this chapter, some analysts view book value as a useful measure of value and therefore treat the ratio of price to book value as an indicator of how aggressively the market values the firm.

Price-to-Cash-Flow Ratio Earnings as reported on the income statement can be affected by the company's choice of accounting practices, and thus are commonly viewed as subject to some imprecision and even manipulation. In contrast, cash flow—which tracks cash actually flowing into or out of the firm—is less affected by accounting decisions. As a result, some analysts prefer to use the ratio of price to cash flow per share rather than price to earnings per share. Some analysts use operating cash flow when calculating this ratio; others prefer "free cash flow," that is, operating cash flow net of new investment.

Price-to-Sales Ratio Many start-up firms have no earnings. As a result, the price–earnings ratio for these firms is meaningless. The price-to-sales ratio (the ratio of stock price to the annual sales per share) has recently become a popular valuation benchmark for these firms. Of course, price-to-sales ratios can vary markedly across industries, because profit margins vary widely.

Be Creative Sometimes a standard valuation ratio will simply not be available, and you will have to devise your own. In the 1990s, some analysts valued retail Internet firms based on the number of Web hits their sites received. As it turns out, they valued these firms using too-generous "price-to-hits" ratios. Nevertheless, in a new investment environment, these analysts used the information available to them to devise the best valuation tools they could.

Figure 18.7 presents the behavior of several valuation measures since 1955. While the levels of these ratios differ considerably, for the most part, they track each other fairly closely, with upturns and downturns at the same times.

18.5 FREE CASH FLOW VALUATION APPROACHES

An alternative approach to the dividend discount model values the firm using free cash flow, that is, cash flow available to the firm or its equityholders net of capital expenditures. This approach is particularly useful for firms that pay no dividends, for which the dividend

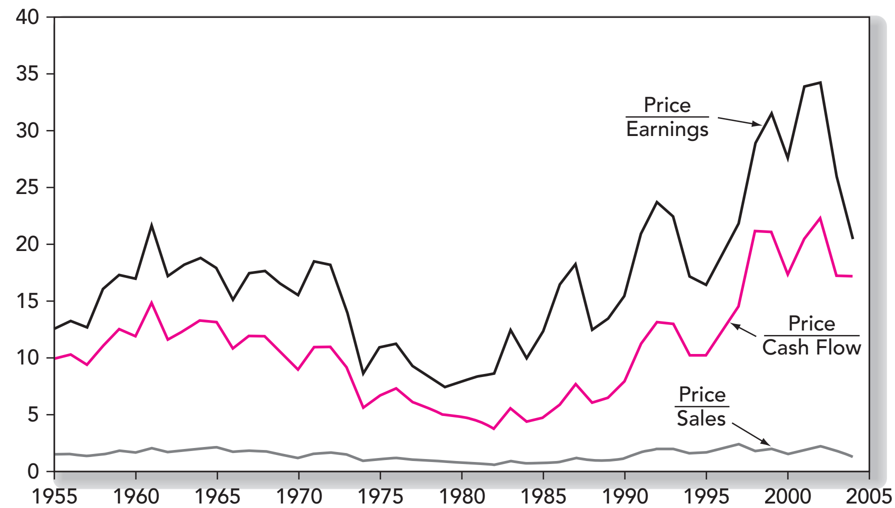

FIGURE 18.7 Market valuation statistics

discount model would be difficult to implement. But free cash flow models may be applied to any firm and can provide useful insights about firm value beyond the DDM.

One approach is to discount the *free cash flow* for the *firm* (FCFF) at the weighted-average cost of capital to obtain the value of the firm, and subtract the then-existing value of debt to find the value of equity. Another is to focus from the start on the free cash flow to *equity holders* (FCFE), discounting those directly at the cost of equity to obtain the market value of equity.

The free cash flow to the firm is the after-tax cash flow that accrues from the firm's operations, net of investments in capital and net working capital. It includes cash flows available to both debt- and equityholders.[9] It is given as follows:

$$\text{FCFF} = \text{EBIT}\,(1 - t_c) + \text{Depreciation} - \text{Capital expenditures} - \text{Increase in NWC} \quad \textbf{(18.9)}$$

where

EBIT = earnings before interest and taxes

t_c = the corporate tax rate

NWC = net working capital

Alternatively, we can focus on cash flow available to equityholders. This will differ from free cash flow to the firm by after-tax interest expenditures, as well as by cash flow associated with net issuance or repurchase of debt (i.e., principal repayments minus proceeds from issuance of new debt).

$$\text{FCFE} = \text{FCFF} - \text{Interest expense} \times (1 - t_c) + \text{Increases in net debt} \quad \textbf{(18.10)}$$

The free cash flow to the firm approach discounts year-by-year cash flows plus some estimate of terminal value, V_T. In Equation 18.11, we use the constant-growth model to estimate terminal value and discount at the weighted-average cost of capital.

[9]This is firm cash flow assuming all-equity financing. Any tax advantage to debt financing is recognized by using an after-tax cost of debt in the computation of weighted-average cost of capital. This issue is discussed in any introductory corporate finance text.

$$\text{Firm value} = \sum_{t=1}^{T} \frac{\text{FCFF}_t}{(1+\text{WACC})^t} + \frac{V_T}{(1+\text{WACC})^T}, \qquad \text{where } V_T = \frac{\text{FCFF}_{T+1}}{\text{WACC} - g} \tag{18.11}$$

To find equity value, we subtract the existing market value of debt from the derived value of the firm.

Alternatively, we can discount free cash flows to *equity* (FCFE) at the cost of *equity*, k_E.

$$\text{Market value of equity} = \sum_{t=1}^{T} \frac{\text{FCFE}_t}{(1+k_E)^t} + \frac{V_T}{(1+k_E)^T}, \qquad \text{where } V_T = \frac{\text{FCFE}_{T+1}}{k_E - g} \tag{18.12}$$

As in the dividend discount model, free cash flow models use a terminal value to avoid adding the present values of an infinite sum of cash flows. That terminal value may simply be the present value of a constant-growth perpetuity (as in the formulas above) or it may be based on a multiple of EBIT, book value, earnings, or free cash flow. As a general rule, estimates of intrinsic value depend critically on terminal value.

Spreadsheet 18.2 presents a free cash flow valuation of Honda using the data supplied by Value Line in Figure 18.2. We start with the free cash flow to the firm approach given in Equation 18.9. Panel A of the spreadsheet lays out values supplied by Value Line. Entries

	A	B	C	D	E	F	G	H	I	J	K	L	M
1		2006	2007	2008	2009	2010	2011						
2	A. Value Line data												
3	P/E	12.70	13.16	13.62	14.08	14.54	15.00						
4	Cap spending/shr		3.55	3.65	3.78	3.92	4.05						
5	LT Debt		16000	15000	13333	11667	10000						
6	Shares		1810	1800	1792	1783	1775						
7	EPS		2.95	3.20	3.53	3.87	4.20						
8	Working Capital		8740	8755	8770	8785	8800						
9													
10	B. Cash flow calculations												
11	Profits (after tax)		5355.0	5775.0	6345.0	6915.0	7485.0						
12	Interest (after tax)		560.9	525.8	467.4	409.0	350.6	= r_debt * (1–tax) * LT Debt					
13	Chg Working Cap			15.0	15.0	15.0	15.0						
14	Depreciation			3575.0	3650.0	3725.0	3800.0						
15	Cap Spending			6570.0	6776.3	6982.5	7188.8						
16								Terminal value					
17	FCFF			3290.8	3671.2	4051.5	4431.8	103528.9					
18	FCFE			1765.0	1537.1	1975.8	2414.6	96092.5		assumes fixed debt ratio after 2011			
19													
20	C. Discount rate calculations												
21	Current beta	0.9								from Value Line			
22	Unlevered beta	0.790								current beta/[1+(1–tax)*debt/equity)]			
23	terminal growth	0.06											
24	tax_rate	0.385								from Value Line			
25	r_debt	0.057								YTM in 2007 on A-rated LT debt			
26	risk-free rate	0.045											
27	market risk prem	0.08											
28	MV equity		70472	78656	89338	100544	112275			Row 3 * Row 11			
29	Debt/Value		0.19	0.16	0.13	0.10	0.08			Row 5/(Row 5+Row 28)			
30	Levered beta		0.900	0.882	0.862	0.846	0.833			unlevered beta * [1+(1–tax)*debt/equity]			
31	k_equity		0.117	0.116	0.114	0.113	0.112	0.112		from CAPM and levered beta			
32	WACC		0.102	0.103	0.104	0.105	0.105	0.105		(1–t)*r_debt*D/V+k_equity*(1–D/V)			
33	PV factor for FCFF		1.000	0.907	0.822	0.744	0.673	0.673		Discount each year at WACC			
34	PV factor for FCFE		1.000	0.896	0.805	0.723	0.651	0.651		Discount each year at k_equity			
35													
36	D. Present values									Intrinsic val	Equity val	Intrin/share	
37	PV(FCFF)			2984	3016	3014	2982	69667		81663	65663	36.28	
38	PV(FCFE)			1582	1237	1429	1571	62513		68332	68332	37.75	

SPREADSHEET 18.2

Free cash flow valuation of Honda Motor Co.

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for middle years are interpolated from beginning and final values. Panel B calculates free cash flow. The sum of after-tax profits in row 11 (from Value Line) plus after-tax interest payments in row 12 [i.e., interest expense × $(1 - t_c)$] equals EBIT$(1 - t_c)$. In row 13 we subtract the change in net working capital, in row 14 we add back depreciation, and in row 15 we subtract capital expenditures. The result in row 17 is the free cash flow to the firm, FCFF, for each year between 2006 and 2009.

To find the present value of these cash flows, we will discount at WACC, which is calculated in panel C. WACC is the weighted average of the after-tax cost of debt and the cost of equity in each year. When computing WACC, we must account for the change in leverage forecast by Value Line. To compute the cost of equity, we will use the CAPM as in our earlier (dividend discount model) valuation exercise, but accounting for the fact that equity beta will decline each year as the firm reduces leverage.[10]

To find Honda's cost of debt, we note that its long-term bonds were rated A in late 2007 and that yields to maturity on A-rated debt at the time were about 5.7%. Honda's debt-to-value ratio is computed in row 29 (assuming that its debt is selling near par value), and WACC is computed in row 32. WACC increases slightly over time as the debt-to-value ratio declines between 2008 and 2011. The present value factor for cash flows accruing in each year is the previous year's factor divided by (1 + WACC) for that year. The present value of each cash flow (row 37) is the free cash flow times the cumulative discount factor.

The terminal value of the firm (cell H17) is computed from the constant-growth model as $FCFF_{2011} \times (1 + g)/(WACC_{2011} - g)$, where g (cell B23) is the assumed value for the steady growth rate. We assume in the spreadsheet that $g = .06$, which is perhaps a bit higher than the long-run growth rate of the broad economy.[11] Terminal value is also discounted back to 2007 (cell H37), and the intrinsic value of the firm is thus found as the sum of discounted free cash flows between 2008 and 2011 plus the discounted terminal value. Finally, the value of debt in 2007 is subtracted from firm value to arrive at the intrinsic value of equity in 2007 (cell K37), and value per share is calculated in cell L37 as equity value divided by number of shares in 2007.

The free cash flow to equity approach yields a similar intrinsic value for the stock.[12] FCFE (row 18) is obtained from FCFF by subtracting after-tax interest expense and net debt repurchases. The cash flows are then discounted at the equity rate. Like WACC, the

[10]Call β_L the firm's equity beta at the initial level of leverage as provided by Value Line. Equity betas reflect both business risk and financial risk. When a firm changes its capital structure (debt/equity mix), it changes financial risk, and therefore equity beta changes. How should we recognize the change in financial risk? As you may remember from an introductory corporate finance class, you must first unleverage beta. This leaves us with business risk. We use the following formula to find unleveraged beta, β_U (where D/E is the firm's current debt-equity ratio):

$$\beta_U = \frac{\beta_L}{1 + (D/E)(1 - t_c)}$$

Then, we re-leverage beta in any particular year using the forecast capital structure for that year (which reintroduces the financial risk associated with that year's capital structure):

$$\beta_L = \beta_U[1 + (D/E)(1 - t_c)]$$

[11]In the long run a firm can't grow forever at a rate higher than the aggregate economy. So by the time we assert that growth is in a stable stage, it seems reasonable that the growth rate should not be significantly greater than that of the overall economy (although it can be less if the firm is in a declining industry).

[12]Over the 2008–2011 period, Value Line predicts that Honda will retire a considerable fraction of its outstanding debt. The implied debt repurchases are a use of cash and reduce the cash flow available to equity. Such repurchases cannot be sustained indefinitely, however, for debt outstanding would soon be run down to zero. Therefore, in our estimate of the terminal value of equity, we compute the final cash flow assuming that starting in 2011 Honda will begin *issuing* enough debt to maintain its debt-to-value ratio. This approach is consistent with the assumption of constant growth and constant discount rates after 2011.

cost of equity changes each period as leverage changes. The present value factor for equity cash flows is presented in row 34. Equity value is reported in cell J38, which is put on a per share basis in cell L38.

Spreadsheet 18.2 is available at the Online Learning Center for this text, **www.mhhe.com/bkm.**

Comparing the Valuation Models

In principle, the free cash flow approach is fully consistent with the dividend discount model and should provide the same estimate of intrinsic value if one can extrapolate to a period in which the firm begins to pay dividends growing at a constant rate. This was demonstrated in two famous papers by Modigliani and Miller.[13] However, in practice, you will find that values from these models may differ, sometimes substantially. This is due to the fact that in practice, analysts are always forced to make simplifying assumptions. For example, how long will it take the firm to enter a constant-growth stage? How should depreciation best be treated? What is the best estimate of ROE? Answers to questions like these can have a big impact on value, and it is not always easy to maintain consistent assumptions across the models.

We have now valued Honda using several approaches, with estimates of intrinsic value as follows:

Model	Intrinsic Value
Two-stage dividend discount model	$34.32
DDM with earnings multiple terminal value	43.28
Three-stage DDM	39.71
Free cash flow to the firm	36.28
Free cash flow to equity	37.75
Market price (from Value Line)	32.10

What should we make of these differences? All of these estimates are somewhat higher than Honda's actual stock price, perhaps indicating that they use an unrealistically high value for the ultimate constant growth rate. In the long run, it seems unlikely that Honda will be able to grow as rapidly as Value Line's forecast for 2011 growth, 9.25%. The two-stage dividend discount model is the most conservative of the estimates, probably because it assumes that Honda's dividend growth rate will fall to its terminal value after only 3 years. In contrast, the three-stage DDM allows growth to taper off over a longer period. The DDM with a terminal value provided by the earnings multiple results in the most extreme estimate of intrinsic value, one that is 35% higher than Honda's actual stock price. Value Line's estimate of the 2011 P/E ratio is higher than recent experience, and its earnings per share estimates also seem on the optimistic side. On the other hand, given the consistency with which these estimates exceed market price, perhaps the stock is indeed underpriced compared to its intrinsic value.

This valuation exercise shows that finding bargains is not as easy as it seems. While these models are easy to apply, establishing proper inputs is more of a challenge. This should not be surprising. In even a moderately efficient market, finding profit opportunities will be more involved than analyzing Value Line data for a few hours. These models are extremely useful to analysts, however, because they provide ballpark estimates of intrinsic value. More than that, they force rigorous thought about underlying assumptions and highlight the variables with the greatest impact on value and the greatest payoff to further analysis.

[13]Franco Modigliani and M. Miller, "The Cost of Capital, Corporation Finance, and the Theory of Investment," *American Economic Review,* June 1958, and "Dividend Policy, Growth, and the Valuation of Shares," *Journal of Business*, October 1961.

18.6 THE AGGREGATE STOCK MARKET

Explaining Past Behavior

It has been well documented that the stock market is a leading economic indicator.[14] This means that it tends to fall before a recession and to rise before an economic recovery. However, the relationship is far from perfectly reliable.

Most scholars and serious analysts would agree that, although the stock market sometimes appears to have a substantial life of its own, responding perhaps to bouts of mass euphoria and then panic, economic events and the anticipation of such events do have a substantial effect on stock prices. Perhaps the two factors with the greatest impact are interest rates and corporate profits.

Figure 18.8 shows the behavior of the earnings-to-price ratio (i.e., the earnings yield) of the S&P 500 stock index versus the yield to maturity on long-term Treasury bonds since 1955. Clearly, the two series track each other quite closely. This is to be expected: The two variables that affect a firm's value are earnings (and implicitly the dividends they can support) and the discount rate, which "translates" future income into present value. Thus, it should not be surprising that the ratio of earnings to stock price (the inverse of the P/E ratio) varies with the interest rate.

Forecasting the Stock Market

The most popular approach to forecasting the overall stock market is the earnings multiplier approach applied at the aggregate level. The first step is to forecast corporate profits for the coming period. Then we derive an estimate of the earnings multiplier, the aggregate

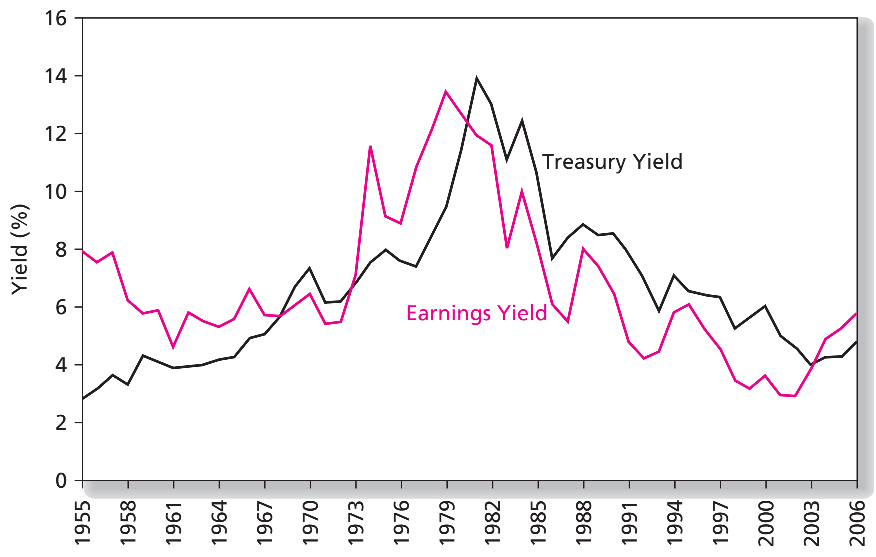

FIGURE 18.8 Earnings yield of S&P 500 versus 10-year Treasury-bond yield

[14] See, for example, Stanley Fischer and Robert C. Merton, "Macroeconomics and Finance: The Role of the Stock Market," *Carnegie-Rochester Conference Series on Public Policy* 21 (1984).

P/E ratio, based on a forecast of long-term interest rates. The product of the two forecasts is the estimate of the end-of-period level of the market.

The forecast of the P/E ratio of the market is sometimes derived from a graph similar to that in Figure 18.8, which plots the *earnings yield* (earnings per share divided by price per share, the reciprocal of the P/E ratio) of the S&P 500 and the yield to maturity on 10-year Treasury bonds. The figure shows that both yields rose dramatically in the 1970s. In the case of Treasury bonds, this was because of an increase in the inflationary expectations built into interest rates. The earnings yield on the S&P 500, however, probably rose because of inflationary distortions that artificially increased reported earnings. We have already seen that P/E ratios tend to fall when inflation rates increase. When inflation moderated in the 1980s, both Treasury and earnings yields fell. For most of the last 30 years, the earnings yield has been within about 1 percentage point of the T-bond rate.

One might use this relationship and the current yield on 10-year Treasury bonds to forecast the earnings yield on the S&P 500. Given that earnings yield, a forecast of earnings could be used to predict the level of the S&P in some future period. Let's consider a simple example of this procedure.

EXAMPLE 18.6 Forecasting the Aggregate Stock Market

A mid-2007 forecast for earnings per share for the S&P 500 portfolio in the coming 12 months was about $97. The 10-year Treasury bond yield was about 4.6%. Because the earnings yield on the S&P 500 has most recently been about 1 percentage point above the 10-year Treasury yield, a first guess for the earnings yield on the S&P 500 might be 5.6%. This would imply a P/E ratio of $1/.056 = 17.86$. Our 1-year-ahead forecast for the S&P 500 index would then be $17.86 \times 97 = 1{,}732$.

Of course, there is uncertainty regarding all three inputs into this analysis: the actual earnings on the S&P 500 stocks, the level of Treasury yields at year-end, and the spread between the Treasury yield and the earnings yield. One would wish to perform sensitivity or scenario analysis to examine the impact of changes in all of these variables. To illustrate, consider Table 18.4, which shows a simple scenario analysis treating possible effects of variation in the Treasury bond yield. The scenario analysis shows that forecast level of the stock market varies inversely and with dramatic sensitivity to interest rate changes.

Some analysts use an aggregate version of the dividend discount model rather than an earnings multiplier approach. All of these models, however, rely heavily on forecasts of

TABLE 18.4

S&P 500 price forecasts under various scenarios

	Most Likely Scenario	Pessimistic Scenario	Optimistic Scenario
Treasury bond yield	4.6%	5.1%	4.1%
Earnings yield	5.6%	6.1%	5.1%
Resulting P/E ratio	17.9	16.4	19.6
EPS forecast	97	97	97
Forecast for S&P 500	1,732	1,590	1,902

Forecast for the earnings yield on the S&P 500 equals Treasury bond yield plus 1%.
The P/E ratio is the reciprocal of the forecast earnings yield.

such macroeconomic variables as GDP, interest rates, and the rate of inflation, which are difficult to predict accurately.

Because stock prices reflect expectations of future dividends, which are tied to the economic fortunes of firms, it is not surprising that the performance of a broad-based stock index like the S&P 500 is taken as a leading economic indicator, that is, a predictor of the performance of the aggregate economy. Stock prices are viewed as embodying consensus forecasts of economic activity and are assumed to move up or down in anticipation of movements in the economy. The government's index of leading economic indicators, which is taken to predict the progress of the business cycle, is made up in part of recent stock market performance. However, the predictive value of the market is far from perfect. A well-known joke, often attributed to Paul Samuelson, is that the market has forecast eight of the last five recessions.

SUMMARY

1. One approach to firm valuation is to focus on the firm's book value, either as it appears on the balance sheet or as adjusted to reflect current replacement cost of assets or liquidation value. Another approach is to focus on the present value of expected future dividends.
2. The dividend discount model holds that the price of a share of stock should equal the present value of all future dividends per share, discounted at an interest rate commensurate with the risk of the stock.
3. Dividend discount models give estimates of the intrinsic value of a stock. If price does not equal intrinsic value, the rate of return will differ from the equilibrium return based on the stock's risk. The actual return will depend on the rate at which the stock price is predicted to revert to its intrinsic value.
4. The constant-growth version of the DDM asserts that if dividends are expected to grow at a constant rate forever, the intrinsic value of the stock is determined by the formula

$$V_0 = \frac{D_1}{k - g}$$

This version of the DDM is simplistic in its assumption of a constant value of g. There are more-sophisticated multistage versions of the model for more-complex environments. When the constant-growth assumption is reasonably satisfied and the stock is selling for its intrinsic value, the formula can be inverted to infer the market capitalization rate for the stock:

$$k = \frac{D_1}{P_0} + g$$

5. The constant-growth dividend discount model is best suited for firms that are expected to exhibit stable growth rates over the foreseeable future. In reality, however, firms progress through life cycles. In early years, attractive investment opportunities are ample and the firm responds with high plowback ratios and rapid dividend growth. Eventually, however, growth rates level off to more sustainable values. Three-stage growth models are well suited to such a pattern. These models allow for an initial period of rapid growth, a final period of steady dividend growth, and a middle, or transition, period in which the dividend growth rate declines from its initial high rate to the lower sustainable rate.
6. Stock market analysts devote considerable attention to a company's price-to-earnings ratio. The P/E ratio is a useful measure of the market's assessment of the firm's growth opportunities. Firms with no growth opportunities should have a P/E ratio that is just the reciprocal of the capitalization rate, *k*. As growth opportunities become a progressively more important component of the total value of the firm, the P/E ratio will increase.

7. The expected growth rate of earnings is related both to the firm's expected profitability and to its dividend policy. The relationship can be expressed as

$$g = (\text{ROE on new investment}) \times (1 - \text{Dividend payout ratio})$$

8. You can relate any DDM to a simple capitalized earnings model by comparing the expected ROE on future investments to the market capitalization rate, *k*. If the two rates are equal, then the stock's intrinsic value reduces to expected earnings per share (EPS) divided by *k*.

9. Many analysts form their estimates of a stock's value by multiplying their forecast of next year's EPS by a predicted P/E multiple. Some analysts mix the P/E approach with the dividend discount model. They use an earnings multiplier to forecast the terminal value of shares at a future date, and add the present value of that terminal value with the present value of all interim dividend payments.

10. The free cash flow approach is the one used most often in corporate finance. The analyst first estimates the value of the entire firm as the present value of expected future free cash flows to the entire firm and then subtracts the value of all claims other than equity. Alternatively, the free cash flows to equity can be discounted at a discount rate appropriate to the risk of the stock.

11. The models presented in this chapter can be used to explain and forecast the behavior of the aggregate stock market. The key macroeconomic variables that determine the level of stock prices in the aggregate are interest rates and corporate profits.

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KEY TERMS

book value
liquidation value
replacement cost
Tobin's *q*
intrinsic value
market capitalization rate
dividend discount model (DDM)
constant-growth DDM
dividend payout ratio
plowback ratio
earnings retention ratio
present value of growth opportunities
price–earnings multiple
earnings management

PROBLEM SETS

Quiz

1. In what circumstances would you choose to use a dividend discount model rather than a free cash flow model to value a firm?
2. In what circumstances is it most important to use multistage dividend discount models rather than constant-growth models?
3. If a security is underpriced (i.e., intrinsic value > price), then what is the relationship between its market capitalization rate and its expected rate of return?

Problems

4. *a.* Computer stocks currently provide an expected rate of return of 16%. MBI, a large computer company, will pay a year-end dividend of $2 per share. If the stock is selling at $50 per share, what must be the market's expectation of the growth rate of MBI dividends?
   b. If dividend growth forecasts for MBI are revised downward to 5% per year, what will happen to the price of MBI stock? What (qualitatively) will happen to the company's price–earnings ratio?
5. *a.* MF Corp. has an ROE of 16% and a plowback ratio of 50%. If the coming year's earnings are expected to be $2 per share, at what price will the stock sell? The market capitalization rate is 12%.
   b. What price do you expect MF shares to sell for in 3 years?
6. The market consensus is that Analog Electronic Corporation has an ROE = 9%, has a beta of 1.25, and plans to maintain indefinitely its traditional plowback ratio of 2/3. This year's earnings were $3 per share. The annual dividend was just paid. The consensus estimate of the coming year's market return is 14%, and T-bills currently offer a 6% return.
   a. Find the price at which Analog stock should sell.
   b. Calculate the P/E ratio.

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c. Calculate the present value of growth opportunities.
d. Suppose your research convinces you Analog will announce momentarily that it will immediately reduce its plowback ratio to 1/3. Find the intrinsic value of the stock. The market is still unaware of this decision. Explain why V_0 no longer equals P_0 and why V_0 is greater or less than P_0.

7. If the expected rate of return of the market portfolio is 15% and a stock with a beta of 1.0 pays a dividend yield of 4%, what must the market believe is the expected rate of price appreciation on that stock?

8. The FI Corporation's dividends per share are expected to grow indefinitely by 5% per year.
a. If this year's year-end dividend is $8 and the market capitalization rate is 10% per year, what must the current stock price be according to the DDM?
b. If the expected earnings per share are $12, what is the implied value of the ROE on future investment opportunities?
c. How much is the market paying per share for growth opportunities (i.e., for an ROE on future investments that exceeds the market capitalization rate)?

9. The stock of Nogro Corporation is currently selling for $10 per share. Earnings per share in the coming year are expected to be $2. The company has a policy of paying out 50% of its earnings each year in dividends. The rest is retained and invested in projects that earn a 20% rate of return per year. This situation is expected to continue indefinitely.
a. Assuming the current market price of the stock reflects its intrinsic value as computed using the constant-growth DDM, what rate of return do Nogro's investors require?
b. By how much does its value exceed what it would be if all earnings were paid as dividends and nothing were reinvested?
c. If Nogro were to cut its dividend payout ratio to 25%, what would happen to its stock price? What if Nogro eliminated the dividend?

10. The risk-free rate of return is 8%, the expected rate of return on the market portfolio is 15%, and the stock of Xyrong Corporation has a beta coefficient of 1.2. Xyrong pays out 40% of its earnings in dividends, and the latest earnings announced were $10 per share. Dividends were just paid and are expected to be paid annually. You expect that Xyrong will earn an ROE of 20% per year on all reinvested earnings forever.
a. What is the intrinsic value of a share of Xyrong stock?
b. If the market price of a share is currently $100, and you expect the market price to be equal to the intrinsic value 1 year from now, what is your expected 1-year holding-period return on Xyrong stock?

11. The Digital Electronic Quotation System (DEQS) Corporation pays no cash dividends currently and is not expected to for the next 5 years. Its latest EPS was $10, all of which was reinvested in the company. The firm's expected ROE for the next 5 years is 20% per year, and during this time it is expected to continue to reinvest all of its earnings. Starting 6 years from now the firm's ROE on new investments is expected to fall to 15%, and the company is expected to start paying out 40% of its earnings in cash dividends, which it will continue to do forever after. DEQS's market capitalization rate is 15% per year.
a. What is your estimate of DEQS's intrinsic value per share?
b. Assuming its current market price is equal to its intrinsic value, what do you expect to happen to its price over the next year? The year after?
c. What effect would it have on your estimate of DEQS's intrinsic value if you expected DEQS to pay out only 20% of earnings starting in year 6?

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12. Recalculate the intrinsic value of Honda in each of the following scenarios by using the three-stage growth model of Spreadsheet 18.1 (available at **www.mhhe.com/bkm;** link to Chapter 18 material). Treat each scenario independently.
a. ROE in the constant-growth period will be 13%.

b. Honda's actual beta is 1.0.
c. The market risk premium is 7.5%.

13. Recalculate the intrinsic value of Honda shares using the free cash flow model of Spreadsheet 18.2 (available at **www.mhhe.com/bkm;** link to Chapter 18 material) under each of the following assumptions. Treat each scenario independently.

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a. Honda's P/E ratio starting in 2011 will be 16.
b. Honda's unlevered beta is 1.0.
c. The market risk premium is 9%.

14. The Duo Growth Company just paid a dividend of $1 per share. The dividend is expected to grow at a rate of 25% per year for the next 3 years and then to level off to 5% per year forever. You think the appropriate market capitalization rate is 20% per year.
a. What is your estimate of the intrinsic value of a share of the stock?
b. If the market price of a share is equal to this intrinsic value, what is the expected dividend yield?
c. What do you expect its price to be 1 year from now? Is the implied capital gain consistent with your estimate of the dividend yield and the market capitalization rate?

15. The Generic Genetic (GG) Corporation pays no cash dividends currently and is not expected to for the next 4 years. Its latest EPS was $5, all of which was reinvested in the company. The firm's expected ROE for the next 4 years is 20% per year, during which time it is expected to continue to reinvest all of its earnings. Starting 5 years from now, the firm's ROE on new investments is expected to fall to 15% per year. GG's market capitalization rate is 15% per year.
a. What is your estimate of GG's intrinsic value per share?
b. Assuming its current market price is equal to its intrinsic value, what do you expect to happen to its price over the next year?

16. The MoMi Corporation's cash flow from operations before interest and taxes was $2 million in the year just ended, and it expects that this will grow by 5% per year forever. To make this happen, the firm will have to invest an amount equal to 20% of pretax cash flow each year. The tax rate is 35%. Depreciation was $200,000 in the year just ended and is expected to grow at the same rate as the operating cash flow. The appropriate market capitalization rate for the unleveraged cash flow is 12% per year, and the firm currently has debt of $4 million outstanding. Use the free cash flow approach to value the firm's equity.

Challenge Problems

17. Chiptech, Inc., is an established computer chip firm with several profitable existing products as well as some promising new products in development. The company earned $1 a share last year, and just paid out a dividend of $.50 per share. Investors believe the company plans to maintain its dividend payout ratio at 50%. ROE equals 20%. Everyone in the market expects this situation to persist indefinitely.
a. What is the market price of Chiptech stock? The required return for the computer chip industry is 15%, and the company has just gone ex-dividend (i.e., the next dividend will be paid a year from now, at $t = 1$).
b. Suppose you discover that Chiptech's competitor has developed a new chip that will eliminate Chiptech's current technological advantage in this market. This new product, which will be ready to come to the market in 2 years, will force Chiptech to reduce the prices of its chips to remain competitive. This will decrease ROE to 15%, and, because of falling demand for its product, Chiptech will decrease the plowback ratio to .40. The plowback ratio will be decreased at the end of the second year, at $t = 2$: The annual year-end dividend for the second year (paid at $t = 2$) will be 60% of that year's earnings. What is your estimate of Chiptech's intrinsic value per share? (*Hint:* Carefully prepare a table of Chiptech's earnings and dividends for each of the next 3 years. Pay close attention to the change in the payout ratio in $t = 2$.)
c. No one else in the market perceives the threat to Chiptech's market. In fact, you are confident that no one else will become aware of the change in Chiptech's competitive status until the

competitor firm publicly announces its discovery near the end of year 2. What will be the rate of return on Chiptech stock in the coming year (i.e., between $t = 0$ and $t = 1$)? In the second year (between $t = 1$ and $t = 2$)? The third year (between $t = 2$ and $t = 3$)? (*Hint:* Pay attention to when the *market* catches on to the new situation. A table of dividends and market prices over time might help.)

CFA® PROBLEMS

1. A common stock pays an annual dividend per share of $2.10. The risk-free rate is 7%, and the risk premium for this stock is 4%. If the annual dividend is expected to remain at $2.10, what is the value of the stock?
2. Which of the following assumptions does the constant-growth dividend discount model require?
   - I. Dividends grow at a constant rate.
   - II. The dividend growth rate continues indefinitely.
   - III. The required rate of return is less than the dividend growth rate.
3. At Litchfield Chemical Corp. (LCC), a director of the company said that the use of dividend discount models by investors is "proof" that the higher the dividend, the higher the stock price.
   - *a.* Using a constant-growth dividend discount model as a basis of reference, evaluate the director's statement.
   - *b.* Explain how an increase in dividend payout would affect each of the following (holding all other factors constant):
     - i. Sustainable growth rate.
     - ii. Growth in book value.
4. Helen Morgan, CFA, has been asked to use the DDM to determine the value of Sundanci, Inc. Morgan anticipates that Sundanci's earnings and dividends will grow at 32% for 2 years and 13% thereafter. Calculate the current value of a share of Sundanci stock by using a two-stage dividend discount model and the data from Tables 18A and 18B.

TABLE 18A
Sundanci actual 2007 and 2008 financial statements for fiscal years ending May 31 ($ million, except per-share data)

Income Statement	2007	2008
Revenue	$ 474	$ 598
Depreciation	20	23
Other operating costs	368	460
Income before taxes	86	115
Taxes	26	35
Net income	60	80
Dividends	18	24
Earnings per share	$0.714	$0.952
Dividend per share	$0.214	$0.286
Common shares outstanding (millions)	84.0	84.0
Balance Sheet	**2007**	**2008**
Current assets	$ 201	$ 326
Net property, plant and equipment	474	489
Total assets	675	815
Current liabilities	57	141
Long-term debt	0	0
Total liabilities	57	141
Shareholders' equity	618	674
Total liabilities and equity	675	815
Capital expenditures	34	38

Required rate of return on equity	14%
Growth rate of industry	13%
Industry P/E ratio	26

TABLE 18B

Selected financial information

5. Abbey Naylor, CFA, has been directed to determine the value of Sundanci's stock using the Free Cash Flow to Equity (FCFE) model. Naylor believes that Sundanci's FCFE will grow at 27% for 2 years and 13% thereafter. Capital expenditures, depreciation, and working capital are all expected to increase proportionately with FCFE.
   a. Calculate the amount of FCFE per share for the year 2008, using the data from Table 18A.
   b. Calculate the current value of a share of Sundanci stock based on the two-stage FCFE model.
   c. i. Describe one limitation of the two-stage DDM model that is addressed by using the two-stage FCFE model.
   ii. Describe one limitation of the two-stage DDM model that is *not* addressed by using the two-stage FCFE model.
6. Christie Johnson, CFA, has been assigned to analyze Sundanci using the constant dividend growth price/earnings (P/E) ratio model. Johnson assumes that Sundanci's earnings and dividends will grow at a constant rate of 13%.
   a. Calculate the P/E ratio based on information in Tables 18A and 18B and on Johnson's assumptions for Sundanci.
   b. Identify, within the context of the constant dividend growth model, how each of the following factors would affect the P/E ratio.
      - Risk (beta) of Sundanci.
      - Estimated growth rate of earnings and dividends.
      - Market risk premium.
7. Dynamic Communication is a U.S. industrial company with several electronics divisions. The company has just released its 2008 annual report. Tables 18C and 18D present a summary of

TABLE 18C

Dynamic Communication balance sheets

	$ Million	
	2008	2007
Cash and equivalents	$ 149	$ 83
Accounts receivable	295	265
Inventory	275	285
Total current assets	$ 719	$ 633
Gross fixed assets	9,350	8,900
Accumulated depreciation	(6,160)	(5,677)
Net fixed assets	$3,190	$3,223
Total assets	$3,909	$3,856
Accounts payable	$ 228	$ 220
Notes payable	0	0
Accrued taxes and expenses	0	0
Total current liabilities	$ 228	$ 220
Long-term debt	$1,650	$1,800
Common stock	50	50
Additional paid-in capital	0	0
Retained earnings	1,981	1,786
Total shareholders' equity	$2,031	$1,836
Total liabilities and shareholders' equity	$3,909	$3,856

	2008	2007
Total revenues	$3,425	$3,300
Operating costs and expenses	2,379	2,319
Earnings before interest, taxes, depreciation and amortization (EBITDA)	$1,046	$ 981
Depreciation and amortization	483	454
Operating income (EBIT)	$ 563	$ 527
Interest expense	104	107
Income before taxes	$ 459	$ 420
Taxes (40%)	184	168
Net income	$ 275	$ 252
Dividends	$ 80	$ 80
Change in retained earnings	$ 195	$ 172
Earnings per share	$ 2.75	$ 2.52
Dividends per share	$ 0.80	$ 0.80
Number of shares outstanding (millions)	100	100

TABLE 18D

Dynamic Communication statements of income (U.S. $ millions except for share data)

	2006	2005	2004
Total revenues	$3,175	$3,075	$3,000
Operating income (EBIT)	495	448	433
Interest expense	104	101	99
Net income	$ 235	$ 208	$ 200
Dividends per share	$ 0.80	$ 0.80	$ 0.80
Total assets	$3,625	$3,414	$3,230
Long-term debt	$1,750	$1,700	$1,650
Total shareholders' equity	$1,664	$1,509	$1,380
Number of shares outstanding (millions)	100	100	100

TABLE 18E

Dynamic Communication selected data from financial statements (U.S. $ millions except for share data)

Dynamic's financial statements for the years 2007 and 2008. Selected data from the financial statements for the years 2004 to 2006 are presented in Table 18E.

a. A group of Dynamic shareholders has expressed concern about the zero growth rate of dividends in the last 4 years and has asked for information about the growth of the company. Calculate Dynamic's sustainable growth rates in 2005 and 2008. Your calculations should use beginning-of-year balance sheet data.

b. Determine how the change in Dynamic's sustainable growth rate (2008 compared to 2005) was affected by changes in its retention ratio and its financial leverage. (*Note:* Your calculations should use beginning-of-year balance sheet data.)

8. Mike Brandreth, an analyst who specializes in the electronics industry, is preparing a research report on Dynamic Communication. A colleague suggests to Brandreth that he may be able to

determine Dynamic's implied dividend growth rate from Dynamic's current common stock price, using the Gordon growth model. Brandreth believes that the appropriate required rate of return for Dynamic's equity is 8 percent.

a. Assume that the firm's current stock price of $58.49 equals intrinsic value. What sustained rate of dividend growth as of December 2008 is implied by this value? Use the constant-growth dividend discount model (i.e., the Gordon growth model).

b. The management of Dynamic has indicated to Brandreth and other analysts that the company's current dividend policy will be continued. Is the use of the Gordon growth model to value Dynamic's common stock appropriate or inappropriate? Justify your response based on the assumptions of the Gordon growth model.

9. Peninsular Research is initiating coverage of a mature manufacturing industry. John Jones, CFA, head of the research department, gathered the following fundamental industry and market data to help in his analysis:

Forecast industry earnings retention rate	40%
Forecast industry return on equity	25%
Industry beta	1.2
Government bond yield	6%
Equity risk premium	5%

a. Compute the price-to-earnings (P_0/E_1) ratio for the industry based on this fundamental data.

b. Jones wants to analyze how fundamental P/E ratios might differ among countries. He gathered the following economic and market data:

Fundamental Factors	Country A	Country B
Forecast growth in real GDP	5%	2%
Government bond yield	10%	6%
Equity risk premium	5%	4%

Determine whether each of these fundamental factors would cause P/E ratios to be generally higher for Country A or higher for Country B.

10. Janet Ludlow's firm requires all its analysts to use a two-stage dividend discount model (DDM) and the capital asset pricing model (CAPM) to value stocks. Using the CAPM and DDM, Ludlow has valued QuickBrush Company at $63 per share. She now must value SmileWhite Corporation.

a. Calculate the required rate of return for SmileWhite by using the information in the following table:

	QuickBrush	SmileWhite
Beta	1.35	1.15
Market price	$45.00	$30.00
Intrinsic value	$63.00	?
Notes:		
Risk-free rate	4.50%	
Expected market return	14.50%	

b. Ludlow estimates the following EPS and dividend growth rates for SmileWhite:

First 3 years	12% per year
Years thereafter	9% per year

Estimate the intrinsic value of SmileWhite by using the table above, and the two-stage DDM. Dividends per share in the most recent year were $1.72.

c. Recommend QuickBrush or SmileWhite stock for purchase by comparing each company's intrinsic value with its current market price.

d. Describe one strength of the two-stage DDM in comparison with the constant-growth DDM. Describe one weakness inherent in all DDMs.

11. Rio National Corp. is a U.S.-based company and the largest competitor in its industry. Tables 18F–18I present financial statements and related information for the company. Table 18J presents relevant industry and market data.

	2008	2007
Cash	$ 13.00	$ 5.87
Accounts receivable	30.00	27.00
Inventory	209.06	189.06
Current assets	$252.06	$221.93
Gross fixed assets	474.47	409.47
Accumulated depreciation	(154.17)	(90.00)
Net fixed assets	320.30	319.47
Total assets	$572.36	$541.40
Accounts payable	$ 25.05	$ 26.05
Notes payable	0.00	0.00
Current portion of long-term debt	0.00	0.00
Current liabilities	$ 25.05	$ 26.05
Long-term debt	240.00	245.00
Total liabilities	$265.05	$271.05
Common stock	160.00	150.00
Retained earnings	147.31	120.35
Total shareholders' equity	$307.31	$270.35
Total liabilities and shareholders' equity	$572.36	$541.40

TABLE 18F

Rio National Corp. summary year-end balance sheets (U.S. $ millions)

TABLE 18G

Rio National Corp. summary income statement for the year ended December 31, 2008 (U.S. $ millions)

Revenue	$300.80
Total operating expenses	(173.74)
Operating profit	127.06
Gain on sale	4.00
Earnings before interest, taxes, depreciation & amortization (EBITDA)	131.06
Depreciation and amortization	(71.17)
Earnings before interest & taxes (EBIT)	59.89
Interest	(16.80)
Income tax expense	(12.93)
Net income	$ 30.16

TABLE 18H

Rio National Corp. supplemental notes for 2008

Note 1:	Rio National had $75 million in capital expenditures during the year.
Note 2:	A piece of equipment that was originally purchased for $10 million was sold for $7 million at year-end, when it had a net book value of $3 million. Equipment sales are unusual for Rio National.
Note 3:	The decrease in long-term debt represents an unscheduled principal repayment; there was no new borrowing during the year.
Note 4:	On January 1, 2008, the company received cash from issuing 400,000 shares of common equity at a price of $25.00 per share.
Note 5:	A new appraisal during the year increased the estimated market value of land held for investment by $2 million, which was not recognized in 2008 income.

TABLE 18I

Rio National Corp. common equity data for 2008

Dividends paid (U.S. $ millions)	$3.20
Weighted-average shares outstanding during 2002	16,000,000
Dividend per share	$0.20
Earnings per share	$1.89
Beta	1.80

Note: The dividend payout ratio is expected to be constant.

TABLE 18J

Industry and market data December 31, 2008

Risk-free rate of return	4.00%
Expected rate of return on market index	9.00%
Median industry price/earnings (P/E) ratio	19.90
Expected industry earnings growth-rate	12.00%

The portfolio manager of a large mutual fund comments to one of the fund's analysts, Katrina Shaar: "We have been considering the purchase of Rio National Corp. equity shares, so I would like you to analyze the value of the company. To begin, based on Rio National's past performance, you can assume that the company will grow at the same rate as the industry."

a. Calculate the value of a share of Rio National equity on December 31, 2008, using the Gordon growth model and the capital asset pricing model.

b. Calculate the sustainable growth rate of Rio National on December 31, 2008. Use 2008 beginning-of-year balance sheet values.

12. While valuing the equity of Rio National Corp. (from the previous problem), Katrina Shaar is considering the use of either cash flow from operations (CFO) or free cash flow to equity (FCFE) in her valuation process.

    a. State two adjustments that Shaar should make to cash flow from operations to obtain free cash flow to equity.

    b. Shaar decides to calculate Rio National's FCFE for the year 2008, starting with net income. Determine for each of the five supplemental notes given in Table 18H whether an adjustment should be made to net income to calculate Rio National's free cash flow to equity for the year 2008, and the dollar amount of any adjustment.

    c. Calculate Rio National's free cash flow to equity for the year 2008.

13. Shaar (from the previous problem) has revised slightly her estimated earnings growth rate for Rio National and, using normalized (underlying) EPS, which is adjusted for temporary impacts on earnings, now wants to compare the current value of Rio National's equity to that of the industry, on a growth-adjusted basis. Selected information about Rio National and the industry is given in Table 18K.

    Compared to the industry, is Rio National's equity overvalued or undervalued on a P/E-to-growth (PEG) basis, using normalized (underlying) earnings per share? Assume that the risk of Rio National is similar to the risk of the industry.

TABLE 18K

Rio National Corp. vs. industry

Rio National	
Estimated earnings growth rate	11.00%
Current share price	$25.00
Normalized (underlying) EPS for 2008	$1.71
Weighted-average shares outstanding during 2008	16,000,000
Industry	
Estimated earnings growth rate	12.00%
Median price/earnings (P/E) ratio	19.90

STANDARD & POOR'S

1. Find the ROEs, the P/E ratios, and the 5-year historical growth rates for 20 of the firms included in the Market Insight Web page at **www.mhhe.com/edumarketinsight.** Select the firms by clicking on the *Population* tab, then use the data in the *Financial Highlights* section in the *Compustat Reports* area. Calculate the plowback ratio for each firm.

a. Compute the sustainable growth rate for each firm, $g = b \times ROE$, where b is the firm's plowback ratio.

b. Compare the growth rates computed in part (*a*) with the P/E ratios of the firms. (It would be useful to plot P/E against g in a scatter diagram. This is easy to do in Excel.) Is there a relationship between g and P/E?

c. What is the average PEG ratio for the firms in your sample? How much variation is there across firms?

d. Find the price-to-book, price-to-sales, and price-to-cash flow ratios for each firm in your sample. Plot a scatter diagram of P/E against these three ratios. What do you conclude?

e. Based on the 5-year historical growth rate of earnings per share for each firm, how is the actual rate of the firm's earnings growth correlated with its sustainable growth rate that you computed in part (*a*)?

f. What factors might affect the future growth rate of earnings? Which of these might be foreseen by investors? Which would be unpredictable?

2. Use the data from Market Insight (**www.mhhe.com/edumarketinsight**) to do the following:

a. Estimate the intrinsic value of one firm from the *Population* sample. You will need to calculate the firm's beta from the historical return series, which is available in the *Excel Analytics* section, under the *Monthly Adjusted Prices* link in the *Market Data* section. Use the monthly returns for the firm and for the S&P 500 index for all months provided. You will also need to make reasonable judgments about the market risk premium, the firm's long-term growth rate based on recent profitability, and the firm's plowback ratio.

b. How does the intrinsic value that you calculated compare to the stock's current price? Is the stock overvalued, undervalued, or correctly priced according to your estimate?

c. How sensitive is your estimate to the assumptions you made? Which assumptions are most critical? You can test this by changing various inputs and checking the results.

d. Redo your analysis using a two-stage growth model and then a three-stage growth model. You will need to make reasonable assumptions about the future growth rates of dividends. Compare the values derived from all three models. Which estimate seems to be most reasonable? Why?

E-Investments

Equity Valuation

Go to the MoneyCentral Investor page at **moneycentral.msn.com/investor/home.asp.** Use the *Research Wizard* function under *Guided Research* to obtain fundamentals, price history, price target, catalysts, and comparison for Wal-Mart (WMT). For comparison, use Target (TGT), BJ's Wholesale Club (BJ), and the Industry.

1. What has been the 1-year sales and income growth for Wal-Mart?
2. What has been the company's 5-year profit margin? How does that compare with the other two firms' profit margins and the industry's profit margin?
3. What have been the percentage price changes for the last 3, 6, and 12 months? How do they compare with the other firms' price changes and the industry's price changes?
4. What are the estimated high and low prices for Wal-Mart for the coming year based on its current P/E multiple?
5. Compare the price performance of Wal-Mart with that of Target and BJ's. Which of the companies appears to be the most expensive in terms of current earnings? Which of the companies is the least expensive in terms of current earnings?
6. What are the firms' ***Stock Scouter Ratings***? How are these ratings interpreted?

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SOLUTIONS TO CONCEPT CHECKS

1. *a.* Dividend yield = \$2.15/\$50 = 4.3%.

   Capital gains yield = (59.77 − 50)/50 = 19.54%.

   Total return = 4.3% + 19.54% = 23.84%.

   b. $k = 6\% + 1.15(14\% - 6\%) = 15.2\%$.

   c. $V_0 = (\$2.15 + \$59.77)/1.152 = \$53.75$, which exceeds the market price. This would indicate a "buy" opportunity.

2. *a.* $D_1/(k - g) = \$2.15/(.152 - .112) = \$53.75$.

   b. $P_1 = P_0(1 + g) = \$53.75(1.112) = \$59.77$.

   c. The expected capital gain equals \$59.77 − \$53.75 = \$6.02, for a percentage gain of 11.2%. The dividend yield is $D_1/P_0 = 2.15/53.75 = 4\%$, for a holding-period return of 4% + 11.2% = 15.2%.

3. *a.* $g = \text{ROE} \times b = 20\% \times .60 = 12\%$.

   $D_1 = .4 \times E_1 = .4 \times \$5 = \$2$.

   $P_0 = 2/(.125 - .12) = 400$.

   b. When the firm invests in projects with ROE less than k, its stock price falls. If $b = 0.60$, then $g = 10\% \times 0.60 = 6\%$ and $P_0 = \$2/(0.125 - 0.06) = \$30.77$. In contrast, if $b = 0$, then $P_0 = \$5/0.125 = \$40$.

4. $$V_{2007} = \frac{.77}{(1.117)} + \frac{.88}{(1.117)^2} + \frac{.99}{(1.117)^3} + \frac{1.10 + P_{2011}}{(1.117)^4}$$

Now compute the sales price in 2011 using the constant-growth dividend discount model. The growth rate will be $g = \text{ROE} \times b = 12\% \times .74 = 8.88\%$.

$$P_{2011} = \frac{1.10 \times (1+g)}{k-g} = \frac{\$1.10 \times 1.0888}{.117 - .0888} = \$42.47$$

Therefore, $V_{2007} = \$30.09$.

5. *a.* ROE = 12%.

$b = \$.50/\$2.00 = .25$.

$g = \text{ROE} \times b = 12\% \times .25 = 3\%$.

$P_0 = D_1/(k - g) = \$1.50/(.10 - .03) = \$21.43$.

$P_0/E_1 = \$21.43/\$2.00 = 10.71$.

b. If $b = .4$, then $.4 \times \$2 = \$.80$ would be reinvested and the remainder of earnings, or \$1.20, would be paid as dividends.

$g = 12\% \times .4 = 4.8\%$.

$P_0 = D_1/(k - g) = \$1.20/(.10 - .048) = \$23.08$.

$P_0/E_1 = \$23.08/\$2.00 = 11.54$.

GLOSSARY

A

abnormal return Return on a stock beyond what would be predicted by market movements alone. Cumulative abnormal return (CAR) is the total abnormal return for the period surrounding an announcement or the release of information.

accounting earnings Earnings of a firm as reported on its income statement.

acid test ratio See quick ratio.

active management Attempts to achieve portfolio returns more than commensurate with risk, either by forecasting broad market trends or by identifying particular mispriced sectors of a market or securities in a market.

active portfolio In the context of the Treynor-Black model, the portfolio formed by mixing analyzed stocks of perceived nonzero alpha values. This portfolio is ultimately mixed with the passive market index portfolio.

adjusted alphas Forecasts for alpha that are modulated to account for statistical imprecision in the analyst's estimate.

agency problem Conflicts of interest among stockholders, bondholders, and managers.

alpha The abnormal rate of return on a security in excess of what would be predicted by an equilibrium model like CAPM or APT.

American depository receipts (ADRs) Domestically traded securities representing claims to shares of foreign stocks.

American option An American option can be exercised before and up to its expiration date. Compare with a *European option*, which can be exercised only on the expiration date.

announcement date Date on which particular news concerning a given company is announced to the public. Used in *event studies*, which researchers use to evaluate the economic impact of events of interest.

annual percentage rate (APR) Interest rate is annualized using simple rather than compound interest.

anomalies Patterns of returns that seem to contradict the efficient market hypothesis.

appraisal ratio The signal-to-noise ratio of an analyst's forecasts. The ratio of alpha to residual standard deviation.

arbitrage A zero-risk, zero-net investment strategy that still generates profits.

arbitrage pricing theory An asset pricing theory that is derived from a factor model, using diversification and arbitrage arguments. The theory describes the relationship between expected returns on securities, given that there are no opportunities to create wealth through risk-free arbitrage investments.

asked price The price at which a dealer will sell a security.

asset allocation Choosing among broad asset classes such as stocks versus bonds.

at the money When the exercise price and asset price of an option are equal.

auction market A market where all traders in a good meet at one place to buy or sell an asset. The NYSE is an example.

average collection period, or **days' receivables** The ratio of accounts receivable to sales, or the total amount of credit extended per dollar of daily sales (average AR/sales × 365).

B

backfill bias Bias in the average returns of a sample of funds induced by including past returns on funds that entered the sample only if they happened to be successful.

balance sheet An accounting statement of a firm's financial position at a specified time.

bank discount yield An annualized interest rate assuming simple interest, a 360-day year, and using the face value of the security rather than purchase price to compute return per dollar invested.

banker's acceptance A money market asset consisting of an order to a bank by a customer to pay a sum of money at a future date.

baseline forecasts Forecast of security returns derived from the assumption that the market is in equilibrium where current prices reflect all available information.

basis The difference between the futures price and the spot price.

basis risk Risk attributable to uncertain movements in the spread between a futures price and a spot price.

behavioral finance Models of financial markets that emphasize implications of psychological factors affecting investor behavior.

benchmark error Use of an inappropriate proxy for the true market portfolio.

benchmark portfolio Portfolio against which a manager is to be evaluated.

beta The measure of the systematic risk of a security. The tendency of a security's returns to respond to swings in the broad market.

bid–asked spread The difference between a dealer's bid and asked price.

bid price The price at which a dealer is willing to purchase a security.

binomial model An option-valuation model predicated on the assumption that stock prices can move to only two values over any short time period.

Black-Scholes formula An equation to value a call option that uses the stock price, the exercise price, the risk-free interest rate, the time to maturity, and the standard deviation of the stock return.

block sale A transaction of more than 10,000 shares of stock.

block transactions Large transactions in which at least 10,000 shares of stock are bought or sold. Brokers or "block houses" often search directly for other large traders rather than bringing the trade to the stock exchange.

bogey The return an investment manager is compared to for performance evaluation.

bond A security issued by a borrower that obligates the issuer to make specified payments to the holder over a specific period. A *coupon bond* obligates the issuer to make interest payments called coupon payments over the life of the bond, then to repay the *face value* at maturity.

bond equivalent yield Bond yield calculated on an annual percentage rate method. Differs from effective annual yield.

bond indenture The contract between the issuer and the bondholder.

bond reconstitution Combining stripped Treasury securities to re-create the original cash flows of a Treasury bond.

bond stripping Selling bond cash flows (either coupon or principal payments) as stand-alone zero-coupon securities.

book-to-market effect The tendency for stocks of firms with high ratios of book-to-market value to generate abnormal returns.

book value An accounting measure describing the net worth of common equity according to a firm's balance sheet.

breadth The extent to which movements in the broad market index are reflected widely in movements of individual stock prices.

brokered market A market where an intermediary (a broker) offers search services to buyers and sellers.

budget deficit The amount by which government spending exceeds government revenues.

bull CD, bear CD A *bull CD* pays its holder a specified percentage of the increase in return on a specified market index while guaranteeing a minimum rate of return. A *bear CD* pays the holder a fraction of any fall in a given market index.

bullish, bearish Words used to describe investor attitudes. *Bullish* means optimistic; *bearish* means pessimistic. Also used in bull market and bear market.

bundling, unbundling A trend allowing creation of securities either by combining primitive and derivative securities into one composite hybrid or by separating returns on an asset into classes.

business cycle Repetitive cycles of recession and recovery.

C

calendar spread Buy one option, and write another with a different expiration date.

callable bond A bond that the issuer may repurchase at a given price in some specified period.

call option The right to buy an asset at a specified exercise price on or before a specified expiration date.

call protection An initial period during which a callable bond may not be called.

capital allocation decision Allocation of invested funds between risk-free assets versus the risky portfolio.

capital allocation line (CAL) A graph showing all feasible risk–return combinations of a risky and risk-free asset.

capital gains The amount by which the sale price of a security exceeds the purchase price.

capital market line (CML) A capital allocation line provided by the market index portfolio.

capital markets Includes longer-term, relatively riskier securities.

cash/bond selection Asset allocation in which the choice is between short-term cash equivalents and longer-term bonds.

cash equivalents Short-term money-market securities.

cash flow matching A form of immunization, matching cash flows from a bond portfolio with an obligation.

cash ratio Measure of liquidity of a firm. Ratio of cash and marketable securities to current liabilities.

cash settlement The provision of some futures contracts that requires not delivery of the underlying assets (as in agricultural futures) but settlement according to the cash value of the asset.

certainty equivalent rate The certain return providing the same utility as a risky portfolio.

certificate of deposit A bank time deposit.

clearinghouse Established by exchanges to facilitate transfer of securities resulting from trades. For options and futures contracts, the clearinghouse may interpose itself as a middleman between two traders.

closed-end (mutual) fund A fund whose shares are traded through brokers at market prices; the fund will not redeem shares at their net asset value. The market price of the fund can differ from the net asset value.

collar An options strategy that brackets the value of a portfolio between two bounds.

collateral A specific asset pledged against possible default on a bond. *Mortgage bonds* are backed by claims on property. *Collateral trust bonds* are backed by claims on other securities. *Equipment obligation bonds* are backed by claims on equipment.

collateralized debt obligation (CDO) A pool of loans sliced into several tranches with different levels of risk.

collateralized mortgage obligation (CMO) A mortgage pass-through security that partitions cash flows from underlying mortgages into classes called *tranches* that receive principal payments according to stipulated rules.

commercial paper Short-term unsecured debt issued by large corporations.

common stock Equities, or equity securities, issued as ownership shares in a publicly held corporation. Shareholders have voting rights and may receive dividends based on their proportionate ownership.

comparison universe The collection of money managers of similar investment style used for assessing relative performance of a portfolio manager.

complete portfolio The entire portfolio, including risky and risk-free assets.

conditional tail expectation Expectation of a random variable conditional on its falling below some threshold value. Often used as a measure of down-side risk.

confidence index Ratio of the yield of top-rated corporate bonds to the yield on intermediate-grade bonds.

conservativism Notion that investors are too slow to update their beliefs in response to new evidence.

constant-growth model A form of the dividend discount model that assumes dividends will grow at a constant rate.

contango theory Holds that the futures price must exceed the expected future spot price.

contingent claim Claim whose value is directly dependent on or is contingent on the value of some underlying assets.

contingent immunization A mixed passive-active strategy that immunizes a portfolio if necessary to guarantee a minimum acceptable return but otherwise allows active management.

convergence arbitrage A bet that two or more prices are out of alignment and that profits can be made when the prices converge back to proper relationship.

convergence property The convergence of futures prices and spot prices at the maturity of the futures contract.

convertible bond A bond with an option allowing the bondholder to exchange the bond for a specified number of shares of common stock in the firm. A *conversion ratio* specifies the number of shares. The *market conversion price* is the current value of the shares for which the bond may be exchanged. The *conversion premium* is the excess of the bond's value over the conversion price.

convexity The curvature of the price-yield relationship of a bond.

corporate bonds Long-term debt issued by private corporations typically paying semiannual coupons and returning the face value of the bond at maturity.

correlation coefficient A statistic in which the covariance is scaled to a value between -1 (perfect negative correlation) and $+1$ (perfect positive correlation).

cost-of-carry relationship See spot-futures parity theorem.

country selection A type of active international management that measures the contribution to performance attributable to investing in the better-performing stock markets of the world.

coupon rate A bond's interest payments per dollar of par value.

covariance A measure of the degree to which returns on two risky assets move in tandem. A positive covariance means that asset returns move together. A negative covariance means they vary inversely.

covered call A combination of selling a call on a stock together with buying the stock.

covered interest arbitrage relationship See interest rate parity theorem.

credit default swap A derivative contract in which one party sells insurance concerning the credit risk of another firm.

credit enhancement Purchase of the financial guarantee of a large insurance company to raise funds.

credit risk Default risk.

cross hedge Hedging a position in one asset using futures on another commodity.

cumulative abnormal return See abnormal return.

currency selection Asset allocation in which the investor chooses among investments denominated in different currencies.

current ratio A ratio representing the ability of the firm to pay off its current liabilities by liquidating current assets (current assets/current liabilities).

current yield A bond's annual coupon payment divided by its price. Differs from yield to maturity.

cyclical industries Industries with above-average sensitivity to the state of the economy.

D

data mining Sorting through large amounts of historical data to uncover systematic patterns that can be exploited.

day order A buy order or a sell order expiring at the close of the trading day.

days' receivables See average collection period.

dealer market A market where traders specializing in particular commodities buy and sell assets for their own accounts. The OTC market is an example.

debenture or unsecured bond A bond not backed by specific collateral.

debt securities Bonds; also called fixed-income securities.

dedication strategy Refers to multiperiod cash flow matching.

default premium A differential in promised yield that compensates the investor for the risk inherent in purchasing a corporate bond that entails some risk of default.

defensive industries Industries with little sensitivity to the state of the economy.

deferred annuities Tax-advantaged life insurance product. Deferred annuities offer deferral of taxes with the option of withdrawing one's funds in the form of a life annuity.

defined benefit plans Pension plans in which retirement benefits are set according to a fixed formula.

defined contribution plans Pension plans in which the employer is committed to making contributions according to a fixed formula.

degree of operating leverage Percentage change in profits for a 1% change in sales.

delta (of option) See hedge ratio.

delta neutral The value of the portfolio is not affected by changes in the value of the asset on which the options are written.

demand shock An event that affects the demand for goods and services in the economy.

derivative asset/contingent claim Securities providing payoffs that depend on or are contingent on the values of other assets such as commodity prices, bond and stock prices, or market index values. Examples are futures and options.

derivative security See primitive security.

direct search market Buyers and sellers seek each other directly and transact directly.

directional strategy Speculation that one sector or another will outperform other sectors of the market.

discount bonds Bonds selling below par value.

discretionary account An account of a customer who gives a broker the authority to make buy and sell decisions on the customer's behalf.

diversifiable risk Risk attributable to firm-specific risk, or nonmarket risk. *Nondiversifiable* risk refers to systematic or market risk.

diversification Spreading a portfolio over many investments to avoid excessive exposure to any one source of risk.

dividend discount model (DDM) A formula stating that the intrinsic value of a firm is the present value of all expected future dividends.

dividend payout ratio Percentage of earnings paid out as dividends.

dividend yield The percent rate of return provided by a stock's dividend payments.

dollar-weighted rate of return The internal rate of return on an investment.

doubling option A sinking fund provision that may allow repurchase of twice the required number of bonds at the sinking fund call price.

Dow theory A technical analysis technique that seeks to discern long- and short-term trends in security prices.

DuPont system Decomposition of firm profitability measures into the underlying factors that determine such profitability.

duration A measure of the average life of a bond, defined as the weighted average of the times until each payment is made, with weights proportional to the present value of the payment.

dynamic hedging Constant updating of hedge positions as market conditions change.

E

EAFE index The European, Australian, Far East index, computed by Morgan Stanley, is a widely used index of non-U.S. stocks.

earnings management The practice of using flexibility in accounting rules to improve the apparent profitability of the firm.

earnings retention ratio Plowback ratio.

earnings yield The ratio of earnings to price, E/P.

economic earnings The real flow of cash that a firm could pay out forever in the absence of any change in the firm's productive capacity.

economic value added (EVA) The spread between ROA and cost of capital multiplied by the capital invested in the firm. It measures the dollar value of the firm's return in excess of its opportunity cost.

effective annual rate (EAR) Interest rate is annualized using compound rather than simple interest.

effective annual yield Annualized interest rate on a security computed using compound interest techniques.

effective duration Percentage change in bond price per change in the level of market interest rates.

efficient diversification The organizing principle of modern portfolio theory, which maintains that any risk-averse investor will search for the highest expected return for any level of portfolio risk.

efficient frontier Graph representing a set of portfolios that maximize expected return at each level of portfolio risk.

efficient frontier of risky assets The portion of the minimum-variance frontier that lies above the global minimum-variance portfolio.

efficient market hypothesis The prices of securities fully reflect available information. Investors buying securities in an efficient market should expect to obtain an equilibrium rate of return. Weak-form EMH asserts that stock prices already reflect all information contained in the history of past prices. The semistrong-form hypothesis asserts that stock prices already reflect all publicly available information. The strong-form hypothesis asserts that stock prices reflect all relevant information including insider information.

elasticity (of an option) Percentage change in the value of an option accompanying a 1% change in the value of a stock.

electronic communication network (ECN) A computer-operated trading network offering an alternative to formal stock exchanges or dealer markets for trading securities.

endowment funds Organizations chartered to invest money for specific purposes.

equities Ownership shares in a firm.

equity Ownership in a firm. Also, the net worth of a margin account.

equivalent taxable yield The pretax yield on a taxable bond providing an after-tax yield equal to the rate on a tax-exempt municipal bond.

Eurodollars Dollar-denominated deposits at foreign banks or foreign branches of American banks.

European, Australian, Far East (EAFE) index A widely used index of non-U.S. stocks computed by Morgan Stanley.

European option A European option can be exercised only on the expiration date. Compare with an American option, which can be exercised before, up to, and on its expiration date.

event study Research methodology designed to measure the impact of an event of interest on stock returns.

event tree Depicts all possible sequences of events.

excess return Rate of return in excess of the risk-free rate.

exchange rate Price of a unit of one country's currency in terms of another country's currency.

exchange rate risk The uncertainty in asset returns due to movements in the exchange rates between the dollar and foreign currencies.

exchange-traded funds (ETFs) Offshoots of mutual funds that allow investors to trade portfolios of securities just as they do shares of stock.

exchanges National or regional auction markets providing a facility for members to trade securities. A seat is a membership on an exchange.

exercise or strike price Price set for calling (buying) an asset or putting (selling) an asset.

expectations hypothesis (of interest rates) Theory that forward interest rates are unbiased estimates of expected future interest rates.

expected return The probability-weighted average of the possible outcomes.

expected return–beta relationship Implication of the CAPM that security risk premiums (expected excess returns) will be proportional to beta.

F

face value The maturity value of a bond.

factor beta Sensitivity of security returns to changes in a systematic factor. Alternatively, factor loading; factor sensitivity.

factor loading See factor beta.

factor model A way of decomposing the factors that influence a security's rate of return into common and firm-specific influences.

factor portfolio A well-diversified portfolio constructed to have a beta of 1.0 on one factor and a beta of 0 on any other factor.

factor sensitivity See factor beta.

fair game An investment prospect that has a zero risk premium.

fair value accounting Use of current values rather than historic cost in the firm's financial statements.

federal funds Funds in a bank's reserve account.

FIFO The first-in first-out accounting method of inventory valuation.

financial assets Financial assets such as stocks and bonds are claims to the income generated by real assets or claims on income from the government.

financial engineering Creating and designing securities with custom-tailored characteristics.

financial intermediary An institution such as a bank, mutual fund, investment company, or insurance company that serves to connect the household and business sectors so households can invest and businesses can finance production.

firm-specific risk See diversifiable risk.

first-pass regression A time series regression to estimate the betas of securities or portfolios.

fiscal policy The use of government spending and taxing for the specific purpose of stabilizing the economy.

fixed annuities Annuity contracts in which the insurance company pays a fixed dollar amount of money per period.

fixed-charge coverage ratio Ratio of earnings to all fixed cash obligations, including lease payments and sinking fund payments.

fixed-income security A security such as a bond that pays a specified cash flow over a specific period.

flight to quality Describes the tendency of investors to require larger default premiums on investments under uncertain economic conditions.

floating-rate bond A bond whose interest rate is reset periodically according to a specified market rate.

forced conversion Use of a firm's call option on a callable convertible bond when the firm knows that bondholders will exercise their option to convert.

forecasting records The historical record of the forecasting errors of a security analyst.

foreign exchange market An informal network of banks and brokers that allows customers to enter forward contracts to purchase or sell currencies in the future at a rate of exchange agreed upon now.

foreign exchange swap An agreement to exchange stipulated amounts of one currency for another at one or more future dates.

forward contract An agreement calling for future delivery of an asset at an agreed-upon price. Also see futures contract.

forward interest rate Rate of interest for a future period that would equate the total return of a long-term bond with that of a strategy of rolling over shorter-term bonds. The forward rate is inferred from the term structure.

framing Decisions are affected by how choices are described, for example, whether uncertainty is posed as potential gains from a low baseline level, or as losses from a higher baseline value.

fully diluted earnings per share Earnings per share expressed as if all outstanding convertible securities and warrants have been exercised.

fundamental analysis Research to predict stock value that focuses on such determinants as earnings and dividends prospects, expectations for future interest rates, and risk evaluation of the firm.

fundamental risk Risk that even if an asset is mispriced, there is still no arbitrage opportunity, because the mispricing can widen before price eventually converges to intrinsic value.

funds of funds Hedge funds that invest in several other hedge funds.

futures contract Obliges traders to purchase or sell an asset at an agreed-upon price on a specified future date. The long position is held by the trader who commits to purchase. The short position is held by the trader who commits to sell. Futures differ from forward contracts in their standardization, exchange trading, margin requirements, and daily settling (marking to market).

futures option The right to enter a specified futures contract at a futures price equal to the stipulated exercise price.

futures price The price at which a futures trader commits to make or take delivery of the underlying asset.

G

gamma The curvature of an option pricing function (as a function of the value of the underlying asset).

geometric average The *n*th root of the product of *n* numbers. It is used to measure the compound rate of return over time.

globalization Tendency toward a worldwide investment environment, and the integration of national capital markets.

gross domestic product (GDP) The market value of goods and services produced over time including the income of foreign corporations and foreign residents working in the United States, but excluding the income of U.S. residents and corporations overseas.

H

hedge fund A private investment pool, open to institutional or wealthy investors, that is largely exempt from SEC regulation and can pursue more speculative policies than mutual funds.

hedge ratio (for an option) The number of stocks required to hedge against the price risk of holding one option. Also called the option's delta.

hedging Investing in an asset to reduce the overall risk of a portfolio.

hedging demands Demands for securities to hedge particular sources of consumption risk, beyond the usual mean variance diversification motivation.

high water mark The previous value of a portfolio that must be reattained before a hedge fund can charge incentive fees.

holding-period return The rate of return over a given period.

homogenous expectations The assumption that all investors use the same expected returns and covariance matrix of security returns as inputs in security analysis.

horizon analysis Forecasting the realized compound yield over various holding periods or investment horizons.

I

illiquidity Difficulty, cost, and/or delay in selling an asset on short notice without offering substantial price concessions.

illiquidity cost Costs due to imperfect liquidity of some security.

illiquidity premium Extra expected return as compensation for limited liquidity.

immunization A strategy that matches durations of assets and liabilities so as to make net worth unaffected by interest rate movements.

implied volatility The standard deviation of stock returns that is consistent with an option's market value.

incentive fee A fee charged by hedge funds equal to a share of any investment returns beyond a stipulated benchmark performance.

in the money In the money describes an option whose exercise would produce profits. Out of the money describes an option where exercise would not be profitable.

income beneficiary One who receives income from a trust.

income statement A financial statement showing a firm's revenues and expenses during a specified period.

indenture The document defining the contract between the bond issuer and the bondholder.

index arbitrage An investment strategy that exploits divergences between actual futures prices and their theoretically correct parity values to make a profit.

index fund A mutual fund holding shares in proportion to their representation in a market index such as the S&P 500.

index model A model of stock returns using a market index such as the S&P 500 to represent common or systematic risk factors.

index option A call or put option based on a stock market index.

indifference curve A curve connecting all portfolios with the same utility according to their means and standard deviations.

industry life cycle Stages through which firms typically pass as they mature.

inflation The rate at which the general level of prices for goods and services is rising.

information ratio Ratio of alpha to the standard deviation of diversifiable risk.

initial public offering Stock issued to the public for the first time by a formerly privately owned company.

input list List of parameters such as expected returns, variances, and covariances necessary to determine the optimal risky portfolio.

inside information Nonpublic knowledge about a corporation possessed by corporate officers, major owners, or other individuals with privileged access to information about a firm.

insider trading Trading by officers, directors, major stockholders, or others who hold private inside information allowing them to benefit from buying or selling stock.

insurance principle The law of averages. The average outcome for many independent trials of an experiment will approach the expected value of the experiment.

interest coverage ratio Measure of financial leverage. Earnings before interest and taxes as a multiple of interest expense.

interest coverage ratio, or **times interest earned** A financial leverage measure (EBIT divided by interest expense).

interest rate The number of dollars earned per dollar invested per period.

interest rate parity relationship (theorem) The spot-futures exchange rate relationship that prevails in well-functioning markets.

interest rate swaps A method to manage interest rate risk where parties trade the cash flows corresponding to different securities without actually exchanging securities directly.

intermarket spread swap Switching from one segment of the bond market to another (from Treasuries to corporates, for example).

intrinsic value (of a firm) The present value of a firm's expected future net cash flows discounted by the required rate of return.

intrinsic value of an option Stock price minus exercise price, or the profit that could be attained by immediate exercise of an in-the-money option.

inventory turnover ratio Cost of goods sold as a multiple of average inventory.

investment Commitment of current resources in the expectation of deriving greater resources in the future.

investment bankers Firms specializing in the sale of new securities to the public, typically by underwriting the issue.

investment company Firm managing funds for investors. An investment company may manage several mutual funds.

investment-grade bond Bond rated BBB and above or Baa and above. Lower-rated bonds are classified as speculative-grade or junk bonds.

investment horizon Time horizon for purposes of investment decisions.

investment portfolio Set of securities chosen by an investor.

J

Jensen's measure The alpha of an investment.

junk bond See speculative-grade bond.

K

kurtosis Measure of the fatness of the tails of a probability distribution. Indicates probability of observing extreme high or low values.

L

Law of One Price The rule stipulating that equivalent securities or bundles of securities must sell at equal prices to preclude arbitrage opportunities.

leading economic indicators Economic series that tend to rise or fall in advance of the rest of the economy.

leverage ratio Ratio of debt to total capitalization of a firm.

LIFO The last-in first-out accounting method of valuing inventories.

limited liability The fact that shareholders have no personal liability to the creditors of the corporation in the event of bankruptcy.

limit order An order specifying a price at which an investor is willing to buy or sell a security.

liquidation value Net amount that could be realized by selling the assets of a firm after paying the debt.

liquidity Liquidity refers to the speed and ease with which an asset can be converted to cash.

liquidity preference theory Theory that the forward rate exceeds expected future interest rates.

liquidity premium Forward rate minus expected future short interest rate.

load Sales charge on the purchase of some mutual funds.

load fund A mutual fund with a sales commission, or load.

lock-up period Period in which investors cannot redeem investments in the hedge fund.

lognormal distribution The log of the variable has a normal (bell-shaped) distribution.

London Interbank Offered Rate (LIBOR) Rate that most creditworthy banks charge one another for large loans of Eurodollars in the London market.

long position hedge Hedging the future cost of a purchase by taking a long futures position to protect against changes in the price of the asset.

lower partial standard deviation Standard deviation computed using only the portion of the probability distribution below the mean of the variable.

M

Macaulay's duration Effective maturity of bond, equal to weighted average of the times until each payment, with weights proportional to the present value of the payment.

maintenance, or variation, margin An established value below which a trader's margin cannot fall. Reaching the maintenance margin triggers a margin call.

margin Describes securities purchased with money borrowed from a broker. Current maximum margin is 50%.

market–book-value ratio Ratio of price per share to book value per share.

market capitalization rate The market-consensus estimate of the appropriate discount rate for a firm's cash flows.

market model Another version of the index model that breaks down return uncertainty into systematic and nonsystematic components.

market neutral A strategy designed to exploit relative mispricing within a market, but which is hedged to avoid taking a stance on the direction of the broad market.

market or systematic risk, firm-specific risk Market risk is risk attributable to common macroeconomic factors. Firm-specific risk reflects risk peculiar to an individual firm that is independent of market risk.

market order A buy or sell order to be executed immediately at current market prices.

market portfolio The portfolio for which each security is held in proportion to its market value.

market price of risk A measure of the extra return, or risk premium, that investors demand to bear risk. The reward-to-risk ratio of the market portfolio.

market risk See systematic risk.

market segmentation or preferred habitat theory The theory that long- and short-maturity bonds are traded in essentially distinct or segmented markets and that prices in one market do not affect those in the other.

market timer An investor who speculates on broad market moves rather than on specific securities.

market timing Asset allocation in which the investment in the market is increased if one forecasts that the market will outperform T-bills.

market-value-weighted index An index of a group of securities computed by calculating a weighted average of the returns of each security in the index, with weights proportional to outstanding market value.

marking to market Describes the daily settlement of obligations on futures positions.

mean-variance analysis Evaluation of risky prospects based on the expected value and variance of possible outcomes.

mean-variance criterion The selection of portfolios based on the means and variances of their returns. The choice of the higher expected return portfolio for a given level of variance or the lower variance portfolio for a given expected return.

mental accounting Individuals mentally segregate assets into independent accounts rather than viewing them as part of a unified portfolio.

minimum-variance frontier Graph of the lowest possible portfolio variance that is attainable for a given portfolio expected return.

minimum-variance portfolio The portfolio of risky assets with lowest variance.

modern portfolio theory (MPT) Principles underlying analysis and evaluation of rational portfolio choices based on risk–return trade-offs and efficient diversification.

modified duration Macaulay's duration divided by 1 + yield to maturity. Measures interest rate sensitivity of bond.

momentum effect The tendency of poorly performing stocks and well-performing stocks in one period to continue that abnormal performance in following periods.

monetary policy Actions taken by the Board of Governors of the Federal Reserve System to influence the money supply or interest rates.

money market Includes short-term, highly liquid, and relatively low-risk debt instruments.

mortality tables Tables of probability that individuals of various ages will die within a year.

mortgage-backed security Ownership claim in a pool of mortgages or an obligation that is secured by such a pool. Also called a *pass-through*, because payments are passed along from the mortgage originator to the purchaser of the mortgage-backed security.

multifactor CAPM Generalization of the basic CAPM that accounts for extra-market hedging demands.

multifactor models Model of security returns positing that returns respond to several systematic factors.

municipal bonds Tax-exempt bonds issued by state and local governments, generally to finance capital improvement projects. General obligation bonds are backed by the general taxing power of the issuer. Revenue bonds are backed by the proceeds from the project or agency they are issued to finance.

mutual fund A firm pooling and managing funds of investors.

mutual fund theorem A result associated with the CAPM, asserting that investors will choose to invest their entire risky portfolio in a market-index mutual fund.

N

NAICS codes North American Industrial Classification System codes that use numerical values to identify industries.

naked option writing Writing an option without an offsetting stock position.

NASDAQ The automated quotation system for the OTC market, showing current bid–asked prices for thousands of stocks.

neglected-firm effect That investments in stock of less well-known firms have generated abnormal returns.

net asset value (NAV) The value of each share expressed as assets minus liabilities on a per-share basis.

nominal interest rate The interest rate in terms of nominal (not adjusted for purchasing power) dollars.

nondirectional strategy A position designed to exploit temporary misalignments in relative pricing. Typically involves a long position in one security hedged with a short position in a related security.

nondiversifiable risk See systematic risk.

nonsystematic risk Nonmarket or firm-specific risk factors that can be eliminated by diversification. Also called unique risk or diversifiable risk. Systematic risk refers to risk factors common to the entire economy.

normal distribution Bell-shaped probability distribution that characterizes many natural phenomena.

notional principal Principal amount used to calculate swap payments.

O

on the run Recently issued bond, selling at or near par value.

on-the-run yield curve Relationship between yield to maturity and time to maturity for newly issued bonds selling at par.

open-end (mutual) fund A fund that issues or redeems its own shares at their net asset value (NAV).

open interest The number of futures contracts outstanding.

optimal risky portfolio An investor's best combination of risky assets to be mixed with safe assets to form the complete portfolio.

option elasticity The percentage increase in an option's value given a 1% change in the value of the underlying security.

original issue discount bond A bond issued with a low coupon rate that sells at a discount from par value.

out of the money Out of the money describes an option where exercise would not be profitable. In the money describes an option where exercise would produce profits.

over-the-counter market An informal network of brokers and dealers who negotiate sales of securities (not a formal exchange).

P

pairs trading Stocks are paired up based on underlying similarities, and longshort positions are established to exploit any relative mispricing between each pair.

par value The face value of the bond.

passive investment strategy See passive management.

passive management Buying a well-diversified portfolio to represent a broad-based market index without attempting to search out mispriced securities.

passive portfolio A market index portfolio.

passive strategy See passive management.

pass-through security Pools of loans (such as home mortgage loans) sold in one package. Owners of pass-throughs receive all principal and interest payments made by the borrowers.

peak The transition from the end of an expansion to the start of a contraction.

P/E effect That portfolios of low P/E stocks have exhibited higher average risk-adjusted returns than high P/E stocks.

personal trust An interest in an asset held by a trustee for the benefit of another person.

plowback ratio The proportion of the firm's earnings that is reinvested in the business (and not paid out as dividends). The plowback ratio equals 1 minus the dividend payout ratio.

political risk Possibility of the expropriation of assets, changes in tax policy, restrictions on the exchange of foreign currency for domestic currency, or other changes in the business climate of a country.

portable alpha; alpha transfer A strategy in which you invest in positive alpha positions, then hedge the systematic risk of that investment, and, finally, establish market exposure where you want it by using passive indexes.

portfolio insurance The practice of using options or dynamic hedge strategies to provide protection against investment losses while maintaining upside potential.

portfolio management Process of combining securities in a portfolio tailored to the investor's preferences and needs, monitoring that portfolio, and evaluating its performance.

portfolio opportunity set The expected return–standard deviation pairs of all portfolios that can be constructed from a given set of assets.

posterior distribution Probability distribution for a variable after adjustment for empirical evidence on its likely value.

preferred habitat theory Holds that investors prefer specific maturity ranges but can be induced to switch if risk premiums are sufficient.

preferred stock Nonvoting shares in a corporation, paying a fixed or variable stream of dividends.

premium The purchase price of an option.

premium bonds Bonds selling above par value.

present value of growth opportunities (PVGO) Net present value of a firm's future investments.

price–earnings multiple See price–earnings ratio.

price–earnings ratio The ratio of a stock's price to its earnings per share. Also referred to as the P/E multiple.

price value of a basis point The change in the value of a fixed-income asset resulting from a 1 basis point change in the asset's yield to maturity.

price-weighted average Weighted average with weights proportional to security prices rather than total capitalization.

primary market New issues of securities are offered to the public here.

primitive security, derivative security A *primitive security* is an instrument such as a stock or bond for which payments depend only on the financial status of its issuer. A *derivative* *security* is created from the set of primitive securities to yield returns that depend on factors beyond the characteristics of the issuer and that may be related to prices of other assets.

principal The outstanding balance on a loan.

prior distribution Probability distribution for a variable before adjusting for empirical evidence on its likely value.

private placement Primary offering in which shares are sold directly to a small group of institutional or wealthy investors.

profit margin See return on sales.

program trading Coordinated buy orders and sell orders of entire portfolios, usually with the aid of computers, often to achieve index arbitrage objectives.

prospect theory Behavioral (as opposed to rational) model of investor utility. Investor utility depends on changes in wealth rather than levels of wealth.

prospectus A final and approved registration statement including the price at which the security issue is offered.

protective covenant A provision specifying requirements of collateral, sinking fund, dividend policy, etc., designed to protect the interests of bondholders.

protective put Purchase of stock combined with a put option that guarantees minimum proceeds equal to the put's exercise price.

proxy An instrument empowering an agent to vote in the name of the shareholder.

prudent investor rule An investment manager must act in accord with the actions of a hypothetical prudent investor.

pseudo-American call option value The maximum of the value derived by assuming that an option will be held until expiration and the value derived by assuming that the option will be exercised just before an ex-dividend date.

public offering, private placement A *public offering* consists of bonds sold in the primary market to the general public; a *private placement* is sold directly to a limited number of institutional investors.

pure plays Bets on particular mispricing across two or more securities, with extraneous sources of risk such as general market exposure hedged away.

pure yield curve Refers to the relationship between yield to maturity and time to maturity for zero-coupon bonds.

pure yield pickup swap Moving to higher-yield bonds.

put bond A bond that the holder may choose either to exchange for par value at some date or to extend for a given number of years.

put/call ratio Ratio of put options to call options outstanding on a stock.

put-call parity theorem An equation representing the proper relationship between put and call prices. Violation of parity allows arbitrage opportunities.

put option The right to sell an asset at a specified exercise price on or before a specified expiration date.

Q

quality of earnings The realism and conservatism of the earnings number and the extent to which we might expect the reported level of earnings to be sustained.

quick ratio A measure of liquidity similar to the current ratio except for exclusion of inventories (cash plus receivables divided by current liabilities).

R

random walk Describes the notion that stock price changes are random and unpredictable.

rate anticipation swap A switch made in response to forecasts of interest rates.

real assets, financial assets *Real assets* are land, buildings, and equipment that are used to produce goods and services. *Financial assets* are claims such as securities to the income generated by real assets.

real interest rate The excess of the interest rate over the inflation rate. The growth rate of purchasing power derived from an investment.

realized compound return Yield assuming that coupon payments are invested at the going market interest rate at the time of their receipt and rolled over until the bond matures.

rebalancing Realigning the proportions of assets in a portfolio as needed.

registered bond A bond whose issuer records ownership and interest payments. Differs from a bearer bond, which is traded without record of ownership and whose possession is its only evidence of ownership.

regression equation An equation that describes the average relationship between a dependent variable and a set of explanatory variables.

regret avoidance Notion from behavioral finance that individuals who make decisions that turn out badly will have more regret when that decision was more unconventional.

reinvestment rate risk The uncertainty surrounding the cumulative future value of reinvested bond coupon payments.

REIT Real estate investment trust, which is similar to a closed-end mutual fund. REITs invest in real estate or loans secured by real estate and issue shares in such investments.

remainderman One who receives the principal of a trust when it is dissolved.

replacement cost Cost to replace a firm's assets. "Reproduction" cost.

representativeness bias People seem to believe that a small sample is just as representative of a broad population as a large one and therefore infer patterns too quickly.

repurchase agreements (repos) Short-term, often overnight, sales of government securities with an agreement to repurchase the securities at a slightly higher price. A *reverse* *repo* is a purchase with an agreement to resell at a specified price on a future date.

residual claim Refers to the fact that shareholders are at the bottom of the list of claimants to assets of a corporation in the event of failure or bankruptcy.

residual income See economic value added (EVA).

residuals Parts of stock returns not explained by the explanatory variable (the market-index return). They measure the impact of firm-specific events during a particular period.

resistance level A price level above which it is supposedly difficult for a stock or stock index to rise.

return on assets (ROA) A profitability ratio; earnings before interest and taxes divided by total assets.

return on equity (ROE) An accounting ratio of net profits divided by equity.

return on sales (ROS), or profit margin The ratio of operating profits per dollar of sales (EBIT divided by sales).

reversal effect The tendency of poorly performing stocks and well-performing stocks in one period to experience reversals in following periods.

reversing trade Entering the opposite side of a currently held futures position to close out the position.

reward-to-volatility ratio Ratio of excess return to portfolio standard deviation.

riding the yield curve Buying long-term bonds in anticipation of capital gains as yields fall with the declining maturity of the bonds.

risk arbitrage Speculation on perceived mispriced securities, usually in connection with merger and acquisition targets.

risk-averse, risk-neutral, risk lover A *risk-averse* investor will consider risky portfolios only if they provide compensation for risk via a risk premium. A *risk-neutral* investor finds the level of risk irrelevant and considers only the expected return of risk prospects. A *risk lover* is willing to accept lower expected returns on prospects with higher amounts of risk.

risk-free asset An asset with a certain rate of return; often taken to be short-term T-bills.

risk-free rate The interest rate that can be earned with certainty.

risk lover See risk-averse.

risk-neutral See risk-averse.

risk premium An expected return in excess of that on risk-free securities. The premium provides compensation for the risk of an investment.

risk–return trade-off If an investor is willing to take on risk, there is the reward of higher expected returns.

risky asset An asset with an uncertain rate of return.

S

scatter diagram Plot of returns of one security versus returns of another security. Each point represents one pair of returns for a given holding period.

seasoned new issue Stock issued by companies that already have stock on the market.

secondary market Already existing securities are bought and sold on the exchanges or in the OTC market.

second-pass regression A cross-sectional regression of portfolio returns on betas. The estimated slope is the measurement of the reward for bearing systematic risk during the period.

sector rotation An investment strategy which entails shifting the portfolio into industry sectors that are forecast to outperform others based on macroeconomic forecasts.

securitization Pooling loans for various purposes into standardized securities backed by those loans, which can then be traded like any other security.

security analysis Determining correct value of a security in the marketplace.

security characteristic line A plot of the excess return on a security over the risk-free rate as a function of the excess return on the market.

security market line Graphical representation of the expected return–beta relationship of the CAPM.

security selection See security selection decision.

security selection decision Choosing the particular securities to include in a portfolio.

semistrong-form EMH See efficient market hypothesis.

separation property The property that portfolio choice can be separated into two independent tasks: (1) determination of the optimal risky portfolio, which is a purely technical problem, and (2) the personal choice of the best mix of the risky portfolio and the risk-free asset.

Sharpe's measure Reward-to-volatility ratio; ratio of portfolio excess return to standard deviation.

shelf registration Advance registration of securities with the SEC for sale up to 2 years following initial registration.

short position or hedge Protecting the value of an asset held by taking a short position in a futures contract.

short rate A one-period interest rate.

short sale The sale of shares not owned by the investor but borrowed through a broker and later repurchased to replace the loan. Profit is earned if the initial sale is at a higher price than the repurchase price.

single-factor model A model of security returns that acknowledges only one common factor. See factor model.

single-index model A model of stock returns that decomposes influences on returns into a systematic factor, as measured by the return on a broad market index, and firm-specific factors.

single-stock futures Futures contracts on single stock rather than an index.

sinking fund A procedure that allows for the repayment of principal at maturity by calling for the bond issuer to repurchase some proportion of the outstanding bonds either in the open market or at a special call price associated with the sinking fund provision.

skew Measure of the asymmetry of a probability distribution.

small-firm effect That investments in stocks of small firms appear to have earned abnormal returns.

soft dollars The value of research services that brokerage houses supply to investment managers "free of charge" in exchange for the investment managers' business.

specialist A trader who makes a market in the shares of one or more firms and who maintains a "fair and orderly market" by dealing personally in the stock.

speculation Undertaking a risky investment with the objective of earning a greater profit than an investment in a risk-free alternative (a risk premium).

speculative-grade bond Bond rated Ba or lower by Moody's, or BB or lower by Standard & Poor's, or an unrated bond.

spot-futures parity theorem, or cost-of-carry relationship Describes the theoretically correct relationship between spot and futures prices. Violation of the parity relationship gives rise to arbitrage opportunities.

spot rate The current interest rate appropriate for discounting a cash flow of some given maturity.

spread (futures) Taking a long position in a futures contract of one maturity and a short position in a contract of different maturity, both on the same commodity.

spread (options) A combination of two or more call options or put options on the same stock with differing exercise prices or times to expiration. A money spread refers to a spread with different exercise price; a time spread refers to differing expiration date.

standard deviation Square root of the variance.

statement of cash flows A financial statement showing a firm's cash receipts and cash payments during a specified period.

statistical arbitrage Use of quantitative systems to uncover many perceived misalignments in relative pricing and ensure profit by averaging over all of these small bets.

stock exchanges Secondary markets where already-issued securities are bought and sold by members.

stock selection An active portfolio management technique that focuses on advantageous selection of particular stocks rather than on broad asset allocation choices.

stock split Issue by a corporation of a given number of shares in exchange for the current number of shares held by stockholders. Splits may go in either direction, either increasing or decreasing the number of shares outstanding. A *reverse split* decreases the number outstanding.

stop-loss order A sell order to be executed if the price of the stock falls below a stipulated level.

stop orders Order to trade contingent on security price designed to limit losses if price moves against the trader.

straddle A combination of buying both a call and a put on the same asset, each with the same exercise price and expiration date. The purpose is to profit from expected volatility.

straight bond A bond with no option features such as callability or convertibility.

street name Describes securities held by a broker on behalf of a client but registered in the name of the firm.

strike price See exercise price.

strip, strap Variants of a straddle. A *strip* is two puts and one call on a stock; a *strap* is two calls and one put, both with the same exercise price and expiration date.

stripped of coupons Describes the practice of some investment banks that sell "synthetic" zero-coupon bonds by marketing the rights to a single payment backed by a coupon-paying Treasury bond.

strong-form EMH See efficient market hypothesis.

subordination clause A provision in a bond indenture that restricts the issuer's future borrowing by subordinating the new leaders' claims on the firm to those of the existing bond holders. Claims of *subordinated* or *junior* debtholders are not paid until the prior debt is paid.

substitution swap Exchange of one bond for a bond with similar attributes but more attractively priced.

supply shock An event that influences production capacity and costs in the economy.

support level A price level below which it is supposedly difficult for a stock or stock index to fall.

survivorship bias Bias in the average returns of a sample of funds induced by excluding past returns on funds that left the sample because they happened to be unsuccessful.

swaption An option on a swap.

systematic risk Risk factors common to the whole economy, nondiversifiable risk; also called market risk.

T

tax anticipation notes Short-term municipal debt to raise funds to pay for expenses before actual collection of taxes.

tax deferral option The feature of the U.S. Internal Revenue Code that the capital gains tax on an asset is payable only when the gain is realized by selling the asset.

tax-deferred retirement plans Employer-sponsored and other plans that allow contributions and earnings to be made and accumulate tax-free until they are paid out as benefits.

tax swap Swapping two similar bonds to receive a tax benefit.

technical analysis Research to identify mispriced securities that focuses on recurrent and predictable stock price patterns and on proxies for buy or sell pressure in the market.

tender offer An offer from an outside investor to shareholders of a company to purchase their shares at a stipulated price, usually substantially above the market price, so that the investor may amass enough shares to obtain control of the company.

term insurance Provides a death benefit only, no build-up of cash value.

term premiums Excess of the yields to maturity on long-term bonds over those of short-term bonds.

term structure of interest rates The pattern of interest rates appropriate for discounting cash flows of various maturities.

times interest earned Ratio of profits to interest expense.

time value (of an option) The part of the value of an option that is due to its positive time to expiration. Not to be confused with present value or the time value of money.

time-weighted average An average of the period-by-period holding-period returns of an investment.

Tobin's q. Ratio of market value of the firm to replacement cost.

total asset turnover The annual sales generated by each dollar of assets (sales/assets).

tracking error The difference between the return on a specified portfolio and that of a benchmark portfolio designed to mimic that portfolio.

tracking portfolio A portfolio constructed to have returns with the highest possible correlation with a systematic risk factor.

tranche See collateralized mortgage obligation.

treasury bill Short-term, highly liquid government securities issued at a discount from the face value and returning the face amount at maturity.

treasury bond or note Debt obligations of the federal government that make semiannual coupon payments and are issued at or near par value.

Treynor's measure Ratio of excess return to beta.

trin statistic Ratio of average trading volume in declining stocks to average volume in advancing stocks. Used in technical analysis.

trough The transition point between recession and recovery.

turnover The ratio of the trading activity of a portfolio to the assets of the portfolio.

U

unbundling See bundling.

underwriters Investment bankers who help companies issue their securities to the public.

underwriting, underwriting syndicate Underwriters (investment bankers) purchase securities from the issuing company and resell them. Usually a syndicate of investment bankers is organized behind a lead firm.

unemployment rate The ratio of the number of people classified as unemployed to the total labor force.

unique risk See diversifiable risk.

unit investment trust Money invested in a portfolio whose composition is fixed for the life of the fund. Shares in a unit trust are called redeemable trust certificates, and they are sold at a premium above net asset value.

universal life policy An insurance policy that allows for a varying death benefit and premium level over the term of the policy, with an interest rate on the cash value that changes with market interest rates.

utility The measure of the welfare or satisfaction of an investor.

utility value The welfare a given investor assigns to an investment with a particular return and risk.

V

value at risk Measure of downside risk. The loss that will be incurred in the event of an extreme adverse price change with some given, typically low, probability.

variable annuities Annuity contracts in which the insurance company pays a periodic amount linked to the investment performance of an underlying portfolio.

variable life policy An insurance policy that provides a fixed death benefit plus a cash value that can be invested in a variety of funds from which the policyholder can choose.

variance A measure of the dispersion of a random variable. Equals the expected value of the squared deviation from the mean.

variation margin See maintenance margin.

views An analyst's opinion on the likely performance of a stock or sector compared to the market-consensus expectation.

volatility risk The risk in the value of options portfolios due to unpredictable changes in the volatility of the underlying asset.

W

warrant An option issued by the firm to purchase shares of the firm's stock.

weak-form EMH See efficient market hypothesis.

well-diversified portfolio A portfolio spread out over many securities in such a way that the weight in any security is close to zero.

whole-life insurance policy Provides a death benefit and a kind of savings plan that builds up cash value for possible future withdrawal.

workout period Realignment period of a temporary misaligned yield relationship.

world investable wealth The part of world wealth that is traded and is therefore accessible to investors.

writing a call Selling a call option.

Y

yield curve A graph of yield to maturity as a function of time to maturity.

yield to maturity A measure of the average rate of return that will be earned on a bond if held to maturity.

Z

zero-beta portfolio The minimum-variance portfolio uncorrelated with a chosen efficient portfolio.

zero-coupon bond A bond paying no coupons that sells at a discount and provides payment of face value only at maturity.

zero-investment portfolio A portfolio of zero net value, established by buying and shorting component securities, usually in the context of an arbitrage strategy.

12b-1 fees Annual fees charged by a mutual fund to pay for marketing and distribution costs.

Useful Formulas

Measures of Risk

Variance of returns: $\sigma^2 = \sum_s p(s)[r(s) - E(r)]^2$

Standard deviation: $\sigma = \sqrt{\sigma^2}$

Covariance between returns: $\text{Cov}(r_i, r_j) = \sum_s p(s)[r_i(s) - E(r_i)]\,[r_j(s) - E(r_j)]$

Beta of security i: $\beta_i = \dfrac{\text{Cov}(r_i, r_M)}{\text{Var}(r_M)}$

Portfolio Theory

Expected rate of return on a portfolio with weights w_i in each security: $E(r_p) = \sum_{i=1}^{n} w_i E(r_i)$

Variance of portfolio rate of return: $\sigma_p^2 = \sum_{j=1}^{n} \sum_{i=1}^{n} w_j w_i \,\text{Cov}(r_i, r_j)$

Market Equilibrium

The security market line: $E(r_i) = r_f + \beta_i[E(r_M) - r_f]$

Fixed-Income Analysis

Present value of $1:

Discrete period compounding: $\text{PV} = 1/(1 + r)^T$

Continuous compounding: $\text{PV} = e^{-rT}$

Forward rate of interest for period T: $f_T = \dfrac{(1 + y_T)^T}{(1 + y_{T-1})^{T-1}} - 1$

Real interest rate: $r = \dfrac{1 + R}{1 + i} - 1$

where R is the nominal interest rate and i is the inflation rate

Duration of a security: $D = \sum_{t=1}^{T} t \times \dfrac{CF_t}{(1 + y)^t} /\text{Price}$

Modified duration: $D^* = D/(1 + y)$

Equity Analysis

Constant growth dividend discount model: $V_0 = \dfrac{D_1}{k - g}$

Sustainable growth rate of dividends: $g = \text{ROE} \times b$

Price/earnings multiple: $P/E = \dfrac{1 - b}{k - \text{ROE} \times b}$

$$\text{ROE} = (1 - \text{Tax rate})\left[\text{ROA} + (\text{ROA} - \text{Interest rate})\frac{\text{Debt}}{\text{Equity}}\right]$$

Derivative Assets

Put-call parity: $P = C - S_0 + PV(X + \text{dividends})$

Black-Scholes formula: $C = Se^{-\delta T}N(d_1) - Xe^{-rT}N(d_2)$

$$d_1 = \frac{\ln(S/X) + (r - \delta + \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

Spot-futures parity: $F_0 = S_0(1 + r - d)^T$

Interest rate parity: $F_0 = E_0\left(\dfrac{1 + r_{\text{US}}}{1 + r_{\text{foreign}}}\right)^T$

Performance Evaluation

Sharpe's measure: $S_p = \dfrac{\bar{r}_p - \bar{r}_f}{\sigma_p}$

Treynor's measure: $T_p = \dfrac{\bar{r}_p - \bar{r}_f}{\beta_p}$

Jensen's measure, or alpha: $\alpha_p = \bar{r}_p - [\bar{r}_f + \beta_p(\bar{r}_M - \bar{r}_f)]$

Geometric average return: $r_G = [(1 + r_1)(1 + r_2)\ldots(1 + r_T)]^{1/T} - 1$

Commonly Used Notation

b	Retention or plowback ratio
C	Call option value
CF	Cash flow
D	Duration
E	Exchange rate
$E(x)$	Expected value of random variable x
F	Futures price
e	2.718, the base for the natural logarithm, used for continuous compounding
e_{it}	The firm-specific return, also called the residual return, of security i in period t
f	Forward rate of interest
g	Growth rate of dividends
H	Hedge ratio for an option, sometimes called the option's delta
i	Inflation rate
k	Market capitalization rate, the required rate of return on a firm's stock
ln	Natural logarithm function
M	The market portfolio
$N(d)$	Cumulative normal function, the probability that a standard normal random variable will have value less than d
p	Probability
P	Put value
PV	Present value
P/E	Price-to-earnings multiple
r	Rate of return on a security; for fixed-income securities, r may denote the rate of interest for a particular period
r_f	The risk-free rate of interest
r_M	The rate of return on the market portfolio
ROE	Return on equity, incremental economic earnings per dollar reinvested in the firm
S_p	Reward-to-volatility ratio of a portfolio, also called Sharpe's measure; the excess expected return divided by the standard deviation
t	Time
T_p	Treynor's measure for a portfolio, excess expected return divided by beta
V	Intrinsic value of a firm, the present value of future dividends per share
X	Exercise price of an option
y	Yield to maturity
α	Rate of return beyond the value that would be forecast from the market's return and the systematic risk of the security
β	Systematic or market risk of a security
ρ_{ij}	Correlation coefficient between returns on securities i and j
σ	Standard deviation
σ^2	Variance
$\text{Cov}(r_i, r_j)$	Covariance between returns on securities i and j

The Tradition Continues...

Investments, 8th edition, by Bodie, Kane, and Marcus keeps the best blend of practical and theoretical coverage while incorporating new and important topics in the world of investments. This text continues the tradition of an appropriate rigor and clear writing style for the MBA investments student. The unifying theme that security markets are nearly efficient, meaning that most securities are usually priced appropriately given their risk and return attributes, is continued in the 8th edition.

New to this edition:

- **Chapter 26 Hedge Funds:** Topics include various hedge fund strategies; market neutral investing and portable alpha; performance evaluation for hedge funds with changing risk exposures; selection bias in hedge fund performance; tail risk in hedge fund portfolios; and hedge fund fees.
- End of chapter problem sets are **now separated by level of difficulty:** quiz, problems, and challenge problems.

Other important content updates include:

- **Chapter 7 Optimal Risky Portfolios:** This chapter contains additional material on the "art" of selecting reasonable parameter values for portfolio construction, and a discussion of what can go wrong when inputs are derived solely from recent historical experience.
- **Chapter 13 Empirical Evidence on Security Returns:** New material on the interpretation of risk premiums has been added. For example, we examine new evidence on the relation between the Fama-French risk factors and more fundamental measures of security risk.
- **Chapter 14 Bond Prices and Yields:** This chapter has been updated with new material explaining collateralized debt obligations (CDOs) as well as the role of credit rating agencies in the recent credit market crisis.
- **Chapter 23 Futures and Swaps:** New material on credit default swaps has been added to this chapter. We show how these securities are constructed, and how they are used to transfer credit risk.
- **Chapter 28 Investment Policy and the Framework of the CFA Institute:** This chapter has been updated to reflect the CFA Institute's expanded rubric for constructing a statement of investment policy.

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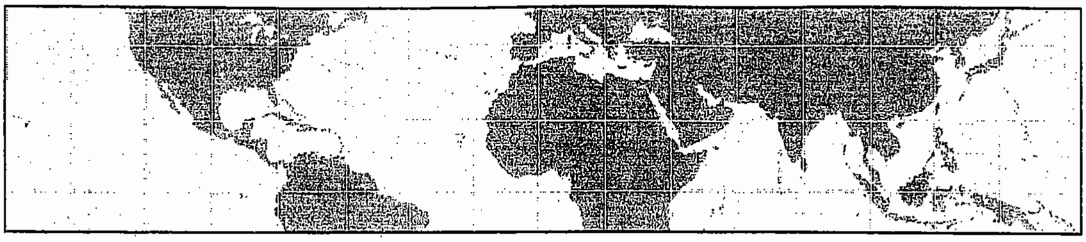

CHAPTER 3 Hedging Strategies Using Futures

Many of the participants in futures markets are hedgers. Their aim is to use futures markets to reduce a particular risk that they face. This risk might relate to fluctuations in the price of oil, a foreign exchange rate, the level of the stock market, or some other variable. A *perfect hedge* is one that completely eliminates the risk. Perfect hedges are rare. For the most part, therefore, a study of hedging using futures contracts is a study of the ways in which hedges can be constructed so that they perform as close to perfect as possible.

In this chapter we consider a number of general issues associated with the way hedges are set up. When is a short futures position appropriate? When is a long futures position appropriate? Which futures contract should be used? What is the optimal size of the futures position for reducing risk? At this stage, we restrict our attention to what might be termed *hedge-and-forget* strategies. We assume that no attempt is made to adjust the hedge once it has been put in place. The hedger simply takes a futures position at the beginning of the life of the hedge and closes out the position at the end of the life of the hedge. In Chapter 17 we will examine dynamic hedging strategies in which the hedge is monitored closely and frequent adjustments are made.

The chapter initially treats futures contracts as forward contracts (that is, it ignores daily settlement). Later it explains an adjustment known as "tailing" that takes account of the difference between futures and forwards.

3.1 BASIC PRINCIPLES

When an individual or company chooses to use futures markets to hedge a risk, the objective is usually to take a position that neutralizes the risk as far as possible. Consider a company that knows it will gain $10,000 for each 1 cent increase in the price of a commodity over the next 3 months and lose $10,000 for each 1 cent decrease in the price during the same period. To hedge, the company's treasurer should take a short futures position that is designed to offset this risk. The futures position should lead to a loss of $10,000 for each 1 cent increase in the price of the commodity over the 3 months and a gain of $10,000 for each 1 cent decrease in the price during this period. If the price of the commodity goes down, the gain on the futures position offsets the loss on the rest of the company's business. If the price of the commodity goes up, the loss on the futures position is offset by the gain on the rest of the company's business.

Short Hedges

A *short hedge* is a hedge, such as the one just described, that involves a short position in futures contracts. A short hedge is appropriate when the hedger already owns an asset and expects to sell it at some time in the future. For example, a short hedge could be used by a farmer who owns some hogs and knows that they will be ready for sale at the local market in two months. A short hedge can also be used when an asset is not owned right now but will be owned at some time in the future. Consider, for example, a US exporter who knows that he or she will receive euros in 3 months. The exporter will realize a gain if the euro increases in value relative to the US dollar and will sustain a loss if the euro decreases in value relative to the US dollar. A short futures position leads to a loss if the euro increases in value and a gain if it decreases in value. It has the effect of offsetting the exporter's risk.

To provide a more detailed illustration of the operation of a short hedge in a specific situation, we assume that it is May 15 today and that an oil producer has just negotiated a contract to sell 1 million barrels of crude oil. It has been agreed that the price that will apply in the contract is the market price on August 15. The oil producer is therefore in the position where it will gain $10,000 for each 1 cent increase in the price of oil over the next 3 months and lose $10,000 for each 1 cent decrease in the price during this period. Suppose that on May 15 the spot price is $60 per barrel and the crude oil futures price on the New York Mercantile Exchange (NYMEX) for August delivery is $59 per barrel. Because each futures contract on NYMEX is for the delivery of 1,000 barrels, the company can hedge its exposure by shorting 1,000 futures contracts. If the oil producer closes out its position on August 15, the effect of the strategy should be to lock in a price close to $59 per barrel.

To illustrate what might happen, suppose that the spot price on August 15 proves to be $55 per barrel. The company realizes $55 million for the oil under its sales contract. Because August is the delivery month for the futures contract, the futures price on August 15 should be very close to the spot price of $55 on that date. The company therefore gains approximately

$$\$59 - \$55 = \$4$$

per barrel, or $4 million in total from the short futures position. The total amount realized from both the futures position and the sales contract is therefore approximately $59 per barrel, or $59 million in total.

For an alternative outcome, suppose that the price of oil on August 15 proves to be $65 per barrel. The company realizes $65 for the oil and loses approximately

$$\$65 - \$59 = \$6$$

per barrel on the short futures position. Again, the total amount realized is approximately $59 million. It is easy to see that in all cases the company ends up with approximately $59 million.

Long Hedges

Hedges that involve taking a long position in a futures contract are known as *long hedges*. A long hedge is appropriate when a company knows it will have to purchase a certain asset in the future and wants to lock in a price now.

Suppose that it is now January 15. A copper fabricator knows it will require 100,000 pounds of copper on May 15 to meet a certain contract. The spot price of copper is 340 cents per pound, and the futures price for May delivery is 320 cents per pound. The fabricator can hedge its position by taking a long position in four futures contracts on the COMEX division of NYMEX and closing its position on May 15. Each contract is for the delivery of 25,000 pounds of copper. The strategy has the effect of locking in the price of the required copper at close to 320 cents per pound.

Suppose that the spot price of copper on May 15 proves to be 325 cents per pound. Because May is the delivery month for the futures contract, this should be very close to the futures price. The fabricator therefore gains approximately

$$100{,}000 \times (\$3.25 - \$3.20) = \$5{,}000$$

on the futures contracts. It pays $100{,}000 \times \$3.25 = \$325{,}000$ for the copper, making the net cost approximately $\$325{,}000 - \$5{,}000 = \$320{,}000$. For an alternative outcome, suppose that the spot price is 305 cents per pound on May 15. The fabricator then loses approximately

$$100{,}000 \times (\$3.20 - \$3.05) = \$15{,}000$$

on the futures contract and pays $100{,}000 \times \$3.05 = \$305{,}000$ for the copper. Again, the net cost is approximately \$320,000, or 320 cents per pound.

Note that it is better for the company to use futures contracts than to buy the copper on January 15 in the spot market. If it does the latter, it will pay 340 cents per pound instead of 320 cents per pound and will incur both interest costs and storage costs. For a company using copper on a regular basis, this disadvantage would be offset by the convenience of having the copper on hand.[1] However, for a company that knows it will not require the copper until May 15, the futures contract alternative is likely to be preferred.

Long hedges can be used to manage an existing short position. Consider an investor who has shorted a certain stock. (See Section 5.2 for a discussion of shorting.) Part of the risk faced by the investor is related to the performance of the whole stock market. The investor can neutralize this risk with a long position in index futures contracts. This type of hedging strategy is discussed further later in the chapter.

The examples we have looked at assume that the futures position is closed out in the delivery month. The hedge has the same basic effect if delivery is allowed to happen. However, making or taking delivery can be costly and inconvenient. For this reason, delivery is not usually made even when the hedger keeps the futures contract until the delivery month. As will be discussed later, hedgers with long positions usually avoid any possibility of having to take delivery by closing out their positions before the delivery period.

We have also assumed in the two examples that there is no daily settlement. In practice, daily settlement does have a small effect on the performance of a hedge. As

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[1] See Section 5.11 for a discussion of convenience yields.

explained in Chapter 2, it means that the payoff from the futures contract is realized day by day throughout the life of the hedge rather than all at the end.

3.2 ARGUMENTS FOR AND AGAINST HEDGING

The arguments in favor of hedging are so obvious that they hardly need to be stated. Most companies are in the business of manufacturing, or retailing or wholesaling, or providing a service. They have no particular skills or expertise in predicting variables such as interest rates, exchange rates, and commodity prices. It makes sense for them to hedge the risks associated with these variables as they arise. The companies can then focus on their main activities—for which presumably they do have particular skills and expertise. By hedging, they avoid unpleasant surprises such as sharp rises in the price of a commodity that is being purchased.

In practice, many risks are left unhedged. In the rest of this section we will explore some of the reasons.

Hedging and Shareholders

One argument sometimes put forward is that the shareholders can, if they wish, do the hedging themselves. They do not need the company to do it for them. This argument is, however, open to question. It assumes that shareholders have as much information about the risks faced by a company as does the company's management. In most instances, this is not the case. The argument also ignores commissions and other transactions costs. These are less expensive per dollar of hedging for large transactions than for small transactions. Hedging is therefore likely to be less expensive when carried out by the company than when it is carried out by individual shareholders. Indeed, the size of futures contracts makes hedging by individual shareholders impossible in many situations.

One thing that shareholders can do far more easily than a corporation is diversify risk. A shareholder with a well-diversified portfolio may be immune to many of the risks faced by a corporation. For example, in addition to holding shares in a company that uses copper, a well-diversified shareholder may hold shares in a copper producer, so that there is very little overall exposure to the price of copper. If companies are acting in the best interests of well-diversified shareholders, it can be argued that hedging is unnecessary in many situations. However, the extent to which managers are in practice influenced by this type of argument is open to question.

Hedging and Competitors

If hedging is not the norm in a certain industry, it may not make sense for one particular company to choose to be different from all others. Competitive pressures within the industry may be such that the prices of the goods and services produced by the industry fluctuate to reflect raw material costs, interest rates, exchange rates, and so on. A company that does not hedge can expect its profit margins to be roughly constant. However, a company that does hedge can expect its profit margins to fluctuate!

To illustrate this point, consider two manufacturers of gold jewelry, SafeandSure

Table 3.1 Danger in hedging when competitors do not hedge.

Change in gold price	*Effect on price of gold jewelry*	*Effect on profits of TakeaChance Co.*	*Effect on profits of SafeandSure Co.*
Increase	Increase	None	Increase
Decrease	Decrease	None	Decrease

Company and TakeaChance Company. We assume that most companies in the industry do not hedge against movements in the price of gold and that TakeaChance Company is no exception. However, SafeandSure Company has decided to be different from its competitors and to use futures contracts to hedge its purchase of gold over the next 18 months. If the price of gold goes up, economic pressures will tend to lead to a corresponding increase in the wholesale price of the jewelry, so that TakeaChance Company's gross profit margin is unaffected. By contrast, SafeandSure Company's profit margin will increase after the effects of the hedge have been taken into account. If the price of gold goes down, economic pressures will tend to lead to a corresponding decrease in the wholesale price of the jewelry. Again, TakeaChance Company's profit margin is unaffected. However, SafeandSure Company's profit margin goes down. In extreme conditions, SafeandSure Company's profit margin could become negative as a result of the "hedging" carried out! The situation is summarized in Table 3.1.

This example emphasizes the importance of looking at the big picture when hedging. All the implications of price changes on a company's profitability should be taken into account in the design of a hedging strategy to protect against the price changes.

Hedging Can Lead to a Worse Outcome

It is important to realize that a hedge using futures contracts can result in a decrease or an increase in a company's profits relative to the position it would be in with no hedging. In the example involving the oil producer considered earlier, if the price of oil goes down, the company loses money on its sale of 1 million barrels of oil, and the futures position leads to an offsetting gain. The treasurer can be congratulated for having had the foresight to put the hedge in place. Clearly, the company is better off than it would be with no hedging. Other executives in the organization, it is hoped, will appreciate the contribution made by the treasurer. If the price of oil goes up, the company gains from its sale of the oil, and the futures position leads to an offsetting loss. The company is in a worse position than it would be with no hedging. Although the hedging decision was perfectly logical, the treasurer may in practice have a difficult time justifying it. Suppose that the price of oil at the end of the hedge is \$69, so that the company loses \$10 per barrel on the futures contract. We can imagine a conversation such as the following between the treasurer and the president:

PRESIDENT: This is terrible. We've lost \$10 million in the futures market in the space of three months. How could it happen? I want a full explanation.

TREASURER: The purpose of the futures contracts was to hedge our exposure to the price of oil, not to make a profit. Don't forget we made \$10 million from the favorable effect of the oil price increases on our business.

Business Snapshot 3.1 Hedging by Gold Mining Companies

It is natural for a gold mining company to consider hedging against changes in the price of gold. Typically it takes several years to extract all the gold from a mine. Once a gold mining company decides to go ahead with production at a particular mine, it has a big exposure to the price of gold. Indeed a mine that looks profitable at the outset could become unprofitable if the price of gold plunges.

Gold mining companies are careful to explain their hedging strategies to potential shareholders. Some gold mining companies do not hedge. They tend to attract shareholders who buy gold stocks because they want to benefit when the price of gold increases and are prepared to accept the risk of a loss from a decrease in the price of gold. Other companies choose to hedge. They estimate the number of ounces they will produce each month for the next few years and enter into short futures or forward contracts to lock in the price that will be received.

Suppose you are Goldman Sachs and have just entered into a forward contract with a hedger whereby you agree to buy a large amount of gold at a fixed price. How do you hedge your risk? The answer is that you borrow gold from a central bank and sell it at the current market price. (The central banks of many countries hold large amounts of gold.) At the end of the life of the forward contract, you buy gold from the gold mining company under the terms of the forward contract and use it to repay the central bank. The central bank charges a fee (perhaps 1.5% per annum), known as the gold lease rate for lending its gold in this way.

PRESIDENT: What's that got to do with it? That's like saying that we do not need to worry when our sales are down in California because they are up in New York.

TREASURER: If the price of oil had gone down...

PRESIDENT: I don't care what would have happened if the price of oil had gone down. The fact is that it went up. I really do not know what you were doing playing the futures markets like this. Our shareholders will expect us to have done particularly well this quarter. I'm going to have to explain to them that your actions reduced profits by $10 million. I'm afraid this is going to mean no bonus for you this year.

TREASURER: That's unfair. I was only...

PRESIDENT: Unfair! You are lucky not to be fired. You lost $10 million.

TREASURER: It all depends on how you look at it...

It is easy to see why many treasurers are reluctant to hedge! Hedging reduces risk for the company. However, it may increase risk for the treasurer if others do not fully understand what is being done. The only real solution to this problem involves ensuring that all senior executives within the organization fully understand the nature of hedging before a hedging program is put in place. Ideally, hedging strategies are set by a company's board of directors and are clearly communicated to both the company's management and the shareholders. (See Business Snapshot 3.1 for a discussion of hedging by gold mining companies.)

3.3 BASIS RISK

The hedges in the examples considered so far have been almost too good to be true. The hedger was able to identify the precise date in the future when an asset would be bought or sold. The hedger was then able to use futures contracts to remove almost all the risk arising from the price of the asset on that date. In practice, hedging is often not quite as straightforward. Some of the reasons are as follows:

1. The asset whose price is to be hedged may not be exactly the same as the asset underlying the futures contract.
2. The hedger may be uncertain as to the exact date when the asset will be bought or sold.
3. The hedge may require the futures contract to be closed out before its delivery month.

These problems give rise to what is termed *basis risk*. This concept will now be explained.

The Basis

The *basis* in a hedging situation is as follows:[2]

$$\text{Basis} = \text{Spot price of asset to be hedged} - \text{Futures price of contract used}$$

If the asset to be hedged and the asset underlying the futures contract are the same, the basis should be zero at the expiration of the futures contract. Prior to expiration, the basis may be positive or negative. From Table 2.2 and Figure 2.2, we see that on January 8, 2007, the basis was negative for gold and positive for orange juice.

As time passes, the spot price and the futures price do not necessarily change by the same amount. As a result, the basis changes. An increase in the basis is referred to as a *strengthening of the basis*; a decrease in the basis is referred to as a *weakening of the basis*. Figure 3.1 illustrates how a basis might change over time in a situation where the basis is positive prior to expiration of the futures contract.

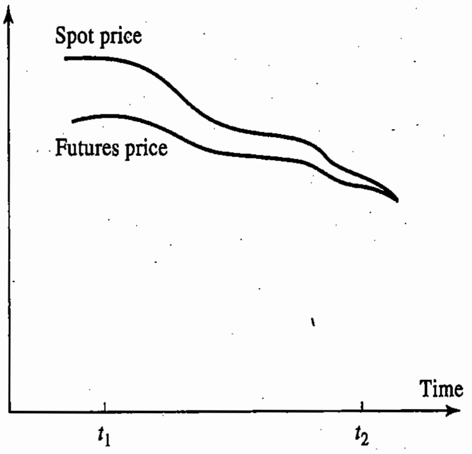
To examine the nature of basis risk, we will use the following notation:

S_1: Spot price at time t_1
S_2: Spot price at time t_2
F_1: Futures price at time t_1
F_2: Futures price at time t_2
b_1: Basis at time t_1
b_2: Basis at time t_2

We will assume that a hedge is put in place at time t_1 and closed out at time t_2. As an example, we will consider the case where the spot and futures prices at the time the hedge is initiated are \$2.50 and \$2.20, respectively, and that at the time the hedge is closed out they are \$2.00 and \$1.90, respectively. This means that $S_1 = 2.50$, $F_1 = 2.20$, $S_2 = 2.00$, and $F_2 = 1.90$.

[2] This is the usual definition. However, the alternative definition Basis = Futures price − Spot price is sometimes used, particularly when the futures contract is on a financial asset.

Figure 3.1 Variation of basis over time.



From the definition of the basis, we have

$$b_1 = S_1 - F_1 \quad \text{and} \quad b_2 = S_2 - F_2$$

so that, in our example, $b_1 = 0.30$ and $b_2 = 0.10$.

Consider first the situation of a hedger who knows that the asset will be sold at time t_2 and takes a short futures position at time t_1. The price realized for the asset is S_2 and the profit on the futures position is $F_1 - F_2$. The effective price that is obtained for the asset with hedging is therefore

$$S_2 + F_1 - F_2 = F_1 + b_2$$

In our example, this is \$2.30. The value of F_1 is known at time t_1. If b_2 were also known at this time, a perfect hedge would result. The hedging risk is the uncertainty associated with b_2 and is known as *basis risk*. Consider next a situation where a company knows it will buy the asset at time t_2 and initiates a long hedge at time t_1. The price paid for the asset is S_2 and the loss on the hedge is $F_1 - F_2$. The effective price that is paid with hedging is therefore

$$S_2 + F_1 - F_2 = F_1 + b_2$$

This is the same expression as before and is \$2.30 in the example. The value of F_1 is known at time t_1, and the term b_2 represents basis risk.

Note that basis risk can lead to an improvement or a worsening of a hedger's position. Consider a short hedge. If the basis strengthens (i.e., increases) unexpectedly, the hedger's position improves; if the basis weakens (i.e., decreases) unexpectedly, the hedger's position worsens. For a long hedge, the reverse holds. If the basis strengthens unexpectedly, the hedger's position worsens; if the basis weakens unexpectedly, the hedger's position improves.

The asset that gives rise to the hedger's exposure is sometimes different from the asset underlying the futures contract that is used for hedging. This increases the basis risk. Define S_2^* as the price of the asset underlying the futures contract at time t_2. As before, S_2 is the price of the asset being hedged at time t_2. By hedging, a company ensures that the price that will be paid (or received) for the asset is

$$S_2 + F_1 - F_2$$

This can be written as

$$F_1 + (S_2^* - F_2) + (S_2 - S_2^*)$$

The terms $S_2^* - F_2$ and $S_2 - S_2^*$ represent the two components of the basis. The $S_2^* - F_2$ term is the basis that would exist if the asset being hedged were the same as the asset underlying the futures contract. The $S_2 - S_2^*$ term is the basis arising from the difference between the two assets.

Choice of Contract

One key factor affecting basis risk is the choice of the futures contract to be used for hedging. This choice has two components:

1. The choice of the asset underlying the futures contract
2. The choice of the delivery month

If the asset being hedged exactly matches an asset underlying a futures contract, the first choice is generally fairly easy. In other circumstances, it is necessary to carry out a careful analysis to determine which of the available futures contracts has futures prices that are most closely correlated with the price of the asset being hedged.

The choice of the delivery month is likely to be influenced by several factors. In the examples given earlier in this chapter, we assumed that, when the expiration of the hedge corresponds to a delivery month, the contract with that delivery month is chosen. In fact, a contract with a later delivery month is usually chosen in these circumstances. The reason is that futures prices are in some instances quite erratic during the delivery month. Moreover, a long hedger runs the risk of having to take delivery of the physical asset if the contract is held during the delivery month. Taking delivery can be expensive and inconvenient. (Long hedgers normally prefer to close out the futures contract and buy the asset from their usual suppliers.)

In general, basis risk increases as the time difference between the hedge expiration and the delivery month increases. A good rule of thumb is therefore to choose a delivery month that is as close as possible to, but later than, the expiration of the hedge. Suppose delivery months are March, June, September, and December for a futures contract on a particular asset. For hedge expirations in December, January, and February, the March contract will be chosen; for hedge expirations in March, April, and May, the June contract will be chosen; and so on. This rule of thumb assumes that there is sufficient liquidity in all contracts to meet the hedger's requirements. In practice, liquidity tends to be greatest in short-maturity futures contracts. Therefore, in some situations, the hedger may be inclined to use short-maturity contracts and roll them forward. This strategy is discussed later in the chapter.

Example 3.1

It is March 1. A US company expects to receive 50 million Japanese yen at the end of July. Yen futures contracts on the Chicago Mercantile Exchange have delivery months of March, June, September, and December. One contract is for the delivery of 12.5 million yen. The company therefore shorts four September yen futures contracts on March 1. When the yen are received at the end of July, the company closes out its position. We suppose that the futures price on March 1 in cents per yen is 0.7800 and that the spot and futures prices when the contract is closed out are 0.7200 and 0.7250, respectively.

The gain on the futures contract is $0.7800 - 0.7250 = 0.0550$ cents per yen. The basis is $0.7200 - 0.7250 = -0.0050$ cents per yen when the contract is closed out. The effective price obtained in cents per yen is the final spot price plus the gain on the futures:

$$0.7200 + 0.0550 = 0.7750$$

This can also be written as the initial futures price plus the final basis:

$$0.7800 + (-0.0050) = 0.7750$$

The total amount received by the company for the 50 million yen is 50×0.00775 million dollars, or \$387,500.

Example 3.2

It is June 8 and a company knows that it will need to purchase 20,000 barrels of crude oil at some time in October or November. Oil futures contracts are currently traded for delivery every month on NYMEX and the contract size is 1,000 barrels. The company therefore decides to use the December contract for hedging and takes a long position in 20 December contracts. The futures price on June 8 is \$68.00 per barrel. The company finds that it is ready to purchase the crude oil on November 10. It therefore closes out its futures contract on that date. The spot price and futures price on November 10 are \$70.00 per barrel and \$69.10 per barrel.

The gain on the futures contract is $69.10 - 68.00 = \$1.10$ per barrel. The basis when the contract is closed out is $70.00 - 69.10 = \$0.90$ per barrel. The effective price paid (in dollars per barrel) is the final spot price less the gain on the futures, or

$$70.00 - 1.10 = 68.90$$

This can also be calculated as the initial futures price plus the final basis,

$$68.00 + 0.90 = 68.90$$

The total price paid is $68.90 \times 20{,}000 = \$1{,}378{,}000$.

3.4 CROSS HEDGING

In the examples considered up to now, the asset underlying the futures contract has been the same as the asset whose price is being hedged. *Cross hedging* occurs when the two assets are different. Consider, for example, an airline that is concerned about the future price of jet fuel. Because there is no futures contract on jet fuel, it might choose to use heating oil futures contracts to hedge its exposure.

The *hedge ratio* is the ratio of the size of the position taken in futures contracts to the size of the exposure. When the asset underlying the futures contract is the same as the asset being hedged, it is natural to use a hedge ratio of 1.0. This is the hedge ratio we have used in the examples considered so far. For instance, in Example 3.2, the hedger's exposure was on 20,000 barrels of oil, and futures contracts were entered into for the delivery of exactly this amount of oil.

When cross hedging is used, setting the hedge ratio equal to 1.0 is not always optimal. The hedger should choose a value for the hedge ratio that minimizes the variance of the value of the hedged position. We now consider how the hedger can do this.

Calculating the Minimum Variance Hedge Ratio

We will use the following notation:

ΔS: Change in spot price, S, during a period of time equal to the life of the hedge
ΔF: Change in futures price, F, during a period of time equal to the life of the hedge
σ_S: Standard deviation of ΔS
σ_F: Standard deviation of ΔF
ρ: Coefficient of correlation between ΔS and ΔF
h^*: Hedge ratio that minimizes the variance of the hedger's position

In the appendix at the end of this chapter, we show that

$$h^* = \rho \frac{\sigma_S}{\sigma_F} \tag{3.1}$$

The optimal hedge ratio is the product of the coefficient of correlation between ΔS and ΔF and the ratio of the standard deviation of ΔS to the standard deviation of ΔF. Figure 3.2 shows how the variance of the value of the hedger's position depends on the hedge ratio chosen.

If $\rho = 1$ and $\sigma_F = \sigma_S$, the hedge ratio, h^*, is 1.0. This result is to be expected, because in this case the futures price mirrors the spot price perfectly. If $\rho = 1$ and $\sigma_F = 2\sigma_S$, the hedge ratio h^* is 0.5. This result is also as expected, because in this case the futures price always changes by twice as much as the spot price.

The optimal hedge ratio, h^*, is the slope of the best-fit line when ΔS is regressed against ΔF, as indicated in Figure 3.3. This is intuitively reasonable, because we require h^* to correspond to the ratio of changes in ΔS to changes in ΔF. The *hedge effectiveness* can be

Figure 3.2 Dependence of variance of hedger's position on hedge ratio.

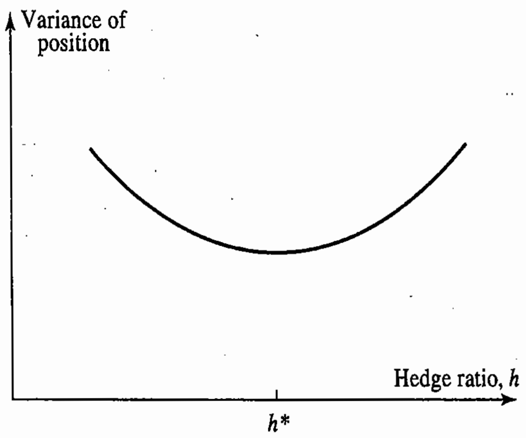
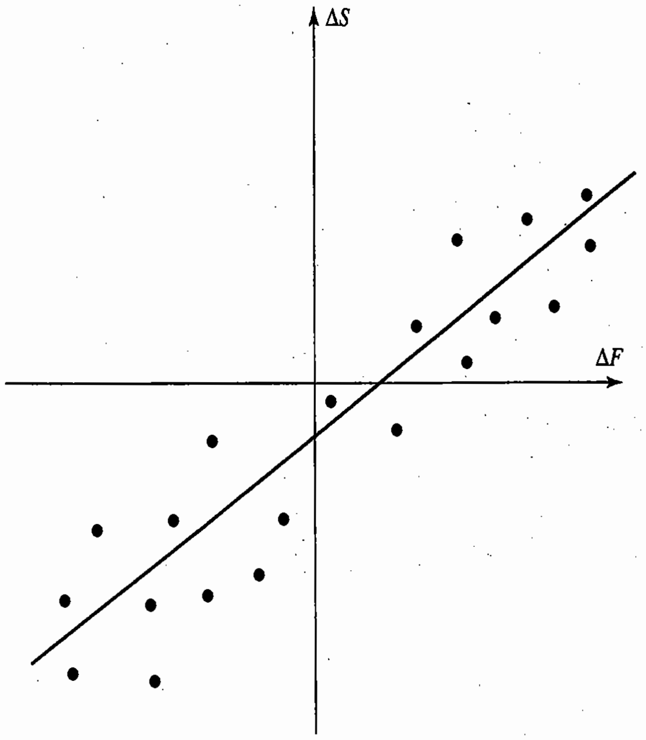


Figure 3.3 Regression of change in spot price against change in futures price.



defined as the proportion of the variance that is eliminated by hedging. This is the R^2 from the regression of ΔS against ΔF and equals ρ^2, or

$$h^{*2}\frac{\sigma_F^2}{\sigma_S^2}$$

The parameters ρ, σ_F, and σ_S in equation (3.1) are usually estimated from historical data on ΔS and ΔF. (The implicit assumption is that the future will in some sense be like the past.) A number of equal nonoverlapping time intervals are chosen, and the values of ΔS and ΔF for each of the intervals are observed. Ideally, the length of each time interval is the same as the length of the time interval for which the hedge is in effect. In practice, this sometimes severely limits the number of observations that are available, and a shorter time interval is used.

Optimal Number of Contracts

Define variables as follows:

Q_A: Size of position being hedged (units)

Q_F: Size of one futures contract (units)

N^*: Optimal number of futures contracts for hedging

The futures contracts should be on h^*Q_A units of the asset. The number of futures contracts required is therefore given by

$$N^* = \frac{h^* Q_A}{Q_F} \tag{3.2}$$

Example 3.3

An airline expects to purchase 2 million gallons of jet fuel in 1 month and decides to use heating oil futures for hedging.[3] We suppose that Table 3.2 gives, for 15 successive months, data on the change, ΔS, in the jet fuel price per gallon and the corresponding change, ΔF, in the futures price for the contract on heating oil that would be used for hedging price changes during the month. The number of observations, which we will denote by n, is 15. We will denote the ith observations on ΔF and ΔS by x_i and y_i, respectively. From Table 3.2, we have

$$\sum x_i = -0.013 \qquad \sum x_i^2 = 0.0138$$
$$\sum y_i = 0.003 \qquad \sum y_i^2 = 0.0097$$
$$\sum x_i y_i = 0.0107$$

Table 3.2 Data to calculate minimum variance hedge ratio when heating oil futures contract is used to hedge purchase of jet fuel.

Month i	*Change in futures price per gallon* $(= x_i)$	*Change in fuel price per gallon* $(= y_i)$
1	0.021	0.029
2	0.035	0.020
3	−0.046	−0.044
4	0.001	0.008
5	0.044	0.026
6	−0.029	−0.019
7	−0.026	−0.010
8	−0.029	−0.007
9	0.048	0.043
10	−0.006	0.011
11	−0.036	−0.036
12	−0.011	−0.018
13	0.019	0.009
14	−0.027	−0.032
15	0.029	0.023

[3] Heating oil futures contracts are more liquid than jet fuel futures contracts. For an account of how Delta Airlines used heating oil futures to hedge its future purchases of jet fuel, see A. Ness, "Delta Wins on Fuel," *Risk*, June 2001, p. 8.

Standard formulas from statistics give the estimate of σ_F as

$$\sqrt{\frac{\sum x_i^2}{n-1} - \frac{(\sum x_i)^2}{n(n-1)}} = 0.0313$$

The estimate of σ_S is

$$\sqrt{\frac{\sum y_i^2}{n-1} - \frac{(\sum y_i)^2}{n(n-1)}} = 0.0263$$

The estimate of ρ is

$$\frac{n\sum x_i y_i - \sum x_i \sum y_i}{\sqrt{[n\sum x_i^2 - (\sum x_i)^2][n\sum y_i^2 - (\sum y_i)^2]}} = 0.928$$

From equation (3.1), the minimum variance hedge ratio, h^*, is therefore

$$0.928 \times \frac{0.0263}{0.0313} = 0.78$$

Each heating oil contract traded on NYMEX is on 42,000 gallons of heating oil. From equation (3.2), the optimal number of contracts is

$$\frac{0.78 \times 2{,}000{,}000}{42{,}000} = 37.14$$

or, rounding to the nearest whole number, 37.

Tailing the Hedge

When futures are used for hedging, a small adjustment, known as *tailing the hedge*, can be made to allow for the impact of daily settlement. In practice this means that equation (3.2) becomes[4]

$$N^* = \frac{h^* V_A}{V_F} \tag{3.3}$$

where V_A is the dollar value of the position being hedged and V_F is the dollar value of one futures contract (the futures price times Q_F). Suppose that in Example 3.3 the spot price and the futures price are 1.94 and 1.99 dollars per gallon. Then $V_A = 2{,}000{,}000 \times 1.94 = 3{,}880{,}0000$ while $V_F = 42{,}000 \times 1.99 = 83{,}580$, so that the optimal number of contracts is

$$\frac{0.78 \times 3,880,000}{83,550} = 36.22$$

If we round this to the nearest whole number, the optimal number of contracts is now 36 rather than 37. The effect of tailing the hedge is to multiply the hedge ratio in equation (3.2) by the ratio of the spot price to the futures price. Ideally the futures position used for hedging should then be adjusted as V_A and V_F change, but in practice this is not usually feasible.

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[4] See Problem 5.23 for an explanation of equation (3.3).

3.5 STOCK INDEX FUTURES

We now move on to consider stock index futures and how they are used to hedge or manage exposures to equity prices.

A *stock index* tracks changes in the value of a hypothetical portfolio of stocks. The weight of a stock in the portfolio equals the proportion of the portfolio invested in the stock. The percentage increase in the stock index over a small interval of time is set equal to the percentage increase in the value of the hypothetical portfolio. Dividends are usually not included in the calculation so that the index tracks the capital gain/loss from investing in the portfolio.[5]

If the hypothetical portfolio of stocks remains fixed, the weights assigned to individual stocks in the portfolio do not remain fixed. When the price of one particular stock in the portfolio rises more sharply than others, more weight is automatically given to that stock. Some indices are constructed from a hypothetical portfolio consisting of one of each of a number of stocks. The weights assigned to the stocks are then proportional to their market prices, with adjustments being made when there are stock splits. Other indices are constructed so that weights are proportional to market capitalization (stock price × number of shares outstanding). The underlying portfolio is then automatically adjusted to reflect stock splits, stock dividends, and new equity issues.

Stock Indices

Table 3.3 shows futures prices for contracts on a number of different stock indices as they were reported in the *Wall Street Journal* of January 9, 2007. The prices refer to the close of trading on January 8, 2007.

The *Dow Jones Industrial Average* is based on a portfolio consisting of 30 blue-chip stocks in the United States. The weights given to the stocks are proportional to their prices. The Chicago Board of Trade trades two futures contracts on the index. One is on $10 times the index. The other (the Mini DJ Industrial Average) is on $5 times the index.

The *Standard & Poor's 500* (S&P 500) *Index* is based on a portfolio of 500 different stocks: 400 industrials, 40 utilities, 20 transportation companies, and 40 financial institutions. The weights of the stocks in the portfolio at any given time are proportional to their market capitalizations. This index accounts for 80% of the market capitalization of all the stocks listed on the New York Stock Exchange. The Chicago Mercantile Exchange (CME) trades two futures contracts on the S&P 500. One is on $250 times the index; the other (the Mini S&P 500 contract) is on $50 times the index.

The *Nasdaq 100* is based on 100 stocks using the National Association of Securities Dealers Automatic Quotations Service. The CME trades two contracts. One is on $100 times the index; the other (the Mini Nasdaq 100 contract) is on $20 times the index.

The *Russell 1000 Index* is an index of the prices of the 1000 largest capitalization stocks in the United States. The *U.S. Dollar Index* is a trade-weighted index of the values of six foreign currencies (the euro, yen, pound, Canadian dollar, Swedish krona, and Swiss franc).

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[5] An exception to this is a *total return index*. This is calculated by assuming that dividends on the hypothetical portfolio are reinvested in the portfolio.

Table 3.3 Index futures quotes from *Wall Street Journal*, January 9, 2007: Columns show month, open, high, low, settle, change, and open interest, respectively.

Index Futures

DJ Industrial Average (CBT)-$10 x Index						
March	12457	12515	12405	**12492**	42	64,772
June	12530	12591	12525	**12591**	42	46
Mini DJ Industrial Average (CBT)-$5 x index						
March	12460	12514	12405	**12492**	42	106,556
June	12570	12577	12540	**12591**	42	21
S&P 500 Index (CME)-$250 x Index						
March	1417.30	1424.50	1413.00	**1422.50**	6.10	601,897
June	1426.10	1437.00	1426.10	**1435.30**	6.20	13,062
Mini S&P 500 (CME)-$50 x Index						
March	1417.25	1424.50	1413.00	**1422.50**	6.00	1,525,973
June	1430.50	1437.00	▼1425.50	**1435.25**	6.25	13,716
Nasdaq 100 (CME)-$100 x index						
March	1797.50	1812.50	1792.00	**1803.50**	6.25	45,550
Mini Nasdaq 100 (CME)-$20 x Index						
March	1798.0	1812.3	1792.3	**1803.5**	6.3	328,990
June	1819.8	1833.3	1814.3	**1825.0**	6.3	92
Russell 1000 (NYBOT)-$500 x index						
March	770.75	773.50	769.15	**773.00**	3.10	70,440
U.S. Dollar Index (NYBOT)-$1,000 x index						
March	84.43	84.62	84.27	**84.37**	-.03	24,181
June	84.10	84.30	84.01	**84.12**	-.03	2,028

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As mentioned in Chapter 2, futures contracts on stock indices are settled in cash, not by delivery of the underlying asset. All contracts are marked to market to either the opening price or the closing price of the index on the last trading day, and the positions are then deemed to be closed. For example, contracts on the S&P 500 are closed out at the opening price of the S&P 500 index on the third Friday of the delivery month.

Hedging an Equity Portfolio

Stock index futures can be used to hedge a well-diversified equity portfolio. Define:

> P: Current value of the portfolio
>
> F: Current value of one futures contract (the futures price times the contract size)

If the portfolio mirrors the index, the optimal hedge ratio, h^*, equals 1.0 and equation (3.3) shows that the number of futures contracts that should be shorted is

$$N^* = \frac{P}{F} \tag{3.4}$$

Suppose, for example, that a portfolio worth $5,050,000 mirrors the S&P 500. The index futures price is 1,010 and each futures contract is on $250 times the index. In this case $P = 5{,}050{,}000$ and $F = 1{,}010 \times 250 = 252{,}500$, so that 20 contracts should be shorted to hedge the portfolio.

When the portfolio does not exactly mirror the index, we can use the parameter beta (β) from the capital asset pricing model to determine the appropriate hedge ratio. Beta is the slope of the best-fit line obtained when excess return on the portfolio over the risk-free rate is regressed against the excess return of the market over the risk-free rate. When $\beta = 1.0$, the return on the portfolio tends to mirror the return on the market; when $\beta = 2.0$, the excess return on the portfolio tends to be twice as great as the excess return on the market; when $\beta = 0.5$, it tends to be half as great; and so on.

A portfolio with a β of 2.0 is twice as sensitive to market movements as a portfolio with a beta 1.0. It is therefore necessary to use twice as many contracts to hedge the portfolio. Similarly, a portfolio with a beta of 0.5 is half as sensitive to market movements as a portfolio with a beta of 1.0 and we should use half as many contracts to hedge it. In general, $h^* = \beta$, so that equation (3.3) gives

$$N^* = \beta \frac{P}{F} \tag{3.5}$$

This formula assumes that the maturity of the futures contract is close to the maturity of the hedge.

We illustrate that this formula gives good results by extending our earlier example. Suppose that a futures contract with 4 months to maturity is used to hedge the value of a portfolio over the next 3 months in the following situation:

Value of S&P 500 index = 1,000
S&P 500 futures price = 1,010
Value of portfolio = \$5,050,000
Risk-free interest rate = 4% per annum
Dividend yield on index = 1% per annum
Beta of portfolio = 1.5

One futures contract is for delivery of \$250 times the index. It follows that $F = 250 \times 1{,}010 = 252{,}500$ and from equation (3.5), the number of futures contracts that should be shorted to hedge the portfolio is

$$1.5 \times \frac{5{,}050{,}000}{252{,}500} = 30$$

Suppose the index turns out to be 900 in 3 months and the futures price is 902. The gain from the short futures position is then

$$30 \times (1010 - 902) \times 250 = \$810{,}000$$

The loss on the index is 10%. The index pays a dividend of 1% per annum, or 0.25% per 3 months. When dividends are taken into account, an investor in the index would therefore earn −9.75% in the 3-month period. Because the portfolio has a β of 1.5, the capital asset pricing model gives

Expected return on portfolio − Risk-free interest rate
= 1.5 × (Return on index − Risk-free interest rate)

The risk-free interest rate is approximately 1% per 3 months. It follows that the expected return (%) on the portfolio during the 3 months when the 3-month return on the index is −9.75% is

$$1.0 + [1.5 \times (-9.75 - 1.0)] = -15.125$$

The expected value of the portfolio (inclusive of dividends) at the end of the 3 months is therefore

$$\$5{,}050{,}000 \times (1 - 0.15125) = \$4{,}286{,}187$$

Table 3.4 Performance of stock index hedge.

Value of index in three months:	900	950	1,000	1,050	1,100
Futures price of index today:	1,010	1,010	1,010	1,010	1,010
Futures price of index in three months:	902	952	1,003	1,053	1,103
Gain on futures position ($):	810,000	435,000	52,500	−322,500	−697,500
Return on market:	−9.750%	−4.750%	0.250%	5.250%	10.250%
Expected return on portfolio:	−15.125%	−7.625%	−0.125%	7.375%	14.875%
Expected portfolio value in three months including dividends ($):	4,286,187	4,664,937	5,043,687	5,422,437	5,801,187
Total value of position in three months ($):	5,096,187	5,099,937	5,096,187	5,099,937	5,103,687

It follows that the expected value of the hedger's position, including the gain on the hedge, is

$$\$4{,}286{,}187 + \$810{,}000 = \$5{,}096{,}187$$

Table 3.4 summarizes these calculations together with similar calculations for other values of the index at maturity. It can be seen that the total expected value of the hedger's position in 3 months is almost independent of the value of the index.

The only thing we have not covered in this example is the relationship between futures prices and spot prices. We will see in Chapter 5 that the 1,010 assumed for the futures price today is roughly what we would expect given the interest rate and dividend we are assuming. The same is true of the futures prices in 3 months shown in Table 3.4.[6]

Reasons for Hedging an Equity Portfolio

Table 3.4 shows that the hedging scheme results in a value for the hedger's position at the end of the 3-month period being about 1% higher than at the beginning of the 3-month period. There is no surprise here. The risk-free rate is 4% per annum, or 1% per 3 months. The hedge results in the investor's position growing at the risk-free rate.

It is natural to ask why the hedger should go to the trouble of using futures contracts. To earn the risk-free interest rate, the hedger can simply sell the portfolio and invest the proceeds in risk-free instruments such as Treasury bills.

One answer to this question is that hedging can be justified if the hedger feels that the stocks in the portfolio have been chosen well. In these circumstances, the hedger might be very uncertain about the performance of the market as a whole, but confident that the stocks in the portfolio will outperform the market (after appropriate adjustments have been made for the beta of the portfolio). A hedge using index futures

[6] The calculations in Table 3.4 assume that the dividend yield on the index is predictable, the risk-free interest rate remains constant, and the return on the index over the 3-month period is perfectly correlated with the return on the portfolio. In practice, these assumptions do not hold perfectly, and the hedge works rather less well than is indicated by Table 3.4.

removes the risk arising from market moves and leaves the hedger exposed only to the performance of the portfolio relative to the market. Another reason for hedging may be that the hedger is planning to hold a portfolio for a long period of time and requires short-term protection in an uncertain market situation. The alternative strategy of selling the portfolio and buying it back later might involve unacceptably high transaction costs.

Changing the Beta of a Portfolio

In the example in Table 3.4, the beta of the hedger's portfolio is reduced to zero. (The hedger's expected return is independent of the performance of the index.) Sometimes futures contracts are used to change the beta of a portfolio to some value other than zero. Continuing with our earlier example:

$$\text{S\&P 500 index} = 1{,}000$$
$$\text{S\&P 500 futures price} = 1{,}010$$
$$\text{Value of portfolio} = \$5{,}050{,}000$$
$$\text{Beta of portfolio} = 1.5$$

As before, $F = 250 \times 1{,}010 = 252{,}500$ and a complete hedge requires

$$1.5 \times \frac{5{,}050{,}000}{252{,}500} = 30$$

contracts to be shorted. To reduce the beta of the portfolio from 1.5 to 0.75, the number of contracts shorted should be 15 rather than 30; to increase the beta of the portfolio to 2.0, a long position in 10 contracts should be taken; and so on. In general, to change the beta of the portfolio from β to β^*, where $\beta > \beta^*$, a short position in

$$(\beta - \beta^*)\frac{P}{F}$$

contracts is required. When $\beta < \beta^*$, a long position in

$$(\beta^* - \beta)\frac{P}{F}$$

contracts is required.

Exposure to the Price of an Individual Stock

Some exchanges do trade futures contracts on selected individual stocks, but in most cases a position in an individual stock can only be hedged using a stock index futures contract.

Hedging an exposure to the price of an individual stock using index futures contracts is similar to hedging a well-diversified stock portfolio. The number of index futures contracts that the hedger should short is given by $\beta P/F$, where β is the beta of the stock, P is the total value of the shares owned, and F is the current value of one index futures contract. Note that although the number of contracts entered into is calculated in the same way as it is when a portfolio of stocks is being hedged, the performance of the hedge is considerably worse. The hedge provides protection only against the risk arising from market movements, and this risk is a relatively small proportion of the total risk in the price movements of individual stocks. The hedge is appropriate when an investor feels that the stock will outperform the market but is unsure about the performance of the market. It can also be used by an investment bank that has underwritten a new issue of the stock and wants protection against moves in the market as a whole.

Consider an investor who in June holds 20,000 IBM shares, each worth \$100. The investor feels that the market will be very volatile over the next month but that IBM has a good chance of outperforming the market. The investor decides to use the August futures contract on the S&P 500 to hedge the position during the 1-month period. The β of IBM is estimated at 1.1. The current futures price for the August contract on the S&P 500 is 900. Each contract is for delivery of \$250 times the index. In this case, $P = 20{,}000 \times 100 = 2{,}000{,}000$ and $F = 900 \times 250 = 225{,}000$. The number of contracts that should be shorted is therefore

$$1.1 \times \frac{2{,}000{,}000}{225{,}000} = 9.78$$

Rounding to the nearest integer, the hedger shorts 10 contracts, closing out the position 1 month later. Suppose IBM rises to \$125 during the month, and the futures price of the S&P 500 rises to 1080. The investor gains $20{,}000 \times (\$125 - \$100) = \$500{,}000$ on IBM while losing $10 \times 250 \times (1080 - 900) = \$450{,}000$ on the futures contracts. .

In this example, the hedge offsets a gain on the underlying asset with a loss on the futures contracts. The offset might seem to be counterproductive. However, it cannot be emphasized often enough that the purpose of a hedge is to reduce risk. A hedge tends to make unfavorable outcomes less unfavorable but also to make favorable outcomes less favorable.

3.6 ROLLING THE HEDGE FORWARD

Sometimes the expiration date of the hedge is later than the delivery dates of all the futures contracts that can be used. The hedger must then roll the hedge forward by closing out one futures contract and taking the same position in a futures contract with a later delivery date. Hedges can be rolled forward many times. Consider a company that wishes to use a short hedge to reduce the risk associated with the price to be received for an asset at time T. If there are futures contracts 1, 2, 3, ..., n (not all necessarily in existence at the present time) with progressively later delivery dates, the company can use the following strategy:

Time t_1: Short futures contract 1

Time t_2: Close out futures contract 1
Short futures contract 2

Time t_3: Close out futures contract 2
Short futures contract 3

⋮

Time t_n: Close out futures contract $n - 1$
Short futures contract n

Time T: Close out futures contract n

Table 3.5 Data for the example on rolling oil hedge forward.

Date	*Apr. 2007*	*Sept. 2007*	*Feb. 2008*	*June 2008*
Oct. 2007 futures price	68.20	67.40		
Mar. 2008 futures price		67.00	66.50	
July 2008 futures price			66.30	65.90
Spot price	69.00			66.00

Suppose that in April 2007 a company realizes that it will have 100,000 barrels of oil to sell in June 2008 and decides to hedge its risk with a hedge ratio of 1.0. (In this example, we do not make the "tailing" adjustment described in Section 3.4.) The current spot price is \$69. Although futures contracts are traded with maturities stretching several years into the future, we suppose that only the first six delivery months have sufficient liquidity to meet the company's needs. The company therefore shorts 100 October 2007 contracts. In September 2007 it rolls the hedge forward into the March 2008 contract. In February 2008 it rolls the hedge forward again into the July 2008 contract.

One possible outcome is shown in Table 3.5. The October 2007 contract is shorted at \$68.20 per barrel and closed out at \$67.40 per barrel for a profit of \$0.80 per barrel; the March 2008 contract is shorted at \$67.00 per barrel and closed out at \$66.50 per barrel for a profit of \$0.50 per barrel. The July 2008 contract is shorted at \$66.30 per barrel and closed out at \$65.90 per barrel for a profit of \$0.40 per barrel. The final spot price is \$66.

The dollar gain per barrel of oil from the short futures contracts is

$$(68.20 - 67.40) + (67.00 - 66.50) + (66.30 - 65.90) = 1.70$$

The oil price declined from \$69 to \$66. Receiving only \$1.70 per barrel compensation for a price decline of \$3.00 may appear unsatisfactory. However, we cannot expect total compensation for a price decline when futures prices are below spot prices. The best we can hope for is to lock in the futures price that would apply to a June 2008 contract if it were actively traded.

The daily settlement of futures contracts can cause a mismatch between the timing of the cash flows on hedge and the timing of the cash flows from the position being hedged. In situations where the hedge is rolled forward so that it lasts a long time this can lead to serious problems (see Business Snapshot 3.2).

SUMMARY

This chapter has discussed various ways in which a company can take a position in futures contracts to offset an exposure to the price of an asset. If the exposure is such that the company gains when the price of the asset increases and loses when the price of the asset decreases, a short hedge is appropriate. If the exposure is the other way round (i.e., the company gains when the price of the asset decreases and loses when the price of the asset increases), a long hedge is appropriate.

Business Snapshot 3.2 Metallgesellschaft: Hedging Gone Awry

Sometimes rolling hedges forward can lead to cash flow pressures. The problem was illustrated dramatically by the activities of a German company, Metallgesellschaft (MG), in the early 1990s.

MG sold a huge volume of 5- to 10-year heating oil and gasoline fixed-price supply contracts to its customers at 6 to 8 cents above market prices. It hedged its exposure with long positions in short-dated futures contracts that were rolled forward. As it turned out, the price of oil fell and there were margin calls on the futures positions. Considerable short-term cash flow pressures were placed on MG. The members of MG who devised the hedging strategy argued that these short-term cash outflows were offset by positive cash flows that would ultimately be realized on the long-term fixed-price contracts. However, the company's senior management and its bankers became concerned about the huge cash drain. As a result, the company closed out all the hedge positions and agreed with its customers that the fixed-price contracts would be abandoned. The outcome was a loss to MG of $1.33 billion.

Hedging is a way of reducing risk. As such, it should be welcomed by most executives. In reality, there are a number of theoretical and practical reasons why companies do not hedge. On a theoretical level, we can argue that shareholders, by holding well-diversified portfolios, can eliminate many of the risks faced by a company. They do not require the company to hedge these risks. On a practical level, a company may find that it is increasing rather than decreasing risk by hedging if none of its competitors does so. Also, a treasurer may fear criticism from other executives if the company makes a gain from movements in the price of the underlying asset and a loss on the hedge.

An important concept in hedging is basis risk. The basis is the difference between the spot price of an asset and its futures price. Basis risk arises from uncertainty as to what the basis will be at maturity of the hedge.

The hedge ratio is the ratio of the size of the position taken in futures contracts to the size of the exposure. It is not always optimal to use a hedge ratio of 1.0. If the hedger wishes to minimize the variance of a position, a hedge ratio different from 1.0 may be appropriate. The optimal hedge ratio is the slope of the best-fit line obtained when changes in the spot price are regressed against changes in the futures price.

Stock index futures can be used to hedge the systematic risk in an equity portfolio. The number of futures contracts required is the beta of the portfolio multiplied by the ratio of the value of the portfolio to the value of one futures contract. Stock index futures can also be used to change the beta of a portfolio without changing the stocks that make up the portfolio.

When there is no liquid futures contract that matures later than the expiration of the hedge, a strategy known as rolling the hedge forward may be appropriate. This involves entering into a sequence of futures contracts. When the first futures contract is near expiration, it is closed out and the hedger enters into a second contract with a later delivery month. When the second contract is close to expiration, it is closed out and the hedger enters into a third contract with a later delivery month; and so on. The result of all this is the creation of a long-dated futures contract by trading a series of short-dated contracts.

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Questions and Problems (Answers in Solutions Manual)

3.1. Under what circumstances are (a) a short hedge and (b) a long hedge appropriate?

3.2. Explain what is meant by *basis risk* when futures contracts are used for hedging.

3.3. Explain what is meant by a *perfect hedge*. Does a perfect hedge always lead to a better outcome than an imperfect hedge? Explain your answer.

3.4. Under what circumstances does a minimum variance hedge portfolio lead to no hedging at all?

3.5. Give three reasons why the treasurer of a company might not hedge the company's exposure to a particular risk.

3.6. Suppose that the standard deviation of quarterly changes in the prices of a commodity is \$0.65, the standard deviation of quarterly changes in a futures price on the commodity is \$0.81, and the coefficient of correlation between the two changes is 0.8. What is the optimal hedge ratio for a 3-month contract? What does it mean?

3.7. A company has a \$20 million portfolio with a beta of 1.2. It would like to use futures contracts on the S&P 500 to hedge its risk. The index futures price is currently standing at 1080, and each contract is for delivery of \$250 times the index. What is the hedge that minimizes risk? What should the company do if it wants to reduce the beta of the portfolio to 0.6?

3.8. In the Chicago Board of Trade's corn futures contract, the following delivery months are available: March, May, July, September, and December. State the contract that should be used for hedging when the expiration of the hedge is in (a) June, (b) July, and (c) January.

3.9. Does a perfect hedge always succeed in locking in the current spot price of an asset for a future transaction? Explain your answer.

3.10. Explain why a short hedger's position improves when the basis strengthens unexpectedly and worsens when the basis weakens unexpectedly.

3.11. Imagine you are the treasurer of a Japanese company exporting electronic equipment to the United States. Discuss how you would design a foreign exchange hedging strategy and the arguments you would use to sell the strategy to your fellow executives.

3.12. Suppose that in Example 3.2 of Section 3.3 the company decides to use a hedge ratio of 0.8. How does the decision affect the way in which the hedge is implemented and the result?

3.13. "If the minimum variance hedge ratio is calculated as 1.0, the hedge must be perfect." Is this statement true? Explain your answer.

3.14. "If there is no basis risk, the minimum variance hedge ratio is always 1.0." Is this statement true? Explain your answer.

3.15. "For an asset where futures prices are usually less than spot prices, long hedges are likely to be particularly attractive." Explain this statement.

3.16. The standard deviation of monthly changes in the spot price of live cattle is (in cents per pound) 1.2. The standard deviation of monthly changes in the futures price of live cattle for the closest contract is 1.4. The correlation between the futures price changes and the spot price changes is 0.7. It is now October 15. A beef producer is committed to purchasing 200,000 pounds of live cattle on November 15. The producer wants to use the December live cattle futures contracts to hedge its risk. Each contract is for the delivery of 40,000 pounds of cattle. What strategy should the beef producer follow?

3.17. A corn farmer argues "I do not use futures contracts for hedging. My real risk is not the price of corn. It is that my whole crop gets wiped out by the weather." Discuss this viewpoint. Should the farmer estimate his or her expected production of corn and hedge to try to lock in a price for expected production?

3.18. On July 1, an investor holds 50,000 shares of a certain stock. The market price is \$30 per share. The investor is interested in hedging against movements in the market over the next month and decides to use the September Mini S&P 500 futures contract. The index futures price is currently 1,500 and one contract is for delivery of \$50 times the index. The beta of the stock is 1.3. What strategy should the investor follow?

3.19. Suppose that in Table 3.5 the company decides to use a hedge ratio of 1.5. How does the decision affect the way the hedge is implemented and the result?

3.20. A futures contract is used for hedging. Explain why the marking to market of the contract can give rise to cash flow problems.

3.21. An airline executive has argued: "There is no point in our using oil futures. There is just as much chance that the price of oil in the future will be less than the futures price as there is that it will be greater than this price." Discuss the executive's viewpoint.

3.22. Suppose that the 1-year gold lease rate is 1.5% and the 1-year risk-free rate is 5.0%. Both rates are compounded annually. Use the discussion in Business Snapshot 3.1 to calculate the maximum 1-year gold forward price Goldman Sachs should quote to the gold-mining company when the spot price is $600.

Assignment Questions

3.23. The following table gives data on monthly changes in the spot price and the futures price for a certain commodity. Use the data to calculate a minimum variance hedge ratio.

Spot price change	+0.50	+0.61	−0.22	−0.35	+0.79
Futures price change	+0.56	+0.63	−0.12	−0.44	+0.60
Spot price change	+0.04	+0.15	+0.70	−0.51	−0.41
Futures price change	−0.06	+0.01	+0.80	−0.56	−0.46

3.24. It is July 16. A company has a portfolio of stocks worth $100 million. The beta of the portfolio is 1.2. The company would like to use the CME December futures contract on the S&P 500 to change the beta of the portfolio to 0.5 during the period July 16 to November 16. The index futures price is currently 1,000, and each contract is on $250 times the index.

(a) What position should the company take?

(b) Suppose that the company changes its mind and decides to increase the beta of the portfolio from 1.2 to 1.5. What position in futures contracts should it take?

3.25. A fund manager has a portfolio worth $50 million with a beta of 0.87. The manager is concerned about the performance of the market over the next 2 months and plans to use 3-month futures contracts on the S&P 500 to hedge the risk. The current level of the index is 1,250, one contract is on 250 times the index, the risk-free rate is 6% per annum, and the dividend yield on the index is 3% per annum. The current 3-month futures price is 1259.

(a) What position should the fund manager take to hedge all exposure to the market over the next 2 months?

(b) Calculate the effect of your strategy on the fund manager's returns if the index in 2 months is 1,000, 1,100, 1,200, 1,300, and 1,400. Assume that the 1-month futures price is 0.25% higher than the index level at this time.

3.26. It is now October 2007. A company anticipates that it will purchase 1 million pounds of copper in each of February 2008, August 2008, February 2009, and August 2009. The company has decided to use the futures contracts traded in the COMEX division of the New York Mercantile Exchange to hedge its risk. One contract is for the delivery of

25,000 pounds of copper. The initial margin is $2,000 per contract and the maintenance margin is $1,500 per contract. The company's policy is to hedge 80% of its exposure. Contracts with maturities up to 13 months into the future are considered to have sufficient liquidity to meet the company's needs. Devise a hedging strategy for the company. Do not make the tailing adjustment described in Section 3.4.

Assume the market prices (in cents per pound) today and at future dates are as follows:

Date	*Oct. 2007*	*Feb. 2008*	*Aug. 2008*	*Feb. 2009*	*Aug. 2009*
Spot price	372.00	369.00	365.00	377.00	388.00
Mar. 2008 futures price	372.30	369.10			
Sept. 2008 futures price	372.80	370.20	364.80		
Mar. 2009 futures price		370.70	364.30	376.70	
Sept. 2009 futures price			364.20	376.50	388.20

What is the impact of the strategy you propose on the price the company pays for copper? What is the initial margin requirement in October 2007? Is the company subject to any margin calls?

APPENDIX

PROOF OF THE MINIMUM VARIANCE HEDGE RATIO FORMULA

Suppose we expect to sell N_A units of an asset at time t_2 and choose to hedge at time t_1 by shorting futures contracts on N_F units of a similar asset. The hedge ratio, which we will denote by h, is

$$h = \frac{N_F}{N_A} \tag{3A.1}$$

We will denote the total amount realized for the asset when the profit or loss on the hedge is taken into account by Y, so that

$$Y = S_2 N_A - (F_2 - F_1) N_F$$

or

$$Y = S_1 N_A + (S_2 - S_1) N_A - (F_2 - F_1) N_F \tag{3A.2}$$

where S_1 and S_2 are the asset prices at times t_1 and t_2, and F_1 and F_2 are the futures prices at times t_1 and t_2. From equation (3A.1), the expression for Y in equation (3A.2) can be written

$$Y = S_1 N_A + N_A(\Delta S - h\,\Delta F) \tag{3A.3}$$

where

$$\Delta S = S_2 - S_1 \quad \text{and} \quad \Delta F = F_2 - F_1$$

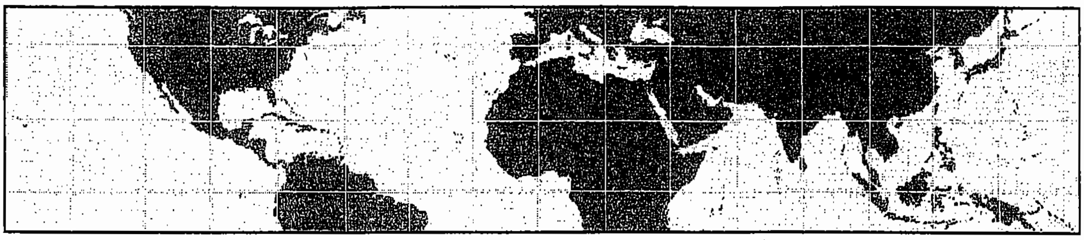
Because S_1 and N_A are known at time t_1, the variance of Y in equation (3A.3) is minimized when the variance of $\Delta S - h\,\Delta F$ is minimized. The variance of $\Delta S - h\,\Delta F$ is

$$v = \sigma_S^2 + h^2\sigma_F^2 - 2h\rho\sigma_S\sigma_F$$

where σ_S, σ_F, and ρ are as defined in Section 3.4, so that

$$\frac{dv}{dh} = 2h\sigma_F^2 - 2\rho\sigma_S\sigma_F$$

Setting this equal to zero, and noting that d^2v/dh^2 is positive, we see that the value of h that minimizes the variance is $h = \rho\sigma_S/\sigma_F$.



CHAPTER 13

The Black–Scholes–Merton Model

In the early 1970s, Fischer Black, Myron Scholes, and Robert Merton achieved a major breakthrough in the pricing of stock options.[1] This involved the development of what has become known as the Black–Scholes (or Black–Scholes–Merton) model. The model has had a huge influence on the way that traders price and hedge options. It has also been pivotal to the growth and success of financial engineering in the last 30 years. In 1997, the importance of the model was recognized when Robert Merton and Myron Scholes were awarded the Nobel prize for economics. Sadly, Fischer Black died in 1995, otherwise he too would undoubtedly have been one of the recipients of this prize.

This chapter shows how the Black–Scholes model for valuing European call and put options on a non-dividend-paying stock is derived. It explains how volatility can be either estimated from historical data or implied from option prices using the model. It shows how the risk-neutral valuation argument introduced in Chapter 11 can be used. It also shows how the Black–Scholes model can be extended to deal with European call and put options on dividend-paying stocks and presents some results on the pricing of American call options on dividend-paying stocks.

13.1 LOGNORMAL PROPERTY OF STOCK PRICES

The model of stock price behavior used by Black, Scholes, and Merton is the model we developed in Chapter 12. It assumes that percentage changes in the stock price in a short period of time are normally distributed. Define

μ: Expected return on stock per year
σ: Volatility of the stock price per year

The mean of the return in time Δt is $\mu\,\Delta t$ and the standard deviation of the return is

[1] See F. Black and M. Scholes, "The Pricing of Options and Corporate Liabilities," *Journal of Political Economy*, 81 (May/June 1973): 637–59; R.C. Merton, "Theory of Rational Option Pricing," *Bell Journal of Economics and Management Science*, 4 (Spring 1973): 141–83.

$\sigma\sqrt{\Delta t}$, so that

$$\frac{\Delta S}{S} \sim \phi(\mu\,\Delta t,\ \sigma^2 \Delta t) \tag{13.1}$$

where ΔS is the change in the stock price S in time Δt, and $\phi(m, v)$ denotes a normal distribution with mean m and variance v.

As shown in Section 12.6, the model implies that

$$\ln S_T - \ln S_0 \sim \phi\left[\left(\mu - \frac{\sigma^2}{2}\right)T,\ \sigma^2 T\right]$$

From this, it follows that

$$\ln\frac{S_T}{S_0} \sim \phi\left[\left(\mu - \frac{\sigma^2}{2}\right)T,\ \sigma^2 T\right] \tag{13.2}$$

and

$$\ln S_T \sim \phi\left[\ln S_0 + \left(\mu - \frac{\sigma^2}{2}\right)T,\ \sigma^2 T\right] \tag{13.3}$$

where S_T is the stock price at a future time T and S_0 is the stock price at time 0. Equation (13.3) shows that $\ln S_T$ is normally distributed, so that S_T has a lognormal distribution. The mean of $\ln S_T$ is $\ln S_0 + (\mu - \sigma^2/2)T$ and the standard deviation is $\sigma\sqrt{T}$.

Example 13.1

Consider a stock with an initial price of \$40, an expected return of 16% per annum, and a volatility of 20% per annum. From equation (13.3), the probability distribution of the stock price S_T in 6 months' time is given by

$$\ln S_T \sim \phi[\ln 40 + (0.16 - 0.2^2/2) \times 0.5,\ 0.2^2 \times 0.5]$$

$$\ln S_T \sim \phi(3.759,\ 0.02)$$

There is a 95% probability that a normally distributed variable has a value within 1.96 standard deviations of its mean. In this case, the standard deviation is $\sqrt{0.02} = 0.141$. Hence, with 95% confidence,

$$3.759 - 1.96 \times 0.141 < \ln S_T < 3.759 + 1.96 \times 0.141$$

This can be written

$$e^{3.759-1.96\times 0.141} < S_T < e^{3.759+1.96\times 0.141}$$

or

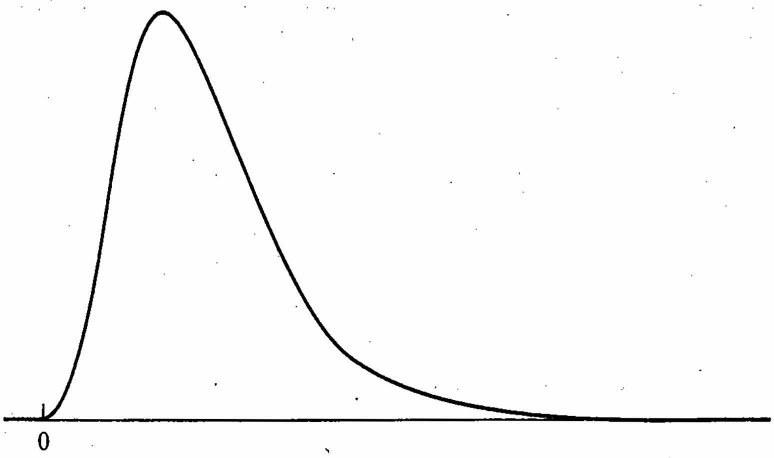
$$32.55 < S_T < 56.56$$

Thus, there is a 95% probability that the stock price in 6 months will lie between 32.55 and 56.56.

A variable that has a lognormal distribution can take any value between zero and infinity. Figure 13.1 illustrates the shape of a lognormal distribution. Unlike the normal distribution, it is skewed so that the mean, median, and mode are all different. From equation (13.3) and the properties of the lognormal distribution, it can be shown that the expected value $E(S_T)$ of S_T is given by

$$E(S_T) = S_0 e^{\mu T} \tag{13.4}$$

Figure 13.1 Lognormal distribution.



This fits in with the definition of μ as the expected rate of return. The variance $\text{var}(S_T)$ of S_T, can be shown to be given by[2]

$$\text{var}(S_T) = S_0^2 e^{2\mu T}(e^{\sigma^2 T} - 1) \tag{13.5}$$

Example 13.2

Consider a stock where the current price is \$20, the expected return is 20% per annum, and the volatility is 40% per annum. The expected stock price, $E(S_T)$, and the variance of the stock price, $\text{var}(S_T)$, in 1 year are given by

$$E(S_T) = 20e^{0.2\times 1} = 24.43 \quad \text{and} \quad \text{var}(S_T) = 400e^{2\times 0.2\times 1}(e^{0.4^2\times 1} - 1) = 103.54$$

The standard deviation of the stock price in 1 year is $\sqrt{103.54}$, or 10.18.

13.2 THE DISTRIBUTION OF THE RATE OF RETURN

The lognormal property of stock prices can be used to provide information on the probability distribution of the continuously compounded rate of return earned on a stock between times 0 and T. If we define the continuously compounded rate of return per annum realized between times 0 and T as x, then

$$S_T = S_0 e^{xT}$$

so that

$$x = \frac{1}{T}\ln\frac{S_T}{S_0} \tag{13.6}$$

From equation (13.2), it follows that

$$x \sim \phi\left(\mu - \frac{\sigma^2}{2}, \frac{\sigma^2}{T}\right) \tag{13.7}$$

[2] See Technical Note 2 on the author's website for a proof of the results in equations (13.4) and (13.5). For a more extensive discussion of the properties of the lognormal distribution, see J. Aitchison and J. A. C. Brown, *The Lognormal Distribution*. Cambridge University Press, 1966.

Thus, the continuously compounded rate of return per annum is normally distributed with mean $\mu - \sigma^2/2$ and standard deviation $\sigma/\sqrt{T}$. As T increases, the standard deviation of x declines. To understand the reason for this, consider two cases: $T = 1$ and $T = 20$. We are more certain about the average return per year over 20 years than we are about the return in any one year.

Example 13.3

Consider a stock with an expected return of 17% per annum and a volatility of 20% per annum. The probability distribution for the average rate of return (continuously compounded) realized over 3 years is normal, with mean

$$0.17 - \frac{0.2^2}{2} = 0.15$$

or 15% per annum, and standard deviation

$$\sqrt{\frac{0.2^2}{3}} = 0.1155$$

or 11.55% per annum. Because there is a 95% chance that a normally distributed variable will lie within 1.96 standard deviations of its mean, we can be 95% confident that the average return realized over 3 years will be between −7.6% and +37.6% per annum.

13.3 THE EXPECTED RETURN

The expected return, μ, required by investors from a stock depends on the riskiness of the stock. The higher the risk, the higher the expected return. It also depends on the level of interest rates in the economy. The higher the level of interest rates, the higher the expected return required on any given stock. Fortunately, we do not have to concern ourselves with the determinants of μ in any detail. It turns out that the value of a stock option, when expressed in terms of the value of the underlying stock, does not depend on μ at all. Nevertheless, there is one aspect of the expected return from a stock that frequently causes confusion and should be explained.

Equation (13.1) shows that $\mu\,\Delta t$ is the expected percentage change in the stock price in a very short period of time, Δt. It is natural to assume from this that μ is the expected continuously compounded return on the stock. However, this is not the case. The continuously compounded return, x, actually realized over a period of time of length T is given by equation (13.6) as

$$x = \frac{1}{T}\ln\frac{S_T}{S_0}$$

and, as indicated in equation (13.7), the expected value $E(x)$ of x is $\mu - \sigma^2/2$.

The reason why the expected continuously compounded return is different from μ is subtle, but important. Suppose we consider a very large number of very short periods of time of length Δt. Define S_i as the stock price at the end of the ith interval and ΔS_i as $S_{i+1} - S_i$. Under the assumptions we are making for stock price behavior, the average of the returns on the stock in each interval is close to μ. In other words, $\mu\,\Delta t$ is close to the arithmetic mean of the $\Delta S_i/S_i$. However, the expected return over the whole period

Business Snapshot 13.1 Mutual Fund Returns Can Be Misleading

The difference between μ and $\mu - \sigma^2/2$ is closely related to an issue in the reporting of mutual fund returns. Suppose that the following is a sequence of returns per annum reported by a mutual fund manager over the last five years (measured using annual compounding):

$$15\%,\quad 20\%,\quad 30\%,\quad -20\%,\quad 25\%$$

The arithmetic mean of the returns, calculated by taking the sum of the returns and dividing by 5, is 14%. However, an investor would actually earn less than 14% per annum by leaving the money invested in the fund for 5 years. The dollar value of \$100 at the end of the 5 years would be

$$100 \times 1.15 \times 1.20 \times 1.30 \times 0.80 \times 1.25 = \$179.40$$

By contrast, a 14% return with annual compounding would give

$$100 \times 1.14^5 = \$192.54$$

The return that gives \$179.40 at the end of five years is 12.4%. This is because

$$100 \times (1.124)^5 = 179.40$$

What average return should the fund manager report? It is tempting for the manager to make a statement such as: "The average of the returns per year that we have realized in the last 5 years is 14%." Although true, this is misleading. It is much less misleading to say: "The average return realized by someone who invested with us for the last 5 years is 12.4% per year." In some jurisdictions, regulations require fund managers to report returns the second way.

This phenomenon is an example of a result that is well known by mathematicians. The geometric mean of a set of numbers (not all the same) is always less than the arithmetic mean. In our example, the return multipliers each year are 1.15, 1.20, 1.30, 0.80, and 1.25. The arithmetic mean of these numbers is 1.140, but the geometric mean is only 1.124.

covered by the data, expressed with a compounding interval of Δt, is close to $\mu - \sigma^2/2$, not μ.[3] Business Snapshot 13.1 provides a numerical example concerning the mutual fund industry to illustrate why this is so. For a mathematical explanation of what is going on, we start with equation (13.4):

$$E(S_T) = S_0 e^{\mu T}$$

Taking logarithms, we get

$$\ln[E(S_T)] = \ln(S_0) + \mu T$$

It is now tempting to set $\ln[E(S_T)] = E[\ln(S_T)]$, so that $E[\ln(S_T)] - \ln(S_0) = \mu T$, or $E[\ln(S_T/S_0)] = \mu T$, which leads to $E(x) = \mu$. However, we cannot do this because ln

[3] The arguments in this section show that the term "expected return" is ambiguous. It can refer either to μ or to $\mu - \sigma^2/2$. Unless otherwise stated, it will be used to refer to μ throughout this book.

is a nonlinear function. In fact, $\ln[E(S_T)] > E[\ln(S_T)]$, so that $E[\ln(S_T/S_0)] < \mu T$, which leads to $E(x) < \mu$. (As pointed out above, $E(x) = \mu - \sigma^2/2$.)

13.4 VOLATILITY

The volatility σ of a stock is a measure of our uncertainty about the returns provided by the stock. Stocks typically have a volatility between 15% and 60%.

From equation (13.7), the volatility of a stock price can be defined as the standard deviation of the return provided by the stock in 1 year when the return is expressed using continuous compounding.

When Δt is small, equation (13.1) shows that $\sigma^2 \Delta t$ is approximately equal to the variance of the percentage change in the stock price in time Δt. This means that $\sigma\sqrt{\Delta t}$ is approximately equal to the standard deviation of the percentage change in the stock price in time Δt. Suppose that $\sigma = 0.3$, or 30%, per annum and the current stock price is \$50. The standard deviation of the percentage change in the stock price in 1 week is approximately

$$30 \times \sqrt{\frac{1}{52}} = 4.16\%$$

A one-standard-deviation move in the stock price in 1 week is therefore 50×0.0416, or \$2.08.

Uncertainty about a future stock price, as measured by its standard deviation, increases—at least approximately—with the square root of how far ahead we are looking. For example, the standard deviation of the stock price in 4 weeks is approximately twice the standard deviation in 1 week.

Estimating Volatility from Historical Data

To estimate the volatility of a stock price empirically, the stock price is usually observed at fixed intervals of time (e.g., every day, week, or month). Define:

$n+1$: Number of observations

S_i: Stock price at end of ith interval, with $i = 0, 1, \dots, n$

τ: Length of time interval in years

and let

$$u_i = \ln\left(\frac{S_i}{S_{i-1}}\right) \quad \text{for } i = 1, 2, \dots, n$$

The usual estimate, s, of the standard deviation of the u_i is given by

$$s = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(u_i - \bar{u})^2}$$

or

$$s = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n} u_i^2 - \frac{1}{n(n-1)}\left(\sum_{i=1}^{n} u_i\right)^2}$$

where $\bar{u}$ is the mean of the u_i.[4]

[4] The mean $\bar{u}$ is often assumed to be zero when estimates of historical volatilities are made.

From equation (13.2), the standard deviation of the u_i is $\sigma\sqrt{\tau}$. The variable s is therefore an estimate of $\sigma\sqrt{\tau}$. It follows that σ itself can be estimated as $\hat{\sigma}$, where

$$\hat{\sigma} = \frac{s}{\sqrt{\tau}}$$

The standard error of this estimate can be shown to be approximately $\hat{\sigma}/\sqrt{2n}$.

Choosing an appropriate value for n is not easy. More data generally lead to more accuracy, but σ does change over time and data that are too old may not be relevant for predicting the future volatility. A compromise that seems to work reasonably well is to use closing prices from daily data over the most recent 90 to 180 days. Alternatively, as a rule of thumb, n can be set equal to the number of days to which the volatility is to be applied. Thus, if the volatility estimate is to be used to value a 2-year option, daily data for the last 2 years are used. More sophisticated approaches to estimating volatility involving GARCH models are discussed in Chapter 21.

Example 13.4

Table 13.1 shows a possible sequence of stock prices during 21 consecutive trading days. In this case,

$$\sum u_i = 0.09531 \text{ and } \sum u_i^2 = 0.00326$$

Table 13.1 Computation of volatility.

Day	*Closing stock price (dollars)*	*Price relative* S_i/S_{i-1}	*Daily return* $u_i = \ln(S_i/S_{i-1})$
0	20.00		
1	20.10	1.00500	0.00499
2	19.90	0.99005	−0.01000
3	20.00	1.00503	0.00501
4	20.50	1.02500	0.02469
5	20.25	0.98780	−0.01227
6	20.90	1.03210	0.03159
7	20.90	1.00000	0.00000
8	20.90	1.00000	0.00000
9	20.75	0.99282	−0.00720
10	20.75	1.00000	0.00000
11	21.00	1.01205	0.01198
12	21.10	1.00476	0.00475
13	20.90	0.99052	−0.00952
14	20.90	1.00000	0.00000
15	21.25	1.01675	0.01661
16	21.40	1.00706	0.00703
17	21.40	1.00000	0.00000
18	21.25	0.99299	−0.00703
19	21.75	1.02353	0.02326
20	22.00	1.01149	0.01143

and the estimate of the standard deviation of the daily return is

$$\sqrt{\frac{0.00326}{19} - \frac{0.09531^2}{380}} = 0.01216$$

or 1.216%. Assuming that there are 252 trading days per year, $\tau = 1/252$ and the data give an estimate for the volatility per annum of $0.01216\sqrt{252} = 0.193$, or 19.3%. The standard error of this estimate is

$$\frac{0.193}{\sqrt{2 \times 20}} = 0.031$$

or 3.1% per annum.

The foregoing analysis assumes that the stock pays no dividends, but it can be adapted to accommodate dividend-paying stocks. The return, u_i, during a time interval that includes an ex-dividend day is given by

$$u_i = \ln\frac{S_i + D}{S_{i-1}}$$

where D is the amount of the dividend. The return in other time intervals is still

$$u_i = \ln\frac{S_i}{S_{i-1}}$$

However, as tax factors play a part in determining returns around an ex-dividend date, it is probably best to discard altogether data for intervals that include an ex-dividend date.

Trading Days vs. Calendar Days

An important issue is whether time should be measured in calendar days or trading days when volatility parameters are being estimated and used. As shown in Business Snapshot 13.2, research shows that volatility is much higher when the exchange is open for trading than when it is closed. As a result, practitioners tend to ignore days when the exchange is closed when estimating volatility from historical data and when calculating the life of an option. The volatility per annum is calculated from the volatility per trading day using the formula

$$\begin{matrix}\text{Volatility} \\ \text{per annum}\end{matrix} = \begin{matrix}\text{Volatility} \\ \text{per trading day}\end{matrix} \times \sqrt{\begin{matrix}\text{Number of trading days} \\ \text{per annum}\end{matrix}}$$

This is what we did in Example 13.4 when calculating volatility from the data in Table 13.1. The number of trading days in a year is usually assumed to be 252 for stocks.

The life of an option is also usually measured using trading days rather than calendar days. It is calculated as T years, where

$$T = \frac{\text{Number of trading days until option maturity}}{252}$$

Business Snapshot 13.2 What Causes Volatility?

It is natural to assume that the volatility of a stock is caused by new information reaching the market. This new information causes people to revise their opinions about the value of the stock. The price of the stock changes and volatility results. This view of what causes volatility is not supported by research. With several years of daily stock price data, researchers can calculate:

1. The variance of stock price returns between the close of trading on one day and the close of trading on the next day when there are no intervening nontrading days
2. The variance of the stock price returns between the close of trading on Friday and the close of trading on Monday

The second of these is the variance of returns over a 3-day period. The first is a variance over a 1-day period. We might reasonably expect the second variance to be three times as great as the first variance. Fama (1965), French (1980), and French and Roll (1986) show that this is not the case. These three research studies estimate the second variance to be, respectively, 22%, 19%, and 10.7% higher than the first variance.

At this stage one might be tempted to argue that these results are explained by more news reaching the market when the market is open for trading. But research by Roll (1984) does not support this explanation. Roll looked at the prices of orange juice futures. By far the most important news for orange juice futures prices is news about the weather and this is equally likely to arrive at any time. When Roll did a similar analysis to that just described for stocks, he found that the second (Friday-to-Monday) variance for orange juice futures is only 1.54 times the first variance.

The only reasonable conclusion from all this is that volatility is to a large extent caused by trading itself. (Traders usually have no difficulty accepting this conclusion!)

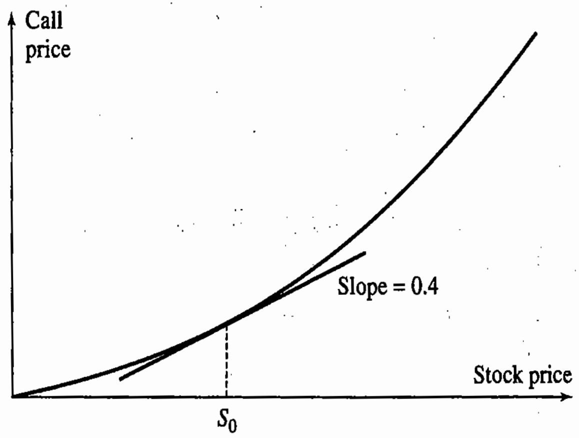
13.5 THE IDEA UNDERLYING THE BLACK–SCHOLES–MERTON DIFFERENTIAL EQUATION

The Black–Scholes–Merton differential equation is an equation that must be satisfied by the price of any derivative dependent on a non-dividend-paying stock. The equation is derived in the next section. Here we consider the nature of the arguments we will use.

These are similar to the no-arbitrage arguments we used to value stock options in Chapter 11 for the situation where stock price movements are binomial. They involve setting up a riskless portfolio consisting of a position in the derivative and a position in the stock. In the absence of arbitrage opportunities, the return from the portfolio must be the risk-free interest rate, r. This leads to the Black-Scholes-Merton differential equation.

The reason a riskless portfolio can be set up is that the stock price and the derivative price are both affected by the same underlying source of uncertainty: stock price movements. In any short period of time, the price of the derivative is perfectly correlated with the price of the underlying stock. When an appropriate portfolio of the stock and the derivative is established, the gain or loss from the stock position always offsets the gain or loss from the derivative position so that the overall value of the portfolio at the end of the short period of time is known with certainty.

Figure 13.2 Relationship between call price and stock price. Current stock price is S_0.



Suppose, for example, that at a particular point in time the relationship between a small change ΔS in the stock price and the resultant small change Δc in the price of a European call option is given by

$$\Delta c = 0.4\,\Delta S$$

This means that the slope of the line representing the relationship between c and S is 0.4, as indicated in Figure 13.2. The riskless portfolio would consist of:

1. A long position in 0.4 shares
2. A short position in one call option

Suppose, for example, that the stock price increases by 10 cents. The option price will increase by 4 cents and the $40 \times 0.10 = \$4$ gain on the shares is equal to the $100 \times 0.04 = \$4$ loss on the short option position.

There is one important difference between the Black–Scholes–Merton analysis and our analysis using a binomial model in Chapter 11. In Black–Scholes–Merton, the position in the stock and the derivative is riskless for only a very short period of time. (Theoretically, it remains riskless only for an instantaneously short period of time.) To remain riskless, it must be adjusted, or *rebalanced*, frequently.[5] For example, the relationship between Δc and ΔS in our example might change from $\Delta c = 0.4\,\Delta S$ today to $\Delta c = 0.5\,\Delta S$ in 2 weeks. This would mean that, in order to maintain the riskless position, an extra 0.1 share would have to be purchased for each call option sold. It is nevertheless true that the return from the riskless portfolio in any very short period of time must be the risk-free interest rate. This is the key element in the Black–Scholes analysis and leads to their pricing formulas.

Assumptions

The assumptions we use to derive the Black–Scholes–Merton differential equation are as follows:

1. The stock price follows the process developed in Chapter 12 with μ and σ constant.
2. The short selling of securities with full use of proceeds is permitted.

[5] We discuss the rebalancing of portfolios in more detail in Chapter 17.

3. There are no transactions costs or taxes. All securities are perfectly divisible.
4. There are no dividends during the life of the derivative.
5. There are no riskless arbitrage opportunities.
6. Security trading is continuous.
7. The risk-free rate of interest, r, is constant and the same for all maturities.

As we discuss in later chapters, some of these assumptions can be relaxed. For example, σ and r can be known functions of t. We can even allow interest rates to be stochastic provided that the stock price distribution at maturity of the option is still lognormal.

13.6 DERIVATION OF THE BLACK–SCHOLES–MERTON DIFFERENTIAL EQUATION

The stock price process we are assuming is the one we developed in Section 12.3:

$$dS = \mu S\,dt + \sigma S\,dz \tag{13.8}$$

Suppose that f is the price of a call option or other derivative contingent on S. The variable f must be some function of S and t. Hence, from equation (12.14),

$$df = \left(\frac{\partial f}{\partial S}\mu S + \frac{\partial f}{\partial t} + \tfrac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right)dt + \frac{\partial f}{\partial S}\sigma S\,dz \tag{13.9}$$

The discrete versions of equations (13.8) and (13.9) are

$$\Delta S = \mu S\,\Delta t + \sigma S\,\Delta z \tag{13.10}$$

and

$$\Delta f = \left(\frac{\partial f}{\partial S}\mu S + \frac{\partial f}{\partial t} + \tfrac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right)\Delta t + \frac{\partial f}{\partial S}\sigma S\,\Delta z \tag{13.11}$$

where Δf and ΔS are the changes in f and S in a small time interval Δt. Recall from the discussion of Itô's lemma in Section 12.5 that the Wiener processes underlying f and S are the same. In other words, the Δz $(= \epsilon\sqrt{\Delta t})$ in equations (13.10) and (13.11) are the same. It follows that a portfolio of the stock and the derivative can be constructed so that the Wiener process is eliminated.

The portfolio is

-1: derivative

$+\partial f/\partial S$: shares

The holder of this portfolio is short one derivative and long an amount $\partial f/\partial S$ of shares. Define Π as the value of the portfolio. By definition

$$\Pi = -f + \frac{\partial f}{\partial S}S \tag{13.12}$$

The change $\Delta\Pi$ in the value of the portfolio in the time interval Δt is given by

$$\Delta\Pi = -\Delta f + \frac{\partial f}{\partial S}\Delta S \tag{13.13}$$

Substituting equations (13.10) and (13.11) into equation (13.13) yields

$$\Delta\Pi = \left(-\frac{\partial f}{\partial t} - \tfrac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right)\Delta t \tag{13.14}$$

Because this equation does not involve Δz, the portfolio must be riskless during time Δt. The assumptions listed in the preceding section imply that the portfolio must instantaneously earn the same rate of return as other short-term risk-free securities. If it earned more than this return, arbitrageurs could make a riskless profit by borrowing money to buy the portfolio; if it earned less, they could make a riskless profit by shorting the portfolio and buying risk-free securities. It follows that

$$\Delta\Pi = r\Pi\,\Delta t \tag{13.15}$$

where r is the risk-free interest rate. Substituting from equations (13.12) and (13.14) into (13.15), we obtain

$$\left(\frac{\partial f}{\partial t} + \tfrac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right)\Delta t = r\left(f - \frac{\partial f}{\partial S}S\right)\Delta t$$

so that

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \tfrac{1}{2}\sigma^2 S^2\frac{\partial^2 f}{\partial S^2} = rf \tag{13.16}$$

Equation (13.16) is the Black–Scholes–Merton differential equation. It has many solutions, corresponding to all the different derivatives that can be defined with S as the underlying variable. The particular derivative that is obtained when the equation is solved depends on the *boundary conditions* that are used. These specify the values of the derivative at the boundaries of possible values of S and t. In the case of a European call option, the key boundary condition is

$$f = \max(S - K,\ 0) \quad \text{when } t = T$$

In the case of a European put option, it is

$$f = \max(K - S,\ 0) \quad \text{when } t = T$$

One point that should be emphasized about the portfolio used in the derivation of equation (13.16) is that it is not permanently riskless. It is riskless only for an infinitesimally short period of time. As S and t change, $\partial f/\partial S$ also changes. To keep the portfolio riskless, it is therefore necessary to frequently change the relative proportions of the derivative and the stock in the portfolio.

Example 13.5

A forward contract on a non-dividend-paying stock is a derivative dependent on the stock. As such, it should satisfy equation (13.16). From equation (5.5), we know that the value of the forward contract, f, at a general time t is given in terms of the stock price S at this time by

$$f = S - Ke^{-r(T-t)}$$

where K is the delivery price. This means that

$$\frac{\partial f}{\partial t} = -rKe^{-r(T-t)}, \qquad \frac{\partial f}{\partial S} = 1, \qquad \frac{\partial^2 f}{\partial S^2} = 0$$

When these are substituted into the left-hand side of equation (13.16), we obtain

$$-rKe^{-r(T-t)} + rS$$

This equals rf, showing that equation (13.16) is indeed satisfied.

The Prices of Tradeable Derivatives

Any function $f(S, t)$ that is a solution of the differential equation (13.16) is the theoretical price of a derivative that could be traded. If a derivative with that price existed, it would not create any arbitrage opportunities. Conversely, if a function $f(S, t)$ does not satisfy the differential equation (13.16), it cannot be the price of a derivative without creating arbitrage opportunities for traders.

To illustrate this point, consider first the function e^S. This does not satisfy the differential equation (13.16). It is therefore not a candidate for being the price of a derivative dependent on the stock price. If an instrument whose price was always e^S existed, there would be an arbitrage opportunity. As a second example, consider the function

$$\frac{e^{(\sigma^2-2r)(T-t)}}{S}$$

This does satisfy the differential equation, and so is, in theory, the price of a tradeable security. (It is the price of a derivative that pays off $1/S_T$ at time T.) For other examples of tradeable derivatives, see Problems 13.11, 13.12, 13.23, and 13.28.

13.7 RISK-NEUTRAL VALUATION

We introduced risk-neutral valuation in connection with the binomial model in Chapter 11. It is without doubt the single most important tool for the analysis of derivatives. It arises from one key property of the Black–Scholes–Merton differential equation (13.16). This property is that the equation does not involve any variables that are affected by the risk preferences of investors. The variables that do appear in the equation are the current stock price, time, stock price volatility, and the risk-free rate of interest. All are independent of risk preferences.

The Black–Scholes–Merton differential equation would not be independent of risk preferences if it involved the expected return, μ, on the stock. This is because the value of μ does depend on risk preferences. The higher the level of risk aversion by investors, the higher μ will be for any given stock. It is fortunate that μ happens to drop out in the derivation of the differential equation.

Because the Black–Scholes–Merton differential equation is independent of risk preferences, an ingenious argument can be used. If risk preferences do not enter the equation, they cannot affect its solution. Any set of risk preferences can, therefore, be used when evaluating f. In particular, the very simple assumption that all investors are risk neutral can be made.

In a world where investors are risk neutral, the expected return on all investment assets is the risk-free rate of interest, r. The reason is that risk-neutral investors do not require a premium to induce them to take risks. It is also true that the present value of any cash flow in a risk-neutral world can be obtained by discounting its expected value at the risk-free rate. The assumption that the world is risk neutral does, therefore, considerably simplify the analysis of derivatives.

Consider a derivative that provides a payoff at one particular time. It can be valued using risk-neutral valuation by using the following procedure:

1. Assume that the expected return from the underlying asset is the risk-free interest rate, r (i.e., assume $\mu = r$).
2. Calculate the expected payoff from the derivative.
3. Discount the expected payoff at the risk-free interest rate.

It is important to appreciate that risk-neutral valuation (or the assumption that all investors are risk neutral) is merely an artificial device for obtaining solutions to the Black–Scholes differential equation. The solutions that are obtained are valid in all worlds, not just those where investors are risk neutral. When we move from a risk-neutral world to a risk-averse world, two things happen. The expected growth rate in the stock price changes and the discount rate that must be used for any payoffs from the derivative changes. It happens that these two changes always offset each other exactly.

Application to Forward Contracts on a Stock

We valued forward contracts on a non-dividend-paying stock in Section 5.7. In Example 13.5, we verified that the pricing formula satisfies the Black–Scholes differential equation. In this section we derive the pricing formula from risk-neutral valuation. We make the assumption that interest rates are constant and equal to r. This is somewhat more restrictive than the assumption in Chapter 5.

Consider a long forward contract that matures at time T with delivery price, K. As indicated in Figure 1.2, the value of the contract at maturity is

$$S_T - K$$

where S_T is the stock price at time T. From the risk-neutral valuation argument, the value of the forward contract at time 0 is its expected value at time T in a risk-neutral world discounted at the risk-free rate of interest. Denoting the value of the forward contract at time zero by f, this means that

$$f = e^{-rT}\hat{E}(S_T - K)$$

where $\hat{E}$ denotes the expected value in a risk-neutral world. Since K is a constant, this equation becomes

$$f = e^{-rT}\hat{E}(S_T) - Ke^{-rT} \tag{13.17}$$

The expected return μ on the stock becomes r in a risk-neutral world. Hence, from equation (13.4), we have

$$\hat{E}(S_T) = S_0 e^{rT} \tag{13.18}$$

Substituting equation (13.18) into equation (13.17) gives

$$f = S_0 - Ke^{-rT} \tag{13.19}$$

This is in agreement with equation (5.5).

13.8 BLACK–SCHOLES PRICING FORMULAS

The Black–Scholes formulas for the prices at time 0 of a European call option on a non-dividend-paying stock and a European put option on a non-dividend-paying stock are

$$c = S_0 N(d_1) - Ke^{-rT} N(d_2) \tag{13.20}$$

and

$$p = Ke^{-rT} N(-d_2) - S_0 N(-d_1) \tag{13.21}$$

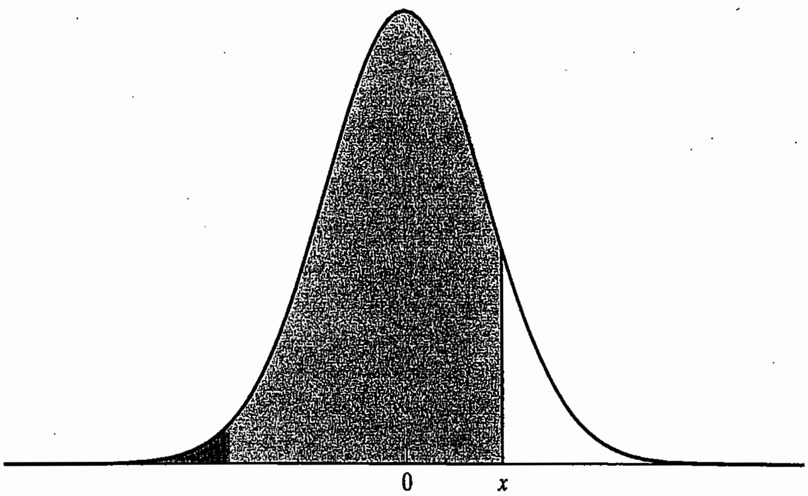
where

$$d_1 = \frac{\ln(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln(S_0/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

The function $N(x)$ is the cumulative probability distribution function for a standardized normal distribution. In other words, it is the probability that a variable with a standard normal distribution, $\phi(0, 1)$, will be less than x. It is illustrated in Figure 13.3. The remaining variables should be familiar. The variables c and p are the European call and European put price, S_0 is the stock price at time zero, K is the strike price, r is the

Figure 13.3 Shaded area represents $N(x)$.



continuously compounded risk-free rate, σ is the stock price volatility, and T is the time to maturity of the option.

One way of deriving the Black–Scholes formulas is by solving the differential equation (13.16) subject to the boundary condition mentioned in Section 13.6.[6] Another approach is to use risk-neutral valuation. Consider a European call option. The expected value of the option at maturity in a risk-neutral world is

$$\hat{E}[\max(S_T - K, 0)]$$

where, as before, $\hat{E}$ denotes the expected value in a risk-neutral world. From the risk-neutral valuation argument, the European call option price c is this expected value discounted at the risk-free rate of interest, that is,

$$c = e^{-rT}\hat{E}[\max(S_T - K, 0)] \tag{13.22}$$

The appendix at the end of this chapter shows that this equation leads to the result in equation (13.20).

To provide an interpretation of the terms in equation (13.20), we note that it can be written

$$c = e^{-rT}[S_0 N(d_1)e^{rT} - KN(d_2)] \tag{13.23}$$

The expression $N(d_2)$ is the probability that the option will be exercised in a risk-neutral world, so that $KN(d_2)$ is the strike price times the probability that the strike price will be paid. The expression $S_0 N(d_1)e^{rT}$ is the expected value in a risk-neutral world of a variable that is equal to S_T if $S_T > K$ and to zero otherwise.

Since it is never optimal to exercise early an American call option on a non-dividend-paying stock (see Section 9.5), equation (13.20) is the value of an American call option on a non-dividend-paying stock. Unfortunately, no exact analytic formula for the value of an American put option on a non-dividend-paying stock has been produced. Numerical procedures for calculating American put values are discussed in Chapter 19.

When the Black–Scholes formula is used in practice the interest rate r is set equal to the zero-coupon risk-free interest rate for a maturity T. As we show in later chapters, this is theoretically correct when r is a known function of time. It is also theoretically correct when the interest rate is stochastic provided that the stock price at time T is lognormal and the volatility parameter is chosen appropriately. As mentioned earlier, time is normally measured as the number of trading days left in the life of the option divided by the number of trading days in 1 year.

Properties of the Black–Scholes Formulas

We now show that the Black–Scholes formulas have the right general properties by considering what happens when some of the parameters take extreme values.

When the stock price, S_0, becomes very large, a call option is almost certain to be exercised. It then becomes very similar to a forward contract with delivery price K.

[6] The differential equation gives the call and put prices at a general time t. For example, the call price that satisfies the differential equation is $c = SN(d_1) - Ke^{-r(T-t)}N(d_2)$, where

$$d_1 = \frac{\ln(S/K) + (r + \sigma^2/2)(T - t)}{\sigma\sqrt{T - t}}$$

and $d_2 = d_1 - \sigma\sqrt{T - t}$. See Problem 13.17 to prove that the differential equation is satisfied.

From equation (5.5), we expect the call price to be

$$S_0 - Ke^{-rT}$$

This is, in fact, the call price given by equation (13.20) because, when S_0 becomes very large, both d_1 and d_2 become very large, and $N(d_1)$ and $N(d_2)$ become close to 1.0. When the stock price becomes very large, the price of a European put option, p, approaches zero. This is consistent with equation (13.21) because $N(-d_1)$ and $N(-d_2)$ are both close to zero in this case.

Consider next what happens when the volatility σ approaches zero. Because the stock is virtually riskless, its price will grow at rate r to S_0e^{rT} at time T and the payoff from a call option is

$$\max(S_0e^{rT} - K,\ 0)$$

Discounting at rate r, the value of the call today is

$$e^{-rT}\max(S_0e^{rT} - K,\ 0) = \max(S_0 - Ke^{-rT},\ 0)$$

To show that this is consistent with equation (13.20), consider first the case where $S_0 > Ke^{-rT}$. This implies that $\ln(S_0/K) + rT > 0$. As σ tends to zero, d_1 and d_2 tend to $+\infty$, so that $N(d_1)$ and $N(d_2)$ tend to 1.0 and equation (13.20) becomes

$$c = S_0 - Ke^{-rT}$$

When $S_0 < Ke^{-rT}$, it follows that $\ln(S_0/K) + rT < 0$. As σ tends to zero, d_1 and d_2 tend to $-\infty$, so that $N(d_1)$ and $N(d_2)$ tend to zero and equation (13.20) gives a call price of zero. The call price is therefore always $\max(S_0 - Ke^{-rT},\ 0)$ as σ tends to zero. Similarly, it can be shown that the put price is always $\max(Ke^{-rT} - S_0,\ 0)$ as σ tends to zero.

13.9 CUMULATIVE NORMAL DISTRIBUTION FUNCTION

The only problem in implementing equations (13.20) and (13.21) is in calculating the cumulative normal distribution function, $N(x)$. Tables for $N(x)$ are provided at the end of this book. The NORMSDIST function calculates $N(x)$ in Excel. A polynomial approximation that gives six-decimal-place accuracy is[7]

$$N(x) = \begin{cases} 1 - N'(x)(a_1k + a_2k^2 + a_3k^3 + a_4k^4 + a_5k^5) & \text{when } x \geqslant 0 \\ 1 - N(-x) & \text{when } x < 0 \end{cases}$$

where

$$k = \frac{1}{1 + \gamma x}, \quad \gamma = 0.2316419$$

$$a_1 = 0.319381530, \quad a_2 = -0.356563782$$

$$a_3 = 1.781477937, \quad a_4 = -1.821255978, \quad a_5 = 1.330274429$$

[7] See M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions*. New York: Dover Publications, 1972.

and

$$N'(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

Example 13.6

The stock price 6 months from the expiration of an option is \$42, the exercise price of the option is \$40, the risk-free interest rate is 10% per annum, and the volatility is 20% per annum. This means that $S_0 = 42$, $K = 40$, $r = 0.1$, $\sigma = 0.2$, $T = 0.5$,

$$d_1 = \frac{\ln(42/40) + (0.1 + 0.2^2/2) \times 0.5}{0.2\sqrt{0.5}} = 0.7693$$

$$d_2 = \frac{\ln(42/40) + (0.1 - 0.2^2/2) \times 0.5}{0.2\sqrt{0.5}} = 0.6278$$

and

$$Ke^{-rT} = 40e^{-0.05} = 38.049$$

Hence, if the option is a European call, its value c is given by

$$c = 42N(0.7693) - 38.049N(0.6278)$$

If the option is a European put, its value p is given by

$$p = 38.049N(-0.6278) - 42N(-0.7693)$$

Using the polynomial approximation just given or the NORMSDIST function in Excel,

$$N(0.7693) = 0.7791, \qquad N(-0.7693) = 0.2209$$

$$N(0.6278) = 0.7349, \qquad N(-0.6278) = 0.2651$$

so that

$$c = 4.76, \qquad p = 0.81$$

Ignoring the time value of money, the stock price has to rise by \$2.76 for the purchaser of the call to break even. Similarly, the stock price has to fall by \$2.81 for the purchaser of the put to break even.

13.10 WARRANTS AND EMPLOYEE STOCK OPTIONS

The exercise of a regular call option on a company has no effect on the number of the company's shares outstanding. If the writer of the option does not own the company's shares, he or she must buy them in the market in the usual way and then sell them to the option holder for the strike price. As explained in Chapter 8, warrants and employee stock options are different from regular call options in that exercise leads to the company issuing more shares and then selling them to the option holder for the strike price. As the strike price is less than the market price, this dilutes the interest of the existing shareholders.

How should potential dilution affect the way we value outstanding warrants and employee stock options? The answer is that it should not! Assuming markets are

Business Snapshot 13.3 Warrants, Employee Stock Options, and Dilution

Consider a company with 100,000 shares each worth \$50. It surprises the market with an announcement that it is granting 100,000 stock options to its employees with a strike price of \$50. If the market sees little benefit to the shareholders from the employee stock options in the form of reduced salaries and more highly motivated managers, the stock price will decline immediately after the announcement of the employee stock options. If the stock price declines to \$45, the dilution cost to the current shareholders is \$5 per share or \$500,000 in total.

Suppose that the company does well so that by the end of three years the share price is \$100. Suppose further that all the options are exercised at this point. The payoff to the employees is \$50 per option. It is tempting to argue that there will be further dilution in that 100,000 shares worth \$100 per share are now merged with 100,000 shares for which only \$50 is paid, so that (a) the share price reduces to \$75 and (b) the payoff to the option holders is only \$25 per option. However, this argument is flawed. The exercise of the options is anticipated by the market and already reflected in the share price. The payoff from each option exercised is \$50.

This example illustrates the general point that when markets are efficient the impact of dilution from executive stock options or warrants is reflected in the stock price as soon as they are announced and does not need to be taken into account again when the options are valued.

efficient the stock price will reflect potential dilution from all outstanding warrants and employee stock options. This is explained in Business Snapshot 13.3.[8]

Consider next the situation a company is in when it is contemplating a new issue of warrants (or employee stock options). We suppose that the company is interested in calculating the cost of the issue assuming that there are no compensating benefits. We assume that the company has N shares worth S_0 each and the number of new options contemplated is M, with each option giving the holder the right to buy one share for K. The value of the company today is NS_0. This value does not change as a result of the warrant issue. Suppose that without the warrant issue the share price will be S_T at the warrant's maturity. This means that (with or without the warrant issue) the total value of the equity and the warrants at time T will NS_T. If the warrants are exercised, there is a cash inflow from the strike price increasing this to $NS_T + MK$. This value is distributed among $N + M$ shares, so that the share price immediately after exercise becomes

$$\frac{NS_T + MK}{N + M}$$

Therefore the payoff to an option holder if the option is exercised is

$$\frac{NS_T + MK}{N + M} - K$$

[8] Analysts sometimes assume that the sum of the values of the warrants and the equity (rather than just the value of the equity) is lognormal. The result is a Black–Scholes type of equation for the value of the warrant in terms of the value of the warrant. See Technical Note 3 on the author's website for an explanation of this model.

or

$$\frac{N}{N+M}(S_T - K)$$

This shows that the value of each option is the value of

$$\frac{N}{N+M}$$

regular call options on the company's stock. Therefore the total cost of the options is M times this.

Example 13.7

A company with 1 million shares worth \$40 each is considering issuing 200,000 warrants each giving the holder the right to buy one share with a strike price of \$60 in 5 years. It wants to know the cost of this. The interest rate is 3% per annum, and the volatility is 30% per annum. The company pays no dividends. From equation (13.20), the value of a 5-year European call option on the stock is \$7.04. In this case, $N = 1{,}000{,}000$ and $M = 200{,}000$, so that the value of each warrant is

$$\frac{1{,}000{,}000}{1{,}000{,}000 + 200{,}000} \times 7.04 = 5.87$$

or \$5.87. The total cost of the warrant issue is $200{,}000 \times 5.87 = \$1.17$ million. Assuming the market perceives no benefits from the warrant issue, we expect the stock price to decline by \$1.17 to \$38.83.

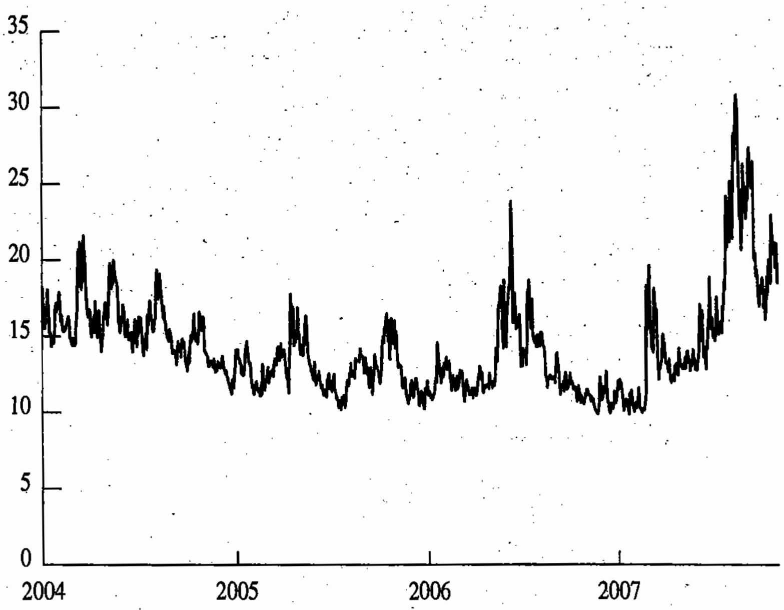
13.11 IMPLIED VOLATILITIES

The one parameter in the Black–Scholes pricing formulas that cannot be directly observed is the volatility of the stock price. In Section 13.4, we discussed how this can be estimated from a history of the stock price. In practice, traders usually work with what are known as *implied volatilities*. These are the volatilities implied by option prices observed in the market.

To illustrate how implied volatilities are calculated, suppose that the value of a European call option on a non-dividend-paying stock is 1.875 when $S_0 = 21$, $K = 20$, $r = 0.1$, and $T = 0.25$. The implied volatility is the value of σ that, when substituted into equation (13.20), gives $c = 1.875$. Unfortunately, it is not possible to invert equation (13.20) so that σ is expressed as a function of S_0, K, r, T, and c. However, an iterative search procedure can be used to find the implied σ. For example, we can start by trying $\sigma = 0.20$. This gives a value of c equal to 1.76, which is too low. Because c is an increasing function of σ, a higher value of σ is required. We can next try a value of 0.30 for σ. This gives a value of c equal to 2.10, which is too high and means that σ must lie between 0.20 and 0.30. Next, a value of 0.25 can be tried for σ. This also proves to be too high, showing that σ lies between 0.20 and 0.25. Proceeding in this way, we can halve the range for σ at each iteration and the correct value of σ can be calculated to any required accuracy.[9] In this example, the implied volatility is 0.235, or 23.5%, per

[9] This method is presented for illustration. Other more powerful methods, such as the Newton–Raphson method, are often used in practice (see footnote 4 of Chapter 4). DerivaGem can be used to calculate implied volatilities.

Figure 13.4 The VIX index, January 2004 to October 2007.



annum. A similar procedure can be used in conjunction with binomial trees to find implied volatilities for American options.

Implied volatilities are used to monitor the market's opinion about the volatility of a particular stock. Whereas historical volatilities (see Section 13.4) are backward looking, implied volatilities are forward looking. Traders often quote the implied volatility of an option rather than its price. This is convenient because the implied volatility tends to be less variable than the option price. As will be explained in Chapter 18, the implied volatilities of actively traded options are used by traders to estimate appropriate implied volatilities for other options.

The VIX Index

The CBOE publishes indices of implied volatility. The most popular index, the SPX VIX, is an index of the implied volatility of 30-day options on the S&P 500 calculated from a wide range of calls and puts.[10] Information on the way the index is calculated is in Section 24.13. Trading in futures on the VIX started in 2004 and trading in options on the VIX started in 2006. A trade involving futures or options on the S&P 500 is a bet on both the future level of the S&P 500 and the volatility of the S&P 500. By contrast, a futures or options contract on the VIX is a bet only on volatility. One contract is on 1,000 times the index. Figure 13.4 shows the VIX index between 2004 and 2007.

Example 13.8

> Suppose that a trader buys an April futures contract on the VIX when the futures price is 18.5 (corresponding to a 30-day S&P 500 volatility of 18.5%) and closes out the contract when the futures price is 19.3 (corresponding to an S&P 500 volatility of 19.3%). The trader makes a gain of $800.

[10] Similarly, the VXN is an index of the volatility of the NASDAQ 100 index and the VXD is an index of the volatility of the Dow Jones Industrial Average.

13.12 DIVIDENDS

Up to now, we have assumed that the stock upon which the option is written pays no dividends. In this section, we modify the Black–Scholes model to take account of dividends. We assume that the amount and timing of the dividends during the life of an option can be predicted with certainty. For short-life options this is not an unreasonable assumption. (For long-life options it is usual to assume that the dividend yield rather the cash dividend payments are known. Options can then be valued as will be described in the next chapter.) The date on which the dividend is paid should be assumed to be the ex-dividend date. On this date the stock price declines by the amount of the dividend.[11]

European Options

European options can be analyzed by assuming that the stock price is the sum of two components: a riskless component that corresponds to the known dividends during the life of the option and a risky component. The riskless component, at any given time, is the present value of all the dividends during the life of the option discounted from the ex-dividend dates to the present at the risk-free rate. By the time the option matures, the dividends will have been paid and the riskless component will no longer exist. The Black–Scholes formula is therefore correct if S_0 is equal to the risky component of the stock price and σ is the volatility of the process followed by the risky component.[12] Operationally, this means that the Black–Scholes formula can be used provided that the stock price is reduced by the present value of all the dividends during the life of the option, the discounting being done from the ex-dividend dates at the risk-free rate. As already mentioned, a dividend is counted as being during the life of the option only if its ex-dividend date occurs during the life of the option.

Example 13.9

Consider a European call option on a stock when there are ex-dividend dates in two months and five months. The dividend on each ex-dividend date is expected to be \$0.50. The current share price is \$40, the exercise price is \$40, the stock price volatility is 30% per annum, the risk-free rate of interest is 9% per annum, and the time to maturity is six months. The present value of the dividends is

$$0.5e^{-0.1667\times0.09} + 0.5e^{-0.4167\times0.09} = 0.9741$$

The option price can therefore be calculated from the Black–Scholes formula,

[11] For tax reasons the stock price may go down by somewhat less than the cash amount of the dividend. To take account of this phenomenon, we need to interpret the word 'dividend' in the context of option pricing as the reduction in the stock price on the ex-dividend date caused by the dividend. Thus, if a dividend of \$1 per share is anticipated and the share price normally goes down by 80% of the dividend on the ex-dividend date, the dividend should be assumed to be \$0.80 for the purposes of the analysis.

[12] In theory, this is not quite the same as the volatility of the stochastic process followed by the whole stock price. The volatility of the risky component is approximately equal to the volatility of the whole stock price multiplied by $S_0/(S_0 - D)$, where D is the present value of the dividends. However, an adjustment is only necessary when volatilities are estimated using historical data. An implied volatility is calculated after the present value of dividends have been subtracted from the stock price and is the volatility of the risky component.

with $S_0 = 40 - 0.9741 = 39.0259$, $K = 40$, $r = 0.09$, $\sigma = 0.3$, and $T = 0.5$:

$$d_1 = \frac{\ln(39.0259/40) + (0.09 + 0.3^2/2) \times 0.5}{0.3\sqrt{0.5}} = 0.2017$$

$$d_2 = \frac{\ln(39.0259/40) + (0.09 - 0.3^2/2) \times 0.5}{0.3\sqrt{0.5}} = -0.0104$$

Using the polynomial approximation in Section 13.9 or the NORMSDIST function in Excel gives

$$N(d_1) = 0.5800, \qquad N(d_2) = 0.4959$$

and, from equation (13.20), the call price is

$$39.0259 \times 0.5800 - 40e^{-0.09\times 0.5} \times 0.4959 = 3.67$$

or \$3.67.

American Options

Consider next American call options. Section 9.5 showed that in the absence of dividends American options should never be exercised early. An extension to the argument shows that, when there are dividends, it can only be optimal to exercise at a time immediately before the stock goes ex-dividend. We assume that n ex-dividend dates are anticipated and that they are at times $t_1, t_2, \dots, t_n$, with $t_1 < t_2 < \dots < t_n$. The dividends corresponding to these times will be denoted by $D_1, D_2, \dots, D_n$, respectively.

We start by considering the possibility of early exercise just prior to the final ex-dividend date (i.e., at time t_n). If the option is exercised at time t_n, the investor receives

$$S(t_n) - K$$

where $S(t)$ denotes the stock price at time t. If the option is not exercised, the stock price drops to $S(t_n) - D_n$. As shown by equation (9.5), the value of the option is then greater than

$$S(t_n) - D_n - Ke^{-r(T-t_n)}$$

It follows that, if

$$S(t_n) - D_n - Ke^{-r(T-t_n)} \geqslant S(t_n) - K$$

that is,

$$D_n \leqslant K\left[1 - e^{-r(T-t_n)}\right] \tag{13.24}$$

it cannot be optimal to exercise at time t_n. On the other hand, if

$$D_n > K\left[1 - e^{-r(T-t_n)}\right] \tag{13.25}$$

for any reasonable assumption about the stochastic process followed by the stock price, it can be shown that it is always optimal to exercise at time t_n for a sufficiently high value of $S(t_n)$. The inequality in (13.25) will tend to be satisfied when the final ex-dividend date is fairly close to the maturity of the option (i.e., $T - t_n$ is small) and the dividend is large.

Consider next time t_{n-1}, the penultimate ex-dividend date. If the option is exercised immediately prior to time t_{n-1}, the investor receives $S(t_{n-1}) - K$. If the option is not exercised at time t_{n-1}, the stock price drops to $S(t_{n-1}) - D_{n-1}$ and the earliest subsequent time at which exercise could take place is t_n. Hence, from equation (9.5), a lower bound to the option price if it is not exercised at time t_{n-1} is

$$S(t_{n-1}) - D_{n-1} - Ke^{-r(t_n - t_{n-1})}$$

It follows that if

$$S(t_{n-1}) - D_{n-1} - Ke^{-r(t_n - t_{n-1})} \geqslant S(t_{n-1}) - K$$

or

$$D_{n-1} \leqslant K\left[1 - e^{-r(t_n - t_{n-1})}\right]$$

it is not optimal to exercise immediately prior to time t_{n-1}. Similarly, for any $i < n$, if

$$D_i \leqslant K\left[1 - e^{-r(t_{i+1} - t_i)}\right] \tag{13.26}$$

it is not optimal to exercise immediately prior to time t_i.

The inequality in (13.26) is approximately equivalent to

$$D_i \leqslant Kr(t_{i+1} - t_i)$$

Assuming that K is fairly close to the current stock price, this inequality is satisfied when the dividend yield on the stock is less than the risk-free rate of interest. This is often the case.

We can conclude from this analysis that, in many circumstances, the most likely time for the early exercise of an American call is immediately before the final ex-dividend date, t_n. Furthermore, if inequality (13.26) holds for $i = 1, 2, \ldots, n-1$ and inequality (13.24) holds, we can be certain that early exercise is never optimal.

Black's Approximation

Black suggests an approximate procedure for taking account of early exercise in call options.[13] This involves calculating, as described earlier in this section, the prices of European options that mature at times T and t_n, and then setting the American price equal to the greater of the two. This approximation seems to work well in most cases.[14]

Example 13.10

Consider the situation in Example 13.9, but suppose that the option is American rather than European. In this case $D_1 = D_2 = 0.5$, $S_0 = 40$, $K = 40$, $r = 0.09$, $t_1 = 2/12$, and $t_2 = 5/12$. Since

$$K\left[1 - e^{-r(t_2 - t_1)}\right] = 40(1 - e^{-0.09 \times 0.25}) = 0.89$$

is greater than 0.5, it follows (see inequality (13.26)) that the option should never be exercised immediately before the first ex-dividend date. In addition, since

$$K\left[1 - e^{-r(T - t_2)}\right] = 40(1 - e^{-0.09 \times 0.0833}) = 0.30$$

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[13] See F. Black, "Fact and Fantasy in the Use of Options," *Financial Analysts Journal*, 31 (July/August 1975): 36–41, 61–72.

[14] For an exact formula, suggested by Roll, Geske, and Whaley, for valuing American calls when there is only one ex-dividend date, see Technical Note 4 on the author's website. This involves the cumulative bivariate normal distribution function. A procedure for calculating this function is given in Technical Note 5 also on the author's website.

is less than 0.5, it follows (see inequality (13.25)) that, when it is sufficiently deep in the money, the option should be exercised immediately before the second ex-dividend date.

We now use Black's approximation to value the option. The present value of the first dividend is

$$0.5e^{-0.1667\times0.09} = 0.4926$$

so that the value of the option, on the assumption that it expires just before the final ex-dividend date, can be calculated using the Black–Scholes formula with $S_0 = 40 - 0.4926 = 39.5074$, $K = 40$, $r = 0.09$, $\sigma = 0.30$, and $T = 0.4167$. It is \$3.52. Black's approximation involves taking the greater of this and the value of the option when it can only be exercised at the end of 6 months. From Example 13.8, we know that the latter is \$3.67. Black's approximation, therefore, gives the value of the American call as \$3.67.

The value of the option given by DerivaGem using "Binomial American" with 500 time steps is \$3.72. (Note that DerivaGem requires dividends to be input in chronological order in the table; the time to a dividend is in the first column and the amount of the dividend is in the second column.) There are two reasons for differences between the Binomial Model (BM) and Black's approximation (BA). The first concerns the timing of the early exercise decision; the second concerns the way volatility is applied. The timing of the early exercise decision tends to make BM greater than BA. In BA, the assumption is that the holder has to decide today whether the option will be exercised after 5 months or after 6 months; BM allows the decision on early exercise at the 5-month point to depend on the stock price at that time. The way in which volatility is applied tends to make BA greater than BM. In BA, when we assume exercise takes place after 5 months, the volatility is applied to the stock price less the present value of the first dividend; when we assume exercise takes place after 6 months, the volatility is applied to the stock price less the present value of both dividends.

SUMMARY

We started this chapter by examining the properties of the process for stock prices introduced in Chapter 12. The process implies that the price of a stock at some future time, given its price today, is lognormal. It also implies that the continuously compounded return from the stock in a period of time is normally distributed. Our uncertainty about future stock prices increases as we look further ahead. The standard deviation of the logarithm of the stock price is proportional to the square root of how far ahead we are looking.

To estimate the volatility σ of a stock price empirically, the stock price is observed at fixed intervals of time (e.g., every day, every week, or every month). For each time period, the natural logarithm of the ratio of the stock price at the end of the time period to the stock price at the beginning of the time period is calculated. The volatility is estimated as the standard deviation of these numbers divided by the square root of the length of the time period in years. Usually, days when the exchanges are closed are ignored in measuring time for the purposes of volatility calculations.

The differential equation for the price of any derivative dependent on a stock can be obtained by creating a riskless portfolio of the option and the stock. Because the derivative and the stock price both depend on the same underlying source of uncertainty, this can always be done. The portfolio that is created remains riskless for only a very short period of time. However, the return on a riskless porfolio must always be the risk-free interest rate if there are to be no arbitrage opportunities.

The expected return on the stock does not enter into the Black–Scholes differential equation. This leads to a useful result known as risk-neutral valuation. This result states that when valuing a derivative dependent on a stock price, we can assume that the world is risk neutral. This means that we can assume that the expected return from the stock is the risk-free interest rate, and then discount expected payoffs at the risk-free interest rate. The Black–Scholes equations for European call and put options can be derived by either solving their differential equation or by using risk-neutral valuation.

An implied volatility is the volatility that, when used in conjunction with the Black–Scholes option pricing formula, gives the market price of the option. Traders monitor implied volatilities. They often quote the implied volatility of an option rather than its price. They have developed procedures for using the volatilities implied by the prices of actively traded options to estimate volatilities for other options.

The Black–Scholes results can be extended to cover European call and put options on dividend-paying stocks. The procedure is to use the Black–Scholes formula with the stock price reduced by the present value of the dividends anticipated during the life of the option, and the volatility equal to the volatility of the stock price net of the present value of these dividends.

In theory, it can be optimal to exercise American call options immediately before any ex-dividend date. In practice, it is often only necessary to consider the final ex-dividend date. Fischer Black has suggested an approximation. This involves setting the American call option price equal to the greater of two European call option prices. The first European call option expires at the same time as the American call option; the second expires immediately prior to the final ex-dividend date.

FURTHER READING

On the Distribution of Stock Price Changes

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On the Black–Scholes Analysis

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On Risk-Neutral Valuation

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French, K. R., and R. Roll "Stock Return Variances: The Arrival of Information and the Reaction of Traders." *Journal of Financial Economics*, 17 (September 1986): 5–26.

Roll R. "Orange Juice and Weather," *American Economic Review*, 74, 5 (December 1984): 861–80.

Questions and Problems (Answers in Solutions Manual)

13.1. What does the Black–Scholes stock option pricing model assume about the probability distribution of the stock price in one year? What does it assume about the continuously compounded rate of return on the stock during the year?

13.2. The volatility of a stock price is 30% per annum. What is the standard deviation of the percentage price change in one trading day?

13.3. Explain the principle of risk-neutral valuation.

13.4. Calculate the price of a 3-month European put option on a non-dividend-paying stock with a strike price of \$50 when the current stock price is \$50, the risk-free interest rate is 10% per annum, and the volatility is 30% per annum.

13.5. What difference does it make to your calculations in Problem 13.4 if a dividend of \$1.50 is expected in 2 months?

13.6. What is *implied volatility*? How can it be calculated?

13.7. A stock price is currently \$40. Assume that the expected return from the stock is 15% and that its volatility is 25%. What is the probability distribution for the rate of return (with continuous compounding) earned over a 2-year period?

13.8. A stock price follows geometric Brownian motion with an expected return of 16% and a volatility of 35%. The current price is \$38.

(a) What is the probability that a European call option on the stock with an exercise price of \$40 and a maturity date in 6 months will be exercised?

(b) What is the probability that a European put option on the stock with the same exercise price and maturity will be exercised?

13.9. Using the notation in this chapter, prove that a 95% confidence interval for S_T is between

$$S_0e^{(\mu-\sigma^2/2)T-1.96\sigma\sqrt{T}} \quad \text{and} \quad S_0e^{(\mu-\sigma^2/2)T+1.96\sigma\sqrt{T}}$$

13.10. A portfolio manager announces that the average of the returns realized in each year of the last 10 years is 20% per annum. In what respect is this statement misleading?

13.11. Assume that a non-dividend-paying stock has an expected return of μ and a volatility of σ. An innovative financial institution has just announced that it will trade a security that pays off a dollar amount equal to $\ln S_T$ at time T, where S_T denotes the value of the stock price at time T.

(a) Use risk-neutral valuation to calculate the price of the security at time t in terms of the stock price, S, at time t.

(b) Confirm that your price satisfies the differential equation (13.16).

13.12. Consider a derivative that pays off S_T^n at time T, where S_T is the stock price at that time. When the stock price follows geometric Brownian motion, it can be shown that its price at time t ($t \leqslant T$) has the form

$$h(t, T)S^n$$

where S is the stock price at time t and h is a function only of t and T.

(a) By substituting into the Black–Scholes–Merton partial differential equation, derive an ordinary differential equation satisfied by $h(t, T)$.

(b) What is the boundary condition for the differential equation for $h(t, T)$?

(c) Show that

$$h(t, T) = e^{[0.5\sigma^2 n(n-1)+r(n-1)](T-t)}$$

where r is the risk-free interest rate and σ is the stock price volatility.

13.13. What is the price of a European call option on a non-dividend-paying stock when the stock price is \$52, the strike price is \$50, the risk-free interest rate is 12% per annum, the volatility is 30% per annum, and the time to maturity is 3 months?

13.14. What is the price of a European put option on a non-dividend-paying stock when the stock price is \$69, the strike price is \$70, the risk-free interest rate is 5% per annum, the volatility is 35% per annum, and the time to maturity is 6 months?

13.15. Consider an American call option on a stock. The stock price is \$70, the time to maturity is 8 months, the risk-free rate of interest is 10% per annum, the exercise price is \$65, and the volatility is 32%. A dividend of \$1 is expected after 3 months and again after 6 months. Show that it can never be optimal to exercise the option on either of the two dividend dates. Use DerivaGem to calculate the price of the option.

13.16. A call option on a non-dividend-paying stock has a market price of $\$2\frac{1}{2}$. The stock price is \$15, the exercise price is \$13, the time to maturity is 3 months, and the risk-free interest rate is 5% per annum. What is the implied volatility?

13.17. With the notation used in this chapter:

(a) What is $N'(x)$?

(b) Show that $SN'(d_1) = Ke^{-r(T-t)}N'(d_2)$, where S is the stock price at time t and

$$d_1 = \frac{\ln(S/K) + (r + \sigma^2/2)(T - t)}{\sigma\sqrt{T-t}}, \quad d_2 = \frac{\ln(S/K) + (r - \sigma^2/2)(T - t)}{\sigma\sqrt{T-t}}$$

(c) Calculate $\partial d_1/\partial S$ and $\partial d_2/\partial S$.

(d) Show that when

$$c = SN(d_1) - Ke^{-r(T-t)}N(d_2)$$

it follows that

$$\frac{\partial c}{\partial t} = -rKe^{-r(T-t)}N(d_2) - SN'(d_1)\frac{\sigma}{2\sqrt{T-t}}$$

where c is the price of a call option on a non-dividend-paying stock.

(e) Show that $\partial c/\partial S = N(d_1)$.
(f) Show that c satisfies the Black–Scholes differential equation.
(g) Show that c satisfies the boundary condition for a European call option, i.e., that $c = \max(S - K, 0)$ as $t \longrightarrow T$.

13.18. Show that the Black–Scholes formulas for call and put options satisfy put–call parity.

13.19. A stock price is currently \$50 and the risk-free interest rate is 5%. Use the DerivaGem software to translate the following table of European call options on the stock into a table of implied volatilities, assuming no dividends. Are the option prices consistent with the assumptions underlying Black–Scholes?

	Maturity (months)		
Strike price (\$)	*3*	*6*	*12*
45	7.0	8.3	10.5
50	3.7	5.2	7.5
55	1.6	2.9	5.1

13.20. Explain carefully why Black's approach to evaluating an American call option on a dividend-paying stock may give an approximate answer even when only one dividend is anticipated. Does the answer given by Black's approach understate or overstate the true option value? Explain your answer.

13.21. Consider an American call option on a stock. The stock price is \$50, the time to maturity is 15 months, the risk-free rate of interest is 8% per annum, the exercise price is \$55, and the volatility is 25%. Dividends of \$1.50 are expected in 4 months and 10 months. Show that it can never be optimal to exercise the option on either of the two dividend dates. Calculate the price of the option.

13.22. Show that the probability that a European call option will be exercised in a risk-neutral world is, with the notation introduced in this chapter, $N(d_2)$. What is an expression for the value of a derivative that pays off \$100 if the price of a stock at time T is greater than K?

13.23. Show that S^{-2r/σ^2} could be the price of a traded security.

13.24. A company has an issue of executive stock options outstanding. Should dilution be taken into account when the options are valued? Explain your answer.

13.25. A company's stock price is \$50 and 10 million shares are outstanding. The company is considering giving its employees 3 million at-the-money 5-year call options. Option exercises will be handled by issuing more shares. The stock price volatility is 25%, the 5-year risk-free rate is 5%, and the company does not pay dividends. Estimate the cost to the company of the employee stock option issue.

Assignment Questions

13.26. A stock price is currently \$50. Assume that the expected return from the stock is 18% and its volatility is 30%. What is the probability distribution for the stock price in 2 years? Calculate the mean and standard deviation of the distribution. Determine the 95% confidence interval.

13.27. Suppose that observations on a stock price (in dollars) at the end of each of 15 consecutive weeks are as follows:

30.2, 32.0, 31.1, 30.1, 30.2, 30.3, 30.6, 33.0, 32.9, 33.0, 33.5, 33.5, 33.7, 33.5, 33.2

Estimate the stock price volatility. What is the standard error of your estimate?

13.28. A financial institution plans to offer a security that pays off a dollar amount equal to S_T^2 at time T.

(a) Use risk-neutral valuation to calculate the price of the security at time t in terms of the stock price S at time t. (*Hint*: The expected value of S_T^2 can be calculated from the mean and variance of S_T given in Section 13.1.)

(b) Confirm that your price satisfies the differential equation (13.16).

13.29. Consider an option on a non-dividend-paying stock when the stock price is \$30, the exercise price is \$29, the risk-free interest rate is 5%, the volatility is 25% per annum, and the time to maturity is 4 months.

(a) What is the price of the option if it is a European call?

(b) What is the price of the option if it is an American call?

(c) What is the price of the option if it is a European put?

(d) Verify that put–call parity holds.

13.30. Assume that the stock in Problem 13.29 is due to go ex-dividend in $1\frac{1}{2}$ months. The expected dividend is 50 cents.

(a) What is the price of the option if it is a European call?

(b) What is the price of the option if it is a European put?

(c) If the option is an American call, are there any circumstances under which it will be exercised early?

13.31. Consider an American call option when the stock price is \$18, the exercise price is \$20, the time to maturity is 6 months, the volatility is 30% per annum, and the risk-free interest rate is 10% per annum. Two equal dividends are expected during the life of the option with ex-dividend dates at the end of 2 months and 5 months. Assume the dividends are 40 cents. Use Black's approximation and the DerivaGem software to value the option. How high can the dividends be without the American option being worth more than the corresponding European option?

APPENDIX

PROOF OF THE BLACK–SCHOLES–MERTON FORMULA

We will prove the Black–Scholes result by first proving another key result that will also be useful in future chapters.

Key Result

If V is lognormally distributed and the standard deviation of $\ln V$ is w, then

$$E[\max(V-K,\ 0)] = E(V)N(d_1) - KN(d_2) \tag{13A.1}$$

where

$$d_1 = \frac{\ln[E(V)/K] + w^2/2}{w}$$

$$d_2 = \frac{\ln[E(V)/K] - w^2/2}{w}$$

and E denotes the expected value.

Proof of Key Result

Define $g(V)$ as the probability density function of V. It follows that

$$E[\max(V-K,\ 0)] = \int_K^\infty (V-K)g(V)\,dV \tag{13A.2}$$

The variable $\ln V$ is normally distributed with standard deviation w. From the properties of the lognormal distribution, the mean of $\ln V$ is m, where[15]

$$m = \ln[E(V)] - w^2/2 \tag{13A.3}$$

Define a new variable

$$Q = \frac{\ln V - m}{w} \tag{13A.4}$$

This variable is normally distributed with a mean of zero and a standard deviation of 1.0. Denote the density function for Q by $h(Q)$ so that

$$h(Q) = \frac{1}{\sqrt{2\pi}} e^{-Q^2/2}$$

Using equation (13A.4) to convert the expression on the right-hand side of equation (13A.2) from an integral over V to an integral over Q, we get

$$E[\max(V-K,\ 0)] = \int_{(\ln K - m)/w}^\infty (e^{Qw+m} - K)\,h(Q)\,dQ$$

or

$$E[\max(V-K,\ 0)] = \int_{(\ln K - m)/w}^\infty e^{Qw+m} h(Q)dQ - K\int_{(\ln K - m)/w}^\infty h(Q)dQ \tag{13A.5}$$

[15] For a proof of this, see Technical Note 2 on the author's website.

Now

$$e^{Qw+m}h(Q) = \frac{1}{\sqrt{2\pi}}e^{(-Q^2+2Qw+2m)/2}$$
$$= \frac{1}{\sqrt{2\pi}}e^{[-(Q-w)^2+2m+w^2]/2}$$
$$= \frac{e^{m+w^2/2}}{\sqrt{2\pi}}e^{[-(Q-w)^2]/2}$$
$$= e^{m+w^2/2}h(Q-w)$$

This means that equation (13A.5) becomes

$$E[\max(V-K,\,0)] = e^{m+w^2/2}\int_{(\ln K-m)/w}^{\infty} h(Q-w)dQ - K\int_{(\ln K-m)/w}^{\infty} h(Q)dQ \qquad \textbf{(13A.6)}$$

If we define $N(x)$ as the probability that a variable with a mean of zero and a standard deviation of 1.0 is less than x, the first integral in equation (13A.6) is

$$1 - N[(\ln K - m)/w - w]$$

or

$$N[(-\ln K + m)/w + w]$$

Substituting for m from equation (13A.3) leads to

$$N\left(\frac{\ln[E(V)/K] + w^2/2}{w}\right) = N(d_1)$$

Similarly the second integral in equation (13A.6) is $N(d_2)$. Equation (13A.6), therefore, becomes

$$E[\max(V-K,\,0)] = e^{m+w^2/2}N(d_1) - KN(d_2)$$

Substituting for m from equation (13A.3) gives the key result.

The Black–Scholes–Merton Result

We now consider a call option on a non-dividend-paying stock maturing at time T. The strike price is K, the risk-free rate is r, the current stock price is S_0, and the volatility is σ. As shown in equation (13.22), the call price c is given by

$$c = e^{-rT}\hat{E}[\max(S_T - K,\,0)] \qquad \textbf{(13A.7)}$$

where S_T is the stock price at time T and $\hat{E}$ denotes the expectation in a risk-neutral world. Under the stochastic process assumed by Black–Scholes, S_T is lognormal. Also, from equations (13.3) and (13.4), $\hat{E}(S_T) = S_0e^{rT}$ and the standard deviation of $\ln S_T$ is $\sigma\sqrt{T}$.

From the key result just proved, equation (13A.7) implies

$$c = e^{-rT}[S_0e^{rT}N(d_1) - KN(d_2)]$$

or

$$c = S_0N(d_1) - Ke^{-rT}N(d_2)$$

where

$$d_1 = \frac{\ln[\hat{E}(S_T)/K] + \sigma^2 T/2}{\sigma\sqrt{T}} = \frac{\ln(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}$$

and

$$d_2 = \frac{\ln[\hat{E}(S_T)/K] - \sigma^2 T/2}{\sigma\sqrt{T}} = \frac{\ln(S_0/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}}$$

This is the Black–Scholes–Merton result.

CHAPTER 15

Options on Stock Indices and Currencies

Options on stock indices and currencies were introduced in Chapter 8. This chapter discusses them in more detail. It explains how they work and reviews some of the ways they can be used. In the second half of the chapter, the valuation results in Chapter 13 are extended to cover European options on a stock paying a known dividend yield. It is then argued that both stock indices and currencies are analogous to stocks paying dividend yields. This enables the results for options on a stock paying a dividend yield to be applied to these types of options as well.

15.1 OPTIONS ON STOCK INDICES

Several exchanges trade options on stock indices. Some of the indices track the movement of the market as a whole. Others are based on the performance of a particular sector (e.g., computer technology, oil and gas, transportation, or telecoms). Among the index options traded on the Chicago Board of Options Exchange are American and European options on the S&P 100 (OEX and XEO), European options on the S&P 500 (SPX), European options on the Dow Jones Industrial Average (DJX), and European options on the Nasdaq 100 (NDX). In Chapter 8, we explained that the CBOE trades LEAPS and flex options on individual stocks. It also offers these option products on indices.

One index option contract is on 100 times the index. (Note that the Dow Jones index used for index options is 0.01 times the usually quoted Dow Jones index.) Index options are settled in cash. This means that, on exercise of the option, the holder of a call option contract receives $(S - K) \times 100$ in cash and the writer of the option pays this amount in cash, where S is the value of the index at the close of trading on the day of the exercise and K is the strike price. Similarly, the holder of a put option contract receives $(K - S) \times 100$ in cash and the writer of the option pays this amount in cash.

Portfolio Insurance

Portfolio managers can use index options to limit their downside risk. Suppose that the value of an index today is S_0. Consider a manager in charge of a well-diversified portfolio whose beta is 1.0. A beta of 1.0 implies that the returns from the portfolio mirror those from the index. Assuming the dividend yield from the portfolio is the same as the dividend yield from the index, the percentage changes in the value of the portfolio can be expected to be approximately the same as the percentage changes in the value of the index. Each contract on the S&P 500 is on 100 times the index. It follows that the value of the portfolio is protected against the possibility of the index falling below K if, for each $100S_0$ dollars in the portfolio, the manager buys one put option contract with strike price K. Suppose that the manager's portfolio is worth $500,000 and the value of the index is 1,000. The portfolio is worth 500 times the index. The manager can obtain insurance against the value of the portfolio dropping below $450,000 in the next three months by buying five three-month put option contracts on the index with a strike price of 900.

To illustrate how the insurance works, consider the situation where the index drops to 880 in three months. The portfolio will be worth about $440,000. The payoff from the options will be $5 \times (900 - 880) \times 100 = \$10{,}000$, bringing the total value of the portfolio up to the insured value of $450,000.

When the Portfolio's Beta Is Not 1.0

If the portfolio's beta (β) is not 1.0, β put options must be purchased for each $100S_0$ dollars in the portfolio, where S_0 is the current value of the index. Suppose that the $500,000 portfolio just considered has a beta of 2.0 instead of 1.0. We continue to assume that the S&P 500 index is 1,000. The number of put options required is

$$2.0 \times \frac{500{,}000}{1{,}000 \times 100} = 10$$

rather than 5 as before.

To calculate the appropriate strike price, the capital asset pricing model can be used. Suppose that the risk free rate is 12%, the dividend yield on both the index and the portfolio is 4%, and protection is required against the value of the portfolio dropping below $450,000 in the next three months. Under the capital asset pricing model, the expected excess return of a portfolio over the risk-free rate is assumed to equal beta

Table 15.1 Calculation of expected value of portfolio when the index is 1,040 in three months and $\beta = 2.0$.

Value of index in three months:	1,040
Return from change in index:	40/1,000, or 4% per three months
Dividends from index:	$0.25 \times 4 = 1\%$ per three months
Total return from index:	$4 + 1 = 5\%$ per three months
Risk-free interest rate:	$0.25 \times 12 = 3\%$ per three months
Excess return from index over risk-free interest rate:	$5 - 3 = 2\%$ per three months
Expected excess return from portfolio over risk-free interest rate:	$2 \times 2 = 4\%$ per three months
Expected return from portfolio:	$3 + 4 = 7\%$ per three months
Dividends from portfolio:	$0.25 \times 4 = 1\%$ per three months
Expected increase in value of portfolio:	$7 - 1 = 6\%$ per three months
Expected value of portfolio:	$\$500{,}000 \times 1.06 = \$530{,}000$

Table 15.2 Relationship between value of index and value of portfolio for $\beta = 2.0$.

Value of index in three months	*Value of portfolio in three months (\$)*
1,080	570,000
1,040	530,000
1,000	490,000
960	450,000
920	410,000
880	370,000

times the excess return of the index portfolio over the risk-free rate. The model enables the expected value of the portfolio to be calculated for different values of the index at the end of three months. Table 15.1 shows the calculations for the case where the index is 1,040. In this case the expected value of the portfolio at the end of the three months is \$530,000. Similar calculations can be carried out for other values of the index at the end of the three months. The results are shown in Table 15.2. The strike price for the options that are purchased should be the index level corresponding to the protection level required on the portfolio. In this case the protection level is \$450,000 and so the correct strike price for the 10 put option contracts that are purchased is 960.[1]

To illustrate how the insurance works, consider what happens if the value of the index falls to 880. As shown in Table 15.2, the value of the portfolio is then about \$370,000. The put options pay off $(960 - 880) \times 10 \times 100 = \$80{,}000$, and this is exactly what is necessary to move the total value of the portfolio manager's position up from \$370,000 to the required level of \$450,000.

The examples in this section show that there are two reasons why the cost of hedging increases as the beta of a portfolio increases. More put options are required and they have a higher strike price.

15.2 CURRENCY OPTIONS

Currency options are primarily traded in the over-the-counter market. The advantage of this market is that large trades are possible, with strike prices, expiration dates, and other features tailored to meet the needs of corporate treasurers. Although European and American currency options do trade on the Philadelphia Stock Exchange in the United States, the exchange-traded market for these options is much smaller than the over-the-counter market.

An example of a European call option is a contract that gives the holder the right to buy one million euros with US dollars at an exchange rate of 1.2000 US dollars per euro. If the actual exchange rate at the maturity of the option is 1.2500, the payoff is

[1] Approximately 1% of \$500,000, or \$5,000, will be earned in dividends over the next three months. If we want the insured level of \$450,000 to include dividends, we can choose a strike price corresponding to \$445,000 rather than \$450,000. This is 955.

1,000,000 × (1.2500 − 1.2000) = \$50,000. Similarly, an example of a European put option is a contract that gives the holder the right to sell ten million Australian dollars for US dollars at an exchange rate of 0.7000 US dollars per Australian dollar. If the actual exchange rate at the maturity of the option is 0.6700, the payoff is 10,000,000 × (0.7000 − 0.6700) = \$300,000.

For a corporation wishing to hedge a foreign exchange exposure, foreign currency options are an interesting alternative to forward contracts. A company due to receive sterling at a known time in the future can hedge its risk by buying put options on sterling that mature at that time. The hedging strategy guarantees that the exchange rate applicable to the sterling will not be less than the strike price, while allowing the company to benefit from any favorable exchange-rate movements. Similarly, a company due to pay sterling at a known time in the future can hedge by buying calls on sterling that mature at that time. This hedging strategy guarantees that the cost of the sterling will not be greater than a certain amount while allowing the company to benefit from favorable exchange-rate movements. Whereas a forward contract locks in the exchange rate for a future transaction, an option provides a type of insurance. This is not free. It costs nothing to enter into a forward transaction, but options require a premium to be paid up front.

Range Forwards

A *range forward contract* is a variation on a standard forward contract for hedging foreign exchange risk. Consider a US company that knows it will receive one million pounds sterling in three months. Suppose that the three-month forward exchange rate is 1.9200 dollars per pound. The company could lock in this exchange rate for the dollars it receives by entering into a short forward contract to sell one million pounds sterling in three months. This would ensure that the amount received for the one million pounds is \$1,920,000.

An alternative is to buy a European put option with a strike price of K_1 and sell a European call option with a strike price K_2, where $K_1 < 1.9200 < K_2$. This is known as a short range forward contract. The payoff is shown in Figure 15.1(a). In both cases the options are on one million pounds. If the exchange rate in three months proves to be less than K_1, the put option is exercised and as a result the company is able to sell the one

Figure 15.1 Payoffs from (a) short and (b) long range-forward contract.

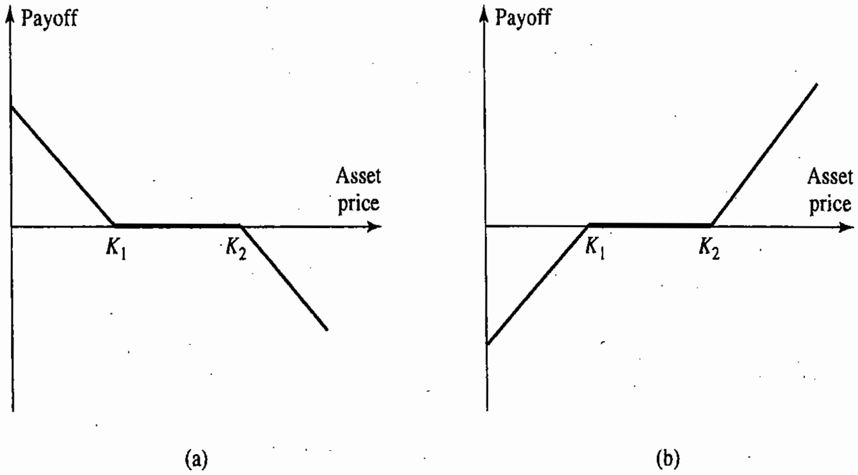
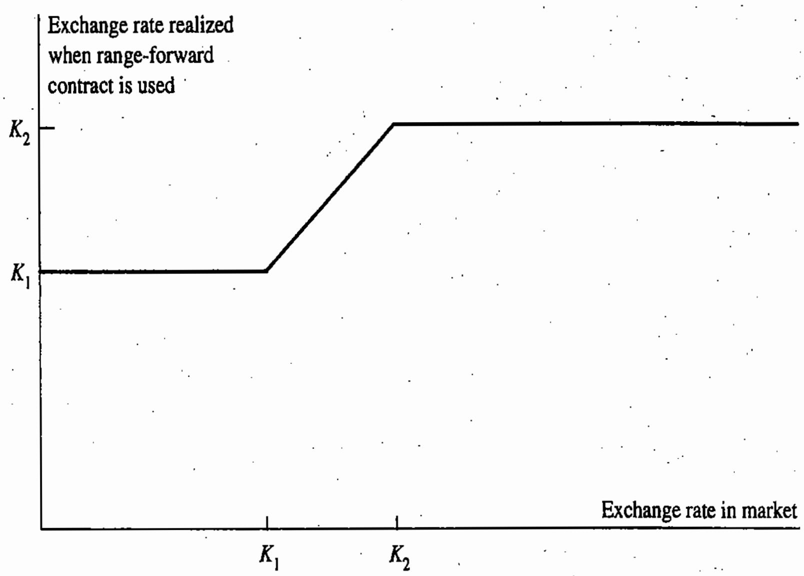


Figure 15.2 Exchange rate realized when either (a) a short range-forward contract is used to hedge a future foreign currency inflow or (b) a long range-forward contract is used to hedge a future foreign currency outflow.



million pounds at an exchange rate of K_1. If the exchange rate is between K_1 and K_2, neither option is exercised and the company gets the current exchange rate for the one million pounds. If the exchange rate is greater than K_2, the call option is exercised against the company with the result that the one million pounds is sold at an exchange rate of K_2. The exchange rate realized for the one million pounds is shown in Figure 15.2.

If the company knew it was due to pay rather than receive one million pounds in three months, it could sell a European put option with strike price K_1 and buy a European call option with strike price K_2. This is known as a long range forward contract and the payoff is shown in Figure 15.1(b). If the exchange rate in three months proves to be less than K_1, the put option is exercised against the company and as a result the company buys the one million pounds it needs at an exchange rate of K_1. If the exchange rate is between K_1 and K_2, neither option is exercised and the company buys the one million pounds at the current exchange rate. If the exchange rate is greater than K_2, the call option is exercised and the company is able to buy the one million pounds at an exchange rate of K_2. The exchange rate paid for the one million pounds is the same as that received for the one million pounds in the earlier example and is shown in Figure 15.2.

In practice, a range forward contract is set up so that the price of the put option equals the price of the call option. This means that it costs nothing to set up the range forward contract, just as it costs nothing to set up a regular forward contract. Suppose that the US and British interest rates are both 5%, so that the spot exchange rate is 1.9200 (the same as the forward exchange rate). Suppose further that the exchange rate volatility is 14%. We can use DerivaGem to show that a put with strike price 1.9000 to sell one pound has the same price as a call option with a strike price of 1.9413 to buy one pound. (Both are worth 0.04338.) Setting $K_1 = 1.9000$ and $K_2 = 1.9413$ therefore leads to a contract with zero cost in our example.

As the strike prices of the call and put options become closer in a range forward contract, the range forward contract becomes a regular forward contract. A short range forward contract becomes a short forward contract and a long range forward contract becomes a long forward contract.

15.3 OPTIONS ON STOCKS PAYING KNOWN DIVIDEND YIELDS

In this section we produce a simple rule that enables valuation results for European options on a non-dividend-paying stock to be extended so that they apply to European options on a stock paying a known dividend yield. Later we show how this enables us to value options on stock indices and currencies.

Dividends cause stock prices to reduce on the ex-dividend date by the amount of the dividend payment. The payment of a dividend yield at rate q therefore causes the growth rate in the stock price to be less than it would otherwise be by an amount q. If, with a dividend yield of q, the stock price grows from S_0 today to S_T at time T, then in the absence of dividends it would grow from S_0 today to $S_T e^{qT}$ at time T. Alternatively, in the absence of dividends it would grow from $S_0 e^{-qT}$ today to S_T at time T.

This argument shows that we get the same probability distribution for the stock price at time T in each of the following two cases:

1. The stock starts at price S_0 and provides a dividend yield at rate q.
2. The stock starts at price $S_0 e^{-qT}$ and pays no dividends.

This leads to a simple rule. When valuing a European option lasting for time T on a stock paying a known dividend yield at rate q, we reduce the current stock price from S_0 to $S_0 e^{-qT}$ and then value the option as though the stock pays no dividends.[2]

Lower Bounds for Option Prices

As a first application of this rule, consider the problem of determining bounds for the price of a European option on a stock paying a dividend yield at rate q. Substituting $S_0 e^{-qT}$ for S_0 in equation (9.1), we see that a lower bound for the European call option price, c, is given by

$$c \geqslant S_0 e^{-qT} - Ke^{-rT} \tag{15.1}$$

We can also prove this directly by considering the following two portfolios:

Portfolio A: one European call option plus an amount of cash equal to Ke^{-rT}

Portfolio B: e^{-qT} shares with dividends being reinvested in additional shares

To obtain a lower bound for a European put option, we can similarly replace S_0 by $S_0 e^{-qT}$ in equation (9.2) to get

$$p \geqslant Ke^{-rT} - S_0 e^{-qT} \tag{15.2}$$

[2] This rule is analogous to the one developed in Section 13.12 for valuing a European option on a stock paying known cash dividends. (In that case we concluded that it is correct to reduce the stock price by the present value of the dividends; in this case we discount the stock price at the dividend yield rate.)

This result can also be proved directly by considering the following portfolios:

Portfolio C: one European put option plus e^{-qT} shares with dividends on the shares being reinvested in additional shares

Portfolio D: an amount of cash equal to Ke^{-rT}

Put–Call Parity

Replacing S_0 by S_0e^{-qT} in equation (9.3) we obtain put–call parity for an option on a stock paying a dividend yield at rate q:

$$c + Ke^{-rT} = p + S_0e^{-qT} \tag{15.3}$$

This result can also be proved directly by considering the following two portfolios:

Portfolio A: one European call option plus an amount of cash equal to Ke^{-rT}

Portfolio C: one European put option plus e^{-qT} shares with dividends on the shares being reinvested in additional shares

Both portfolios are both worth $\max(S_T, K)$ at time T. They must therefore be worth the same today, and the put–call parity result in equation (15.3) follows. For American options, the put–call parity relationship is (see Problem 15.12)

$$S_0e^{-qT} - K \leqslant C - P \leqslant S_0 - Ke^{-rT}$$

Pricing Formulas

By replacing S_0 by S_0e^{-qT} in the Black–Scholes formulas, equations (13.20) and (13.21), we obtain the price, c, of a European call and the price, p, of a European put on a stock paying a dividend yield at rate q as

$$c = S_0e^{-qT}N(d_1) - Ke^{-rT}N(d_2) \tag{15.4}$$

$$p = Ke^{-rT}N(-d_2) - S_0e^{-qT}N(-d_1) \tag{15.5}$$

Since

$$\ln\frac{S_0e^{-qT}}{K} = \ln\frac{S_0}{K} - qT$$

it follows that d_1 and d_2 are given by

$$d_1 = \frac{\ln(S_0/K) + (r - q + \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln(S_0/K) + (r - q - \sigma^2/2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

These results were first derived by Merton.[3] As discussed in Chapter 13, the word *dividend* should, for the purposes of option valuation, be defined as the reduction in the stock price on the ex-dividend date arising from any dividends declared. If the dividend yield rate is known but not constant during the life of the option, equations (15.4)

[3] See R. C. Merton, "Theory of Rational Option Pricing," *Bell Journal of Economics and Management Science*, 4 (Spring 1973): 141–83.

and (15.5) are still true, with q equal to the average annualized dividend yield during the option's life.

Differential Equation and Risk-Neutral Valuation

To prove the results in equations (15.4) and (15.5) more formally, we can either solve the differential equation that the option price must satisfy or use risk-neutral valuation.

When we include a dividend yield of q in the analysis in Section 13.6, the differential equation (13.16) becomes[4]

$$\frac{\partial f}{\partial t} + (r-q)S\frac{\partial f}{\partial S} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf \tag{15.6}$$

Like equation (13.16), this does not involve any variable affected by risk preferences. Therefore the risk-neutral valuation procedure described in Section 13.7 can be used.

In a risk-neutral world, the total return from the stock must be r. The dividends provide a return of q. The expected growth rate in the stock price must therefore be $r - q$. It follows that the risk-neutral process for the stock price is

$$dS = (r-q)S\,dt + \sigma S\,dz \tag{15.7}$$

To value a derivative dependent on a stock that provides a dividend yield equal to q, we set the expected growth rate of the stock equal to $r - q$ and discount the expected payoff at rate r. When the expected growth rate in the stock price is $r - q$, the expected stock price at time T is $S_0e^{(r-q)T}$. A similar analysis to that in the appendix to Chapter 13 gives the expected payoff for a call option in a risk-neutral world as

$$e^{(r-q)T}S_0N(d_1) - KN(d_2)$$

where d_1 and d_2 are defined as above. Discounting at rate r for time T leads to equation (15.4).

15.4 VALUATION OF EUROPEAN STOCK INDEX OPTIONS

In valuing index futures in Chapter 5, we assumed that the index could be treated as an asset paying a known yield. In valuing index options, we make similar assumptions. This means that equations (15.1) and (15.2) provide a lower bound for European index options; equation (15.3) is the put–call parity result for European index options; equations (15.4) and (15.5) can be used to value European options on an index; and the binomial tree approach can be used for American options. In all cases, S_0 is equal to the value of the index, σ is equal to the volatility of the index, and q is equal to the average annualized dividend yield on the index during the life of the option.

Example 15.1

Consider a European call option on the S&P 500 that is two months from maturity. The current value of the index is 930, the exercise price is 900, the risk-free interest rate is 8% per annum, and the volatility of the index is 20% per annum. Dividend

[4] See Technical Note 6 on the author's website for a proof of this.

yields of 0.2% and 0.3% are expected in the first month and the second month, respectively. In this case $S_0 = 930$, $K = 900$, $r = 0.08$, $\sigma = 0.2$, and $T = 2/12$. The total dividend yield during the option's life is 0.2% + 0.3% = 0.5%. This is 3% per annum. Hence, $q = 0.03$ and

$$d_1 = \frac{\ln(930/900) + (0.08 - 0.03 + 0.2^2/2) \times 2/12}{0.2\sqrt{2/12}} = 0.5444$$

$$d_2 = \frac{\ln(930/900) + (0.08 - 0.03 - 0.2^2/2) \times 2/12}{0.2\sqrt{2/12}} = 0.4628$$

$$N(d_1) = 0.7069, \qquad N(d_2) = 0.6782$$

so that the call price, c, is given by equation (15.4) as

$$c = 930 \times 0.7069e^{-0.03\times 2/12} - 900 \times 0.6782e^{-0.08\times 2/12} = 51.83$$

One contract would cost \$5,183.

The calculation of q should include only dividends whose ex-dividend date occurs during the life of the option. In the United States ex-dividend dates tend to occur during the first week of February, May, August, and November. At any given time the correct value of q is therefore likely to depend on the life of the option. This is even more true for some foreign indices. In Japan, for example, all companies tend to use the same ex-dividend dates.

If the absolute amount of the dividend that will be paid on the stocks underlying the index (rather than the dividend yield) is assumed to be known, the basic Black–Scholes formula can be used with the initial stock price being reduced by the present value of the dividends. This is the approach recommended in Chapter 13 for a stock paying known dividends. However, it may be difficult to implement for a broadly based stock index because it requires a knowledge of the dividends expected on every stock underlying the index.

It is sometimes argued that the return from a portfolio of stocks is certain to beat the return from a bond portfolio in the long run when both have the same initial value. If this were so, a long-dated put option on the stock portfolio where the strike price equaled the future value of the bond portfolio would not cost very much. In fact, as indicated by Business Snapshot 15.1, it is quite expensive.

Forward Prices

Define F_0 as the forward price of the index for a contract with maturity T. As shown by equation (5.3), $F_0 = S_0e^{(r-q)T}$. This means that the equations for the European call price c and the European put price p in equations (15.4) and (15.5) can be written

$$c = F_0e^{-rT}N(d_1) - Ke^{-rT}N(d_2) \tag{15.8}$$

$$p = Ke^{-rT}N(-d_2) - F_0e^{-rT}N(-d_1) \tag{15.9}$$

where

$$d_1 = \frac{\ln(F/K) + \sigma^2T/2}{\sigma\sqrt{T}} \quad \text{and} \quad d_2 = \frac{\ln(F/K) - \sigma^2T/2}{\sigma\sqrt{T}}$$

Business Snapshot 15.1 Can We Guarantee that Stocks Will Beat Bonds in the Long Run?

It is often said that if you are a long-term investor you should buy stocks rather than bonds. Consider a US fund manager who is trying to persuade investors to buy, as a long-term investment, an equity fund that is expected to mirror the S&P 500. The manager might be tempted to offer purchasers of the fund a guarantee that their return will be at least as good as the return on risk-free bonds over the next 10 years. Historically stocks have outperformed bonds in the United States over almost any 10-year period. It appears that the fund manager would not be giving much away.

In fact, this type of guarantee is surprisingly expensive. Suppose that an equity index is 1,000 today, the dividend yield on the index is 1% per annum, the volatility of the index is 15% per annum, and the 10-year risk-free rate is 5% per annum. To outperform bonds, the stocks underlying the index must earn more than 5% per annum. The dividend yield will provide 1% per annum. The capital gains on the stocks must therefore provide 4% per annum. This means that we require the index level to be at least $1{,}000e^{0.04\times 10} = 1{,}492$ in 10 years.

A guarantee that the return on \$1,000 invested in the index will be greater than the return on \$1,000 invested in bonds over the next 10 years is therefore equivalent to the right to sell the index for 1,492 in 10 years. This is a European put option on the index and can be valued from equation (15.5) with $S_0 = 1{,}000$, $K = 1{,}492$, $r = 5\%$, $\sigma = 15\%$, $T = 10$, and $q = 1\%$. The value of the put option is 169.7. This shows that the guarantee contemplated by the fund manager is worth about 17% of the fund—hardly something that should be given away!

The put–call parity relationship in equation (15.3) can be written

$$c + Ke^{-rT} = p + F_0e^{-rT}$$

or

$$F_0 = K + (c - p)e^{rT} \tag{15.10}$$

If, as is not uncommon in the exchange-traded markets, pairs of puts and calls with the same strike price are traded actively for a particular maturity date, this equation can be used to estimate the forward price of the index for that maturity date. Once the forward prices of the index for a number of different maturity dates have been obtained, the term structure of forward rates can be estimated, and other options can be valued using equations (15.8) and (15.9). The advantage of this approach is that the dividend yield on the index does not have to be estimated explicitly.

Implied Dividend Yields

If estimates of the dividend yield are required (e.g because an American option is being valued), calls and puts with the same strike price and time to maturity can be used. From equation (15.3),

$$q = -\frac{1}{T}\ln\frac{c - p + Ke^{-rT}}{S_0}$$

For a particular strike price and time to maturity, the estimates of q calculated from this equation are liable to be unreliable. But when the results from many matched pairs of calls and puts are combined, a clear picture of the dividend yield being assumed by the market emerges.

15.5 VALUATION OF EUROPEAN CURRENCY OPTIONS

To value currency options, we define S_0 as the spot exchange rate. To be precise, S_0 is the value of one unit of the foreign currency in US dollars. As explained in Section 5.10, a foreign currency is analogous to a stock paying a known dividend yield. The owner of foreign currency receives a yield equal to the risk-free interest rate, r_f, in the foreign currency. Equations (15.1) and (15.2), with q replaced by r_f, provide bounds for the European call price, c, and the European put price, p:

$$c \geqslant S_0 e^{-r_f T} - Ke^{-rT}$$

$$p \geqslant Ke^{-rT} - S_0 e^{-r_f T}$$

Equation (15.3), with q replaced by r_f, provides the put–call parity result for currency options:

$$c + Ke^{-rT} = p + S_0 e^{-r_f T}$$

Finally, equations (15.4) and (15.5) provide the pricing formulas for currency options when q is replaced by r_f:

$$c = S_0 e^{-r_f T} N(d_1) - Ke^{-rT} N(d_2) \tag{15.11}$$

$$p = Ke^{-rT} N(-d_2) - S_0 e^{-r_f T} N(-d_1) \tag{15.12}$$

where

$$d_1 = \frac{\ln(S_0/K) + (r - r_f + \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln(S_0/K) + (r - r_f - \sigma^2/2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

Both the domestic interest rate, r, and the foreign interest rate, r_f, are the rates for a maturity T. Put and call options on a currency are symmetrical in that a put option to sell currency A for currency B at strike price K is the same as a call option to buy B with currency A at strike price $1/K$ (see Problem 15.8).

Example 15.2

Consider a four-month European call option on the British pound. Suppose that the current exchange rate is 1.6000, the exercise price is 1.6000, the risk-free interest rate in the United States is 8% per annum, the risk-free interest rate in Britain is 11% per annum, and the option price is 4.3 cents. In this case, $S_0 = 1.6$, $K = 1.6$, $r = 0.08$, $r_f = 0.11$, $T = 0.3333$, and $c = 0.043$. The implied volatility can be calculated by trial and error. A volatility of 20% gives an option price of 0.0639; a volatility of 10% gives an option price of 0.0285; and so on. The implied volatility is 14.1%.

Using Forward Exchange Rates

Because banks and other financial institutions trade forward contracts on foreign exchange rates actively, foreign exchange rates are often used for valuing options.

From equation (5.9), the forward rate, F_0, for a maturity T is given by

$$F_0 = S_0 e^{(r-r_f)T}$$

This relationship allows equations (15.11) and (15.12) to be simplified to

$$c = e^{-rT}[F_0 N(d_1) - KN(d_2)] \tag{15.13}$$

$$p = e^{-rT}[KN(-d_2) - F_0 N(-d_1)] \tag{15.14}$$

where

$$d_1 = \frac{\ln(F_0/K) + \sigma^2 T/2}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln(F_0/K) - \sigma^2 T/2}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

These equations are the same as equations (15.8) and (15.9). As we shall see in Chapter 16, a European option on any asset can be valued in terms of forward or futures contracts on the asset using equations (15.13) and (15.14). Note that the maturity of the forward or futures contract must be the same as the maturity of the European option.

15.6 AMERICAN OPTIONS

As described in Chapter 11, binomial trees can be used to value American options on indices and currencies. As in the case of American options on a non-dividend-paying stock, the parameter determining the size of up movements, u, is set equal to $e^{\sigma\sqrt{\Delta t}}$, where σ is the volatility and Δt is the length of time steps. The parameter determining the size of down movements, d, is set equal to $1/u$, or $e^{-\sigma\sqrt{\Delta t}}$. For a non-dividend-paying stock, the probability of an up movement is

$$p = \frac{a-d}{u-d}$$

where $a = e^{r\Delta t}$. For options on indices and currencies, the formula for p is the same, but a is defined differently. In the case of options on an index,

$$a = e^{(r-q)\Delta t}$$

where q is the dividend yield on the index. In the case of options on a currency,

$$a = e^{(r-r_f)\Delta t}$$

where r_f is the foreign risk-free rate. Example 11.1 in Section 11.9 shows how a two-step tree can be constructed to value an option on an index. Example 11.2 shows how a three-step tree can be constructed to value an option on a currency. Further examples of the use of binomial trees to value options on indices and currencies are given in Chapter 19.

In some circumstances it is optimal to exercise American currency options prior to maturity. Thus, American currency options are worth more than their European counterparts. In general, call options on high-interest currencies and put options on low-interest currencies are the most likely to be exercised prior to maturity. The reason is that a high-interest currency is expected to depreciate and a low-interest currency is expected to appreciate.

SUMMARY

The index options that trade on exchanges are settled in cash. On exercise of an index call option, the holder receives 100 times the amount by which the index exceeds the strike price. Similarly, on exercise of an index put option contract, the holder receives 100 times the amount by which the strike price exceeds the index. Index options can be used for portfolio insurance. If the value of the portfolio mirrors the index, it is appropriate to buy one put option contract for each $100S_0$ dollars in the portfolio, where S_0 is the value of the index. If the portfolio does not mirror the index, β put option contracts should be purchased for each $100S_0$ dollars in the portfolio, where β is the beta of the portfolio calculated using the capital asset pricing model. The strike price of the put options purchased should reflect the level of insurance required.

Most currency options are traded in the over-the-counter market. They can be used by corporate treasurers to hedge a foreign exchange exposure. For example, a US corporate treasurer who knows that the company will be receiving sterling at a certain time in the future can hedge by buying put options that mature at that time. Similarly, a US corporate treasurer who knows that the company will be paying sterling at a certain time in the future can hedge by buying call options that mature at that time. Currency options can also be used to create a range forward contract. This is a zero-cost contract that provides downside protection while giving up some of the upside.

The Black–Scholes formula for valuing European options on a non-dividend-paying stock can be extended to cover European options on a stock paying a known dividend yield. The extension can be used to value European options on stock indices and currencies because:

1. A stock index is analogous to a stock paying a dividend yield. The dividend yield is the dividend yield on the stocks that make up the index.
2. A foreign currency is analogous to a stock paying a dividend yield. The foreign risk-free interest rate plays the role of the dividend yield.

Binomial trees can be used to value American options on stock indices and currencies.

FURTHER READING

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Questions and Problems (Answers in Solutions Manual)

15.1. A portfolio is currently worth \$10 million and has a beta of 1.0. An index is currently standing at 800. Explain how a put option on the index with a strike price of 700 can be used to provide portfolio insurance.

15.2. "Once we know how to value options on a stock paying a dividend yield, we know how to value options on stock indices and currencies." Explain this statement.

15.3. A stock index is currently 300, the dividend yield on the index is 3% per annum, and the risk-free interest rate is 8% per annum. What is a lower bound for the price of a six-month European call option on the index when the strike price is 290?

15.4. A currency is currently worth \$0.80 and has a volatility of 12%. The domestic and foreign risk-free interest rates are 6% and 8%, respectively. Use a two-step binomial tree to value (a) a European four-month call option with a strike price of 0.79 and (b) an American four-month call option with the same strike price.

15.5. Explain how corporations can use range forward contracts to hedge their foreign exchange risk.

15.6. Calculate the value of a three-month at-the-money European call option on a stock index when the index is at 250, the risk-free interest rate is 10% per annum, the volatility of the index is 18% per annum, and the dividend yield on the index is 3% per annum.

15.7. Calculate the value of an eight-month European put option on a currency with a strike price of 0.50. The current exchange rate is 0.52, the volatility of the exchange rate is 12%, the domestic risk-free interest rate is 4% per annum, and the foreign risk-free interest rate is 8% per annum.

15.8. Show that the formula in equation (15.12) for a put option to sell one unit of currency A for currency B at strike price K gives the same value as equation (15.11) for a call option to buy K units of currency B for currency A at strike price $1/K$.

15.9. A foreign currency is currently worth \$1.50. The domestic and foreign risk-free interest rates are 5% and 9%, respectively. Calculate a lower bound for the value of a six-month call option on the currency with a strike price of \$1.40 if it is (a) European and (b) American.

15.10. Consider a stock index currently standing at 250. The dividend yield on the index is 4% per annum, and the risk-free rate is 6% per annum. A three-month European call option on the index with a strike price of 245 is currently worth \$10. What is the value of a three-month put option on the index with a strike price of 245?

15.11. An index currently stands at 696 and has a volatility of 30% per annum. The risk-free rate of interest is 7% per annum and the index provides a dividend yield of 4% per annum. Calculate the value of a three-month European put with an exercise price of 700.

15.12. Show that, if C is the price of an American call with exercise price K and maturity T on a stock paying a dividend yield of q, and P is the price of an American put on the same stock with the same strike price and exercise date, then

$$S_0e^{-qT} - K < C - P < S_0 - Ke^{-rT},$$

where S_0 is the stock price, r is the risk-free rate, and $r > 0$. (*Hint*: To obtain the first half of the inequality, consider possible values of:

Portfolio A: a European call option plus an amount K invested at the risk-free rate
Portfolio B: an American put option plus e^{-qT} of stock with dividends being re-invested in the stock

To obtain the second half of the inequality, consider possible values of:

Portfolio C: an American call option plus an amount Ke^{-rT} invested at the risk-free rate
Portfolio D: a European put option plus one stock with dividends being reinvested in the stock.)

15.13. Show that a European call option on a currency has the same price as the corresponding European put option on the currency when the forward price equals the strike price.

15.14. Would you expect the volatility of a stock index to be greater or less than the volatility of a typical stock? Explain your answer.

15.15. Does the cost of portfolio insurance increase or decrease as the beta of a portfolio increases? Explain your answer.

15.16. Suppose that a portfolio is worth \$60 million and the S&P 500 is at 1,200. If the value of the portfolio mirrors the value of the index, what options should be purchased to provide protection against the value of the portfolio falling below \$54 million in one year's time?

15.17. Consider again the situation in Problem 15.16. Suppose that the portfolio has a beta of 2.0, the risk-free interest rate is 5% per annum, and the dividend yield on both the portfolio and the index is 3% per annum. What options should be purchased to provide protection against the value of the portfolio falling below \$54 million in one year's time?

15.18. An index currently stands at 1,500. European call and put options with a strike price of 1,400 and time to maturity of six months have market prices of 154.00 and 34.25, respectively. The six-month risk-free rate is 5%. What is the implied dividend yield?

15.19. A total return index tracks the return, including dividends, on a certain portfolio. Explain how you would value (a) forward contracts and (b) European options on the index.

15.20. What is the put–call parity relationship for European currency options?

15.21. Can an option on the yen–euro exchange rate be created from two options, one on the dollar–euro exchange rate, and the other on the dollar–yen exchange rate? Explain your answer.

15.22. Prove the results in equations (15.1), (15.2), and (15.3) using the portfolios indicated.

Assignment Questions

15.23. The Dow Jones Industrial Average on January 12, 2007, was 12,556 and the price of the March 126 call was \$2.25. Use the DerivaGem software to calculate the implied volatility of this option. Assume the risk-free rate was 5.3% and the dividend yield was 3%. The option expires on March 20, 2007. Estimate the price of a March 126 put. What is the volatility implied by the price you estimate for this option? (Note that options are on the Dow Jones index divided by 100.)

15.24. A stock index currently stands at 300 and has a volatility of 20%. The risk-free interest rate is 8% and the dividend yield on the index is 3%. Use a three-step binomial tree to value a six-month put option on the index with a strike price of 300 if it is (a) European and (b) American?

15.25. Suppose that the spot price of the Canadian dollar is US \$0.85 and that the Canadian dollar/US dollar exchange rate has a volatility of 4% per annum. The risk-free rates of interest in Canada and the United States are 4% and 5% per annum, respectively. Calculate the value of a European call option to buy one Canadian dollar for US \$0.85 in nine months. Use put–call parity to calculate the price of a European put option to sell one Canadian dollar for US \$0.85 in nine months. What is the price of a call option to buy US \$0.85 with one Canadian dollar in nine months?

15.26. A mutual fund announces that the salaries of its fund managers will depend on the performance of the fund. If the fund loses money, the salaries will be zero. If the fund makes a profit, the salaries will be proportional to the profit. Describe the salary of a fund manager as an option. How is a fund manager motivated to behave with this type of remuneration package?

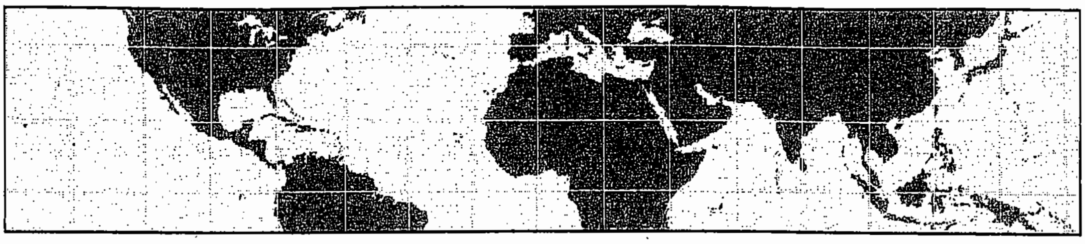
15.27. Assume that the price of currency A expressed in terms of the price of currency B follows the process

$$dS = (r_B - r_A)S\,dt + \sigma S\,dz$$

where r_A is the risk-free interest rate in currency A and r_B is the risk-free interest rate in currency B. What is the process followed by the price of currency B expressed in terms of currency A?

15.28. The three-month forward USD/euro exchange rate is 1.3000. The exchange rate volatility is 15%. A US company will have to pay 1 million euros in three months. The euro and USD risk-free rates are 5% and 4%, respectively. The company decides to use a range forward contract with the lower strike price equal to 1.2500.

(a) What should the higher strike price be to create a zero-cost contract?
(b) What position in calls and puts should the company take?
(c) Does your answer depend on the euro risk-free rate? Explain.
(d) Does your answer depend on the USD risk-free rate? Explain.



Glossary of Terms

ABS *See* Asset-Backed Security

Accrual Swap An interest rate swap where interest on one side accrues only when a certain condition is met.

Accrued Interest The interest earned on a bond since the last coupon payment date.

Adaptive Mesh Model A model developed by Figlewski and Gao that grafts a high-resolution tree on to a low-resolution tree so that there is more detailed modeling of the asset price in critical regions.

Agency Costs Costs arising from a situation where the agent (e.g., manager) is not motivated to act in the best interests of the principal (e.g., shareholder).

American Option An option that can be exercised at any time during its life.

Amortizing Swap A swap where the notional principal decreases in a predetermined way as time passes.

Analytic Result Result where answer is in the form of an equation.

Arbitrage A trading strategy that takes advantage of two or more securities being mispriced relative to each other.

Arbitrageur An individual engaging in arbitrage.

Asian Option An option with a payoff dependent on the average price of the underlying asset during a specified period.

Ask Price The price that a dealer is offering to sell an asset.

Asked Price *See* Ask Price.

Asset-Backed Security Security created from a portfolio of loans, bonds, credit card receivables, or other assets.

Asset-or-Nothing Call Option An option that provides a payoff equal to the asset price if the asset price is above the strike price and zero otherwise.

Asset-or-Nothing Put Option An option that provides a payoff equal to the asset price if the asset price is below the strike price and zero otherwise.

Asset Swap Exchanges the coupon on a bond for LIBOR plus a spread.

As-You-Like-It Option *See* Chooser Option.

At-the-Money Option An option in which the strike price equals the price of the underlying asset.

Average Price Call Option An option giving a payoff equal to the greater of zero and the amount by which the average price of the asset exceeds the strike price.

Average Price Put Option An option giving a payoff equal to the greater of zero and the amount by which the strike price exceeds the average price of the asset.

Average Strike Option An option that provides a payoff dependent on the difference between the final asset price and the average asset price.

Backdating Practice (often illegal) of marking a document with a date that precedes the current date.

Back Testing Testing a value-at-risk or other model using historical data.

Backwards Induction A procedure for working from the end of a tree to its beginning in order to value an option.

Barrier Option An option whose payoff depends on whether the path of the underlying asset has reached a barrier (i.e., a certain predetermined level).

Base Correlation Correlation that leads to the price of a 0% to X% CDO tranche being consistent with the market for a particular value of X.

Basel II New international regulations for calculating bank capital expected to come into effect in about 2007.

Basis The difference between the spot price and the futures price of a commodity.

Basis Point When used to describe an interest rate, a basis point is one hundredth of one percent (= 0.01%)

Basis Risk The risk to a hedger arising from uncertainty about the basis at a future time.

Basis Swap A swap where cash flows determined by one floating reference rate are exchanged for cash flows determined by another floating reference rate.

Basket Credit Default Swap Credit default swap where there are several reference entities.

Basket Option An option that provides a payoff dependent on the value of a portfolio of assets.

Bear Spread A short position in a put option with strike price K_1 combined with a long position in a put option with strike price K_2 where $K_2 > K_1$. (A bear spread can also be created with call options.)

Bermudan Option An option that can be exercised on specified dates during its life.

Beta A measure of the systematic risk of an asset.

Bid–Ask Spread The amount by which the ask price exceeds the bid price.

Bid–Offer Spread *See* Bid–Ask Spread.

Bid Price The price that a dealer is prepared to pay for an asset.

Binary Credit Default Swap Instrument where there is a fixed dollar payoff in the event of a default by a particular company.

Binary Option Option with a discontinuous payoff, e.g., a cash-or-nothing option or an asset-or-nothing option.

Binomial Model A model where the price of an asset is monitored over successive short periods of time. In each short period it is assumed that only two price movements are possible.

Binomial Tree A tree that represents how an asset price can evolve under the binomial model.

Bivariate Normal Distribution A distribution for two correlated variables, each of which is normal.

Black's Approximation An approximate procedure developed by Fischer Black for valuing a call option on a dividend-paying stock.

Black's Model An extension of the Black–Scholes model for valuing European options on futures contracts. As described in Chapter 26, it is used extensively in practice to value European options when the distribution of the asset price at maturity is assumed to be lognormal.

Black–Scholes Model A model for pricing European options on stocks, developed by Fischer Black, Myron Scholes, and Robert Merton.

Board Broker The individual who handles limit orders in some exchanges. The board broker makes information on outstanding limit orders available to other traders.

Bond Option An option where a bond is the underlying asset.

Bond Yield Discount rate which, when applied to all the cash flows of a bond, causes the present value of the cash flows to equal the bond's market price.

Bootstrap Method A procedure for calculating the zero-coupon yield curve from market data.

Boston Option *See* Deferred Payment Option.

Box Spread A combination of a bull spread created from calls and a bear spread created from puts.

Break Forward *See* Deferred Payment Option.

Brownian Motion *See* Wiener Process.

Bull Spread A long position in a call with strike price K_1 combined with a short position in a call with strike price K_2, where $K_2 > K_1$. (A bull spread can also be created with put options.)

Butterfly Spread A position that is created by taking a long position in a call with strike price K_1, a long position in a call with strike price K_3, and a short position in two calls with strike price K_2, where $K_3 > K_2 > K_1$ and $K_2 = 0.5(K_1 + K_3)$. (A butterfly spread can also be created with put options.)

Calendar Spread A position that is created by taking a long position in a call option that matures at one time and a short position in a similar call option that matures at a different time. (A calendar spread can also be created using put options.)

Calibration Method for implying a model's parameters from the prices of actively traded options.

Callable Bond A bond containing provisions that allow the issuer to buy it back at a predetermined price at certain times during its life.

Call Option An option to buy an asset at a certain price by a certain date.

Cancelable Swap Swap that can be canceled by one side on prespecified dates.

Cap *See* Interest Rate Cap.

Cap Rate The rate determining payoffs in an interest rate cap.

Capital Asset Pricing Model A model relating the expected return on an asset to its beta.

Caplet One component of an interest rate cap.

Cash Flow Mapping A procedure for representing an instrument as a portfolio of zero-coupon bonds for the purpose of calculating value at risk.

Cash-or-Nothing Call Option An option that provides a fixed predetermined payoff if the final asset price is above the strike price and zero otherwise.

Cash-or-Nothing Put Option An option that provides a fixed predetermined payoff if the final asset price is below the strike price and zero otherwise.

Cash Settlement Procedure for settling a futures contract in cash rather than by delivering the underlying asset.

CAT Bond Bond where the interest and, possibly, the principal paid are reduced if a particular category of "catastrophic" insurance claims exceed a certain amount.

CDD Cooling degree days. The maximum of zero and the amount by which the daily average temperature is greater than 65° Fahrenheit. The average temperature is the average of the highest and lowest temperatures (midnight to midnight).

CDO *See* Collateralized Debt Obligation.

CDO Squared An instrument in which the default risks in a portfolio of CDO tranches are allocated to new securities.

CDX NA IG Portfolio of 125 North American companies.

Cheapest-to-Deliver Bond The bond that is cheapest to deliver in the Chicago Board of Trade bond futures contract.

Cholesky Decomposition A method of sampling from a multivariate normal distribution.

Chooser Option An option where the holder has the right to choose whether it is a call or a put at some point during its life.

Class of Options *See* Option Class.

Clean Price of Bond The quoted price of a bond. The cash price paid for the bond (or dirty price) is calculated by adding the accrued interest to the clean price.

Clearinghouse A firm that guarantees the performance of the parties in an exchange-traded derivatives transaction (also referred to as a clearing corporation).

Clearing Margin A margin posted by a member of a clearinghouse.

CMO Collateralized Mortgage Obligation.

Collar *See* Interest Rate Collar.

Collateralization A system for posting collateral by one or both parties in a derivatives transaction.

Collateralized Debt Obligation A way of packaging credit risk. Several classes of securities (known as tranches) are created from a portfolio of bonds and there are rules for determining how the cost of defaults are allocated to classes.

Collateralized Mortgage Obligation (CMO) A mortgage-backed security where investors are divided into classes and there are rules for determining how principal repayments are channeled to the classes.

Combination A position involving both calls and puts on the same underlying asset.

Commission Brokers Individuals who execute trades for other people and charge a commission for doing so.

Commodity Futures Trading Commission A body that regulates trading in futures contracts in the United States.

Commodity Swap A swap where cash flows depend on the price of a commodity.

Compound Correlation Correlation implied from the market price of a CDO tranche.

Compound Option An option on an option.

Compounding Frequency This defines how an interest rate is measured.

Compounding Swap Swap where interest compounds instead of being paid.

Conditional Value at Risk (C-VaR) Expected loss during N days conditional on being in the $(100 - X)\%$ tail of the distribution of profits/losses. The variable N is the time horizon and $X\%$ is the confidence level.

Confirmation Contract confirming verbal agreement between two parties to a trade in the over-the-counter market.

Constant Elasticity of Variance (CEV) Model Model where the variance of the change in a variable in a short period of time is proportional to the value of the variable.

Constant Maturity Swap (CMS) A swap where a swap rate is exchanged for either a fixed rate or a floating rate on each payment date.

Constant Maturity Treasury Swap A swap where the yield on a Treasury bond is exchanged for either a fixed rate or a floating rate on each payment date.

Consumption Asset An asset held for consumption rather than investment.

Contango A situation where the futures price is above the expected future spot price.

Continuous Compounding A way of quoting interest rates. It is the limit as the assumed compounding interval is made smaller and smaller.

Control Variate Technique A technique that can sometimes be used for improving the accuracy of a numerical procedure.

Convenience Yield A measure of the benefits from ownership of an asset that are not obtained by the holder of a long futures contract on the asset.

Conversion Factor A factor used to determine the number of bonds that must be delivered in the Chicago Board of Trade bond futures contract.

Convertible Bond A corporate bond that can be converted into a predetermined amount of the company's equity at certain times during its life.

Convexity A measure of the curvature in the relationship between bond prices and bond yields.

Convexity Adjustment An overworked term. For example, it can refer to the adjustment necessary to convert a futures interest rate to a forward interest rate. It can also refer to the adjustment to a forward rate that is sometimes necessary when Black's model is used.

Copula A way of defining the correlation between variables with known distributions.

Cornish–Fisher Expansion An approximate relationship between the fractiles of a probability distribution and its moments.

Cost of Carry The storage costs plus the cost of financing an asset minus the income earned on the asset.

Counterparty The opposite side in a financial transaction.

Coupon Interest payment made on a bond.

Covariance Measure of the linear relationship between two variables (equals the correlation between the variables times the product of their standard deviations).

Covered Call A short position in a call option on an asset combined with a long position in the asset.

Credit Contagion The tendency of a default by one company to lead to defaults by other companies.

Credit Default Swap An instrument that gives the holder the right to sell a bond for its face value in the event of a default by the issuer.

Credit Derivative A derivative whose payoff depends on the creditworthiness of one or more companies or countries.

Credit Rating A measure of the creditworthiness of a bond issue.

Credit Ratings Transition Matrix A table showing the probability that a company will move from one credit rating to another during a certain period of time.

Credit Risk The risk that a loss will be experienced because of a default by the counterparty in a derivatives transaction.

Credit Spread Option Option whose payoff depends on the spread between the yields earned on two assets.

Credit Value at Risk The credit loss that will not be exceeded at some specified confidence level.

CreditMetrics A procedure for calculating credit value at risk.

Cross Hedging Hedging an exposure to the price of one asset with a contract on another asset.

Cumulative Distribution Function The probability that a variable will be less than x as a function of x.

Currency Swap A swap where interest and principal in one currency are exchanged for interest and principal in another currency.

Day Count A convention for quoting interest rates.

Day Trade A trade that is entered into and closed out on the same day.

Default Correlation Measures the tendency of two companies to default at about the same time.

Default Intensity *See* Hazard Rate.

Default Probability Density Measures the unconditional probability of default in a future short period of time.

Deferred Payment Option An option where the price paid is deferred until the end of the option's life.

Deferred Swap An agreement to enter into a swap at some time in the future (also called a forward swap).

Delivery Price Price agreed to (possibly some time in the past) in a forward contract.

Delta The rate of change of the price of a derivative with the price of the underlying asset.

Delta Hedging A hedging scheme that is designed to make the price of a portfolio of derivatives insensitive to small changes in the price of the underlying asset.

Delta-Neutral Portfolio A portfolio with a delta of zero so that there is no sensitivity to small changes in the price of the underlying asset.

DerivaGem The software accompanying this book.

Derivative An instrument whose price depends on, or is derived from, the price of another asset.

Deterministic Variable A variable whose future value is known.

Diagonal Spread A position in two calls where both the strike prices and times to maturity are different. (A diagonal spread can also be created with put options.)

Differential Swap A swap where a floating rate in one currency is exchanged for a floating rate in another currency and both rates are applied to the same principal.

Diffusion Process Model where value of asset changes continuously (no jumps).

Dirty Price of Bond Cash price of bond.

Discount Bond *See* Zero-Coupon Bond.

Discount Instrument An instrument, such as a Treasury bill, that provides no coupons.

Discount Rate The annualized dollar return on a Treasury bill or similar instrument expressed as a percentage of the final face value.

Dividend A cash payment made to the owner of a stock.

Dividend Yield The dividend as a percentage of the stock price.

Dollar Duration The product of a bond's modified duration and the bond price.

Down-and-In Option An option that comes into existence when the price of the underlying asset declines to a prespecified level.

Down-and-Out Option An option that ceases to exist when the price of the underlying asset declines to a prespecified level.

Downgrade Trigger A clause in a contract that states that the contract will be terminated with a cash settlement if the credit rating of one side falls below a certain level.

Drift Rate The average increase per unit of time in a stochastic variable.

Duration A measure of the average life a bond. It is also an approximation to the ratio of the proportional change in the bond price to the absolute change in its yield.

Duration Matching A procedure for matching the durations of assets and liabilities in a financial institution.

Dynamic Hedging A procedure for hedging an option position by periodically changing the position held in the underlying asset. The objective is usually to maintain a delta-neutral position.

Early Exercise Exercise prior to the maturity date.

Efficient Market Hypothesis A hypothesis that asset prices reflect relevant information.

Electronic Trading System of trading where a computer is used to match buyers and sellers.

Embedded Option An option that is an inseparable part of another instrument.

Empirical Research Research based on historical market data.

Employee Stock Option A stock option issued by company on its own stock and given to its employees as part of their remuneration.

Equilibrium Model A model for the behavior of interest rates derived from a model of the economy.

Equity Swap A swap where the return on an equity portfolio is exchanged for either a fixed or a floating rate of interest.

Eurocurrency A currency that is outside the formal control of the issuing country's monetary authorities.

Eurodollar A dollar held in a bank outside the United States.

Eurodollar Futures Contract A futures contract written on a Eurodollar deposit.

Eurodollar Interest Rate The interest rate on a Eurodollar deposit.

European Option An option that can be exercised only at the end of its life.

EWMA Exponentially weighted moving average.

Exchange Option An option to exchange one asset for another.

Ex-dividend Date When a dividend is declared, an ex-dividend date is specified. Investors who own shares of the stock just before the ex-dividend date receive the dividend.

Exercise Limit Maximum number of option contracts that can be exercised within a five-day period.

Exercise Multiple Ratio of stock price to strike price at time of exercise for employee stock option.

Exercise Price The price at which the underlying asset may be bought or sold in an option contract (also called the strike price).

Exotic Option A nonstandard option.

Expectations Theory The theory that forward interest rates equal expected future spot interest rates.

Expected Shortfall *See* Conditional Value at Risk.

Expected Value of a Variable The average value of the variable obtained by weighting the alternative values by their probabilities.

Expiration Date The end of life of a contract.

Explicit Finite Difference Method A method for valuing a derivative by solving the underlying differential equation. The value of the derivative at time t is related to three values at time $t + \Delta t$. It is essentially the same as the trinomial tree method.

Exponentially Weighted Moving Average Model A model where exponential weighting is used to provide forecasts for a variable from historical data. It is sometimes applied to variances and covariances in value at risk calculations.

Exponential Weighting A weighting scheme where the weight given to an observation depends on how recent it is. The weight given to an observation i time periods ago is λ times the weight given to an observation $i - 1$ time periods ago where $\lambda < 1$.

Exposure The maximum loss from default by a counterparty.

Extendable Bond A bond whose life can be extended at the option of the holder.

Extendable Swap A swap whose life can be extended at the option of one side to the contract.

Factor Source of uncertainty.

Factor analysis An analysis aimed at finding a small number of factors that describe most of the variation in a large number of correlated variables (similar to a principal components analysis).

FAS 123 Accounting standard in United States relating to employee stock options.

FAS 133 Accounting standard in United States relating to instruments used for hedging.

FASB Financial Accounting Standards Board.

Financial Intermediary A bank or other financial institution that facilitates the flow of funds between different entities in the economy.

Finite Difference Method A method for solving a differential equation.

Flat Volatility The name given to volatility used to price a cap when the same volatility is used for each caplet.

Flex Option An option traded on an exchange with terms that are different from the standard options traded by the exchange.

Flexi Cap Interest rate cap where there is a limit on the total number of caplets that can be exercised.

Floor *See* Interest Rate Floor.

Floor–Ceiling Agreement *See* Collar.

Floorlet One component of a floor.

Floor Rate The rate in an interest rate floor agreement.

Foreign Currency Option An option on a foreign exchange rate.

Forward Contract A contract that obligates the holder to buy or sell an asset for a predetermined delivery price at a predetermined future time.

Forward Exchange Rate The forward price of one unit of a foreign currency.

Forward Interest Rate The interest rate for a future period of time implied by the rates prevailing in the market today.

Forward Price The delivery price in a forward contract that causes the contract to be worth zero.

Forward Rate Rate of interest for a period of time in the future implied by today's zero rates.

Forward Rate Agreement (FRA) Agreement that a certain interest rate will apply to a certain principal amount for a certain time period in the future.

Forward Risk-Neutral World A world is forward risk-neutral with respect to a certain asset when the market price of risk equals the volatility of that asset.

Forward Start Option An option designed so that it will be at-the-money at some time in the future.

Forward Swap *See* Deferred Swap.

Futures Contract A contract that obligates the holder to buy or sell an asset at a predetermined delivery price during a specified future time period. The contract is settled daily.

Futures Option An option on a futures contract.

Futures Price The delivery price currently applicable to a futures contract.

Futures-Style Option Futures contract on the payoff from an option.

Gamma The rate of change of delta with respect to the asset price.

Gamma-Neutral Portfolio A portfolio with a gamma of zero.

GARCH Model A model for forecasting volatility where the variance rate follows a mean-reverting process.

Gaussian Copula Model A model for defining a correlation structure between two or more variables. In some credit derivatives models, it is used to define a correlation structure for times to default.

Gaussian Quadrature Procedure for integrating over a normal distribution.

Generalized Wiener Process A stochastic process where the change in a variable in time t has a normal distribution with mean and variance both proportional to t.

Geometric Average The nth root of the product of n numbers.

Geometric Brownian Motion A stochastic process often assumed for asset prices where the logarithm of the underlying variable follows a generalized Wiener process.

Girsanov's Theorem Result showing that when we change the measure (e.g., move from real world to risk-neutral world) the expected return of a variable changes but the volatility remains the same.

Greeks Hedge parameters such as delta, gamma, vega, theta, and rho.

Haircut Discount applied to the value of an asset for collateral purposes.

Hazard Rate Measures probability of default in a short period of time conditional on no earlier default.

HDD Heating degree days. The maximum of zero and the amount by which the daily average temperature is less than 65° Fahrenheit. The average temperature is the average of the highest and lowest temperatures (midnight to midnight).

Hedge A trade designed to reduce risk.

Hedger An individual who enters into hedging trades.

Hedge Ratio The ratio of the size of a position in a hedging instrument to the size of the position being hedged.

Historical Simulation A simulation based on historical data.

Historic Volatility A volatility estimated from historical data.

Holiday Calendar Calendar defining which days are holidays for the purposes of determining payment dates in a swap.

IMM Dates Third Wednesday in March, June, September, and December.

Implicit Finite Difference Method A method for valuing a derivative by solving the underlying differential equation. The value of the derivative at time $t + \Delta t$ is related to three values at time t.

Implied Correlation Correlation number implied from the price of a credit derivative using the Gaussian copula or similar model.

Implied Distribution A distribution for a future asset price implied from option prices.

Implied Tree A tree describing the movements of an asset price that is constructed to be consistent with observed option prices.

Implied Volatility Volatility implied from an option price using the Black–Scholes or a similar model.

Implied Volatility Function (IVF) Model Model designed so that it matches the market prices of all European options.

Inception Profit Profit created by selling a derivative for more than its theoretical value.

Index Amortizing Swap See indexed principal swap.

Index Arbitrage An arbitrage involving a position in the stocks comprising a stock index and a position in a futures contract on the stock index.

Index Futures A futures contract on a stock index or other index.

Index Option An option contract on a stock index or other index.

Indexed Principal Swap A swap where the principal declines over time. The reduction in the principal on a payment date depends on the level of interest rates.

Initial Margin The cash required from a futures trader at the time of the trade.

Instantaneous Forward Rate Forward rate for a very short period of time in the future.

Interest Rate Cap An option that provides a payoff when a specified interest rate is above a certain level. The interest rate is a floating rate that is reset periodically.

Interest Rate Collar A combination of an interest-rate cap and an interest rate floor.

Interest Rate Derivative A derivative whose payoffs are dependent on future interest rates.

Interest Rate Floor An option that provides a payoff when an interest rate is below a certain level. The interest rate is a floating rate that is reset periodically.

Interest Rate Option An option where the payoff is dependent on the level of interest rates.

Interest Rate Swap An exchange of a fixed rate of interest on a certain notional principal for a floating rate of interest on the same notional principal.

International Swaps and Derivatives Association Trade Association for over-the-counter derivatives and developer of master agreements used in over-the-counter contracts.

In-the-Money Option Either (a) a call option where the asset price is greater than the strike price or (b) a put option where the asset price is less than the strike price.

Intrinsic Value For a call option, this is the greater of the excess of the asset price over the strike price and zero. For a put option, it is the greater of the excess of the strike price over the asset price and zero.

Inverted Market A market where futures prices decrease with maturity.

Investment Asset An asset held by at least some individuals for investment purposes.

IO Interest Only. A mortgage-backed security where the holder receives only interest cash flows on the underlying mortgage pool.

ISDA *See* International Swaps and Derivatives Association.

Itô Process A stochastic process where the change in a variable during each short period of time of length Δt has a normal distribution. The mean and variance of the distribution are proportional to Δt and are not necessarily constant.

Itô's Lemma A result that enables the stochastic process for a function of a variable to be calculated from the stochastic process for the variable itself.

ITraxx Europe Portfolio of 125 investment-grade European companies.

Jump–Diffusion Model Model where asset price has jumps superimposed on to a diffusion process such as geometric Brownian motion.

Kurtosis A measure of the fatness of the tails of a distribution.

LEAPS Long-term equity anticipation securities. These are relatively long-term options on individual stocks or stock indices.

LIBID London interbank bid rate. The rate bid by banks on Eurocurrency deposits (i.e., the rate at which a bank is willing to borrow from other banks).

LIBOR London interbank offered rate. The rate offered by banks on Eurocurrency deposits (i.e., the rate at which a bank is willing to lend to other banks).

LIBOR Curve LIBOR zero-coupon interest rates as a function of maturity.

LIBOR-in-Arrears Swap Swap where the interest paid on a date is determined by the interest rate observed on that date (not by the interest rate observed on the previous payment date).

Limit Move The maximum price move permitted by the exchange in a single trading session.

Limit Order An order that can be executed only at a specified price or one more favorable to the investor.

Liquidity Preference Theory A theory leading to the conclusion that forward interest rates are above expected future spot interest rates.

Liquidity Premium The amount that forward interest rates exceed expected future spot interest rates.

Liquidity Risk Risk that it will not be possible to sell a holding of a particular instrument at its theoretical price.

Locals Individuals on the floor of an exchange who trade for their own account rather than for someone else.

Lognormal Distribution A variable has a lognormal distribution when the logarithm of the variable has a normal distribution.

Long Hedge A hedge involving a long futures position.

Long Position A position involving the purchase of an asset.

Lookback Option An option whose payoff is dependent on the maximum or minimum of the asset price achieved during a certain period.

Low Discrepancy Sequence *See* Quasi-random Sequence.

Maintenance Margin When the balance in a trader's margin account falls below the maintenance margin level, the trader receives a margin call requiring the account to be topped up to the initial margin level.

Margin The cash balance (or security deposit) required from a futures or options trader.

Margin Call A request for extra margin when the balance in the margin account falls below the maintenance margin level.

Market Maker A trader who is willing to quote both bid and offer prices for an asset.

Market Model A model most commonly used by traders.

Market Price of Risk A measure of the trade-offs investors make between risk and return.

Market Segmentation Theory A theory that short interest rates are determined independently of long interest rates by the market.

Marking to Market The practice of revaluing an instrument to reflect the current values of the relevant market variables.

Markov Process A stochastic process where the behavior of the variable over a short period of time depends solely on the value of the variable at the beginning of the period, not on its past history.

Martingale A zero drift stochastic process.

Maturity Date The end of the life of a contract.

Maximum Likelihood Method A method for choosing the values of parameters by maximizing the probability of a set of observations occurring.

Mean Reversion The tendency of a market variable (such as an interest rate) to revert back to some long-run average level.

Measure Sometimes also called a probability measure, it defines the market price of risk.

Modified Duration A modification to the standard duration measure so that it more accurately describes the relationship between proportional changes in a bond price and actual changes in its yield. The modification takes account of the compounding frequency with which the yield is quoted.

Money Market Account An investment that is initially equal to $1 and, at time t, increases at the very short-term risk-free interest rate prevailing at that time.

Monte Carlo Simulation A procedure for randomly sampling changes in market variables in order to value a derivative.

Mortgage-Backed Security A security that entitles the owner to a share in the cash flows realized from a pool of mortgages.

Naked Position A short position in a call option that is not combined with a long position in the underlying asset.

Netting The ability to offset contracts with positive and negative values in the event of a default by a counterparty.

Newton–Raphson Method An iterative procedure for solving nonlinear equations.

No-Arbitrage Assumption The assumption that there are no arbitrage opportunities in market prices.

No-Arbitrage Interest Rate Model A model for the behavior of interest rates that is exactly consistent with the initial term structure of interest rates.

Nonstationary Model A model where the volatility parameters are a function of time.

Nonsystematic Risk Risk that can be diversified away.

Normal Backwardation A situation where the futures price is below the expected future spot price.

Normal Distribution The standard bell-shaped distribution of statistics.

Normal Market A market where futures prices increase with maturity.

Notional Principal The principal used to calculate payments in an interest rate swap. The principal is "notional" because it is neither paid nor received.

Numeraire Defines the units in which security prices are measured. For example, if the price of IBM is the numeraire, all security prices are measured relative to IBM. If IBM is $80 and a particular security price is $50, the security price is 0.625 when IBM is the numeraire.

Numerical Procedure A method of valuing an option when no formula is available.

OCC Options Clearing Corporation. *See* Clearinghouse.

Offer Price *See* Ask Price.

Open Interest The total number of long positions outstanding in a futures contract (equals the total number of short positions).

Open Outcry System of trading where traders meet on the floor of the exchange

Option The right to buy or sell an asset.

Option-Adjusted Spread The spread over the Treasury curve that makes the theoretical price of an interest rate derivative equal to the market price.

Option Class All options of the same type (call or put) on a particular stock.

Option Series All options of a certain class with the same strike price and expiration date.

Order Book Official *See* Board Broker.

Out-of-the-Money Option Either (a) a call option where the asset price is less than the strike price or (b) a put option where the asset price is greater than the strike price.

Over-the-Counter Market A market where traders deal by phone. The traders are usually financial institutions, corporations, and fund managers.

Package A derivative that is a portfolio of standard calls and puts, possibly combined with a position in forward contracts and the asset itself.

Par Value The principal amount of a bond.

Par Yield The coupon on a bond that makes its price equal the principal.

Parallel Shift A movement in the yield curve where each point on the curve changes by the same amount.

Path-Dependent Option An option whose payoff depends on the whole path followed by the underlying variable—not just its final value.

Payoff The cash realized by the holder of an option or other derivative at the end of its life.

Plain Vanilla A term used to describe a standard deal.

P-Measure Real-world measure.

PO Principal Only. A mortgage-backed security where the holder receives only principal cash flows on the underlying mortgage pool.

Poisson Process A process describing a situation where events happen at random. The probability of an event in time Δt is $\lambda \Delta t$, where λ is the intensity of the process.

Portfolio Immunization Making a portfolio relatively insensitive to interest rates.

Portfolio Insurance Entering into trades to ensure that the value of a portfolio will not fall below a certain level.

Position Limit The maximum position a trader (or group of traders acting together) is allowed to hold.

Premium The price of an option.

Prepayment function A function estimating the prepayment of principal on a portfolio of mortgages in terms of other variables.

Principal The par or face value of a debt instrument.

Principal Components Analysis An analysis aimed at finding a small number of factors that describe most of the variation in a large number of correlated variables (similar to a factor analysis).

Program Trading A procedure where trades are automatically generated by a computer and transmitted to the trading floor of an exchange.

Protective Put A put option combined with a long position in the underlying asset.

Pull-to-Par The reversion of a bond's price to its par value at maturity.

Put–Call Parity The relationship between the price of a European call option and the price of a European put option when they have the same strike price and maturity date.

Put Option An option to sell an asset for a certain price by a certain date.

Puttable Bond A bond where the holder has the right to sell it back to the issuer at certain predetermined times for a predetermined price.

Puttable Swap A swap where one side has the right to terminate early.

Q-Measure Risk-neutral measure.

Quanto A derivative where the payoff is defined by variables associated with one currency but is paid in another currency.

Quasi-random Sequences A sequences of numbers used in a Monte Carlo simulation that are representative of alternative outcomes rather than random.

Rainbow Option An option whose payoff is dependent on two or more underlying variables.

Range Forward Contract The combination of a long call and short put or the combination of a short call and long put.

Ratchet Cap Interest rate cap where the cap rate applicable to an accrual period equals the rate for the previous accrual period plus a spread.

Real Option Option involving real (as opposed to financial) assets. Real assets include land, plant, and machinery.

Rebalancing The process of adjusting a trading position periodically. Usually the purpose is to maintain delta neutrality.

Recovery Rate Amount recovered in the event of a default as a percent of the face value.

Reference Entity Company for which default protection is bought in a credit default swap.

Repo Repurchase agreement. A procedure for borrowing money by selling securities to a counterparty and agreeing to buy them back later at a slightly higher price.

Repo Rate The rate of interest in a repo transaction.

Reset Date The date in a swap or cap or floor when the floating rate for the next period is set.

Reversion Level The level that the value of a market variable (e.g., an interest rate) tends to revert.

Rho Rate of change of the price of a derivative with the interest rate.

Rights Issue An issue to existing shareholders of a security giving them the right to buy new shares at a certain price.

Risk-Free Rate The rate of interest that can be earned without assuming any risks.

Risk-Neutral Valuation The valuation of an option or other derivative assuming the world is risk neutral. Risk-neutral valuation gives the correct price for a derivative in all worlds, not just in a risk-neutral world.

Risk-Neutral World A world where investors are assumed to require no extra return on average for bearing risks.

Roll Back *See* Backwards Induction.

Scalper A trader who holds positions for a very short period of time.

Scenario Analysis An analysis of the effects of possible alternative future movements in market variables on the value of a portfolio.

SEC Securities and Exchange Commission.

Settlement Price The average of the prices that a contract trades for immediately before the bell signaling the close of trading for a day. It is used in mark-to-market calculations.

Short Hedge A hedge where a short futures position is taken.

Short Position A position assumed when traders sell shares they do not own.

Short Rate The interest rate applying for a very short period of time.

Short Selling Selling in the market shares that have been borrowed from another investor.

Short-Term Risk-Free Rate *See* Short Rate.

Shout Option An option where the holder has the right to lock in a minimum value for the payoff at one time during its life.

Simulation *See* Monte Carlo Simulation.

Specialist An individual responsible for managing limit orders on some exchanges. The specialist does not make the information on outstanding limit orders available to other traders.

Speculator An individual who is taking a position in the market. Usually the individual is betting that the price of an asset will go up or that the price of an asset will go down.

Spot Interest Rate *See* Zero-Coupon Interest Rate.

Spot Price The price for immediate delivery.

Spot Volatilities The volatilities used to price a cap when a different volatility is used for each caplet.

Spread Option An option where the payoff is dependent on the difference between two market variables.

Spread Transaction A position in two or more options of the same type.

Static Hedge A hedge that does not have to be changed once it is initiated.

Static Options Replication A procedure for hedging a portfolio that involves finding another portfolio of approximately equal value on some boundary.

Step-up Swap A swap where the principal increases over time in a predetermined way.

Sticky Cap Interest rate cap where the cap rate applicable to an accrual period equals the capped rate for the previous accrual period plus a spread.

Stochastic Process An equation describing the probabilistic behavior of a stochastic variable.

Stochastic Variable A variable whose future value is uncertain.

Stock Dividend A dividend paid in the form of additional shares.

Stock Index An index monitoring the value of a portfolio of stocks.

Stock Index Futures Futures on a stock index.

Stock Index Option An option on a stock index.

Stock Option Option on a stock.

Stock Split The conversion of each existing share into more than one new share.

Storage Costs The costs of storing a commodity.

Straddle A long position in a call and a put with the same strike price.

Strangle A long position in a call and a put with different strike prices.

Strap A long position in two call options and one put option with the same strike price.

Stress Testing Testing of the impact of extreme market moves on the value of a portfolio.

Strike Price The price at which the asset may be bought or sold in an option contract (also called the exercise price).

Strip A long position in one call option and two put options with the same strike price.

Strip Bonds Zero-coupon bonds created by selling the coupons on Treasury bonds separately from the principal.

Subprime Mortgage Mortgage granted to borrower with a poor credit history or no credit history.

Swap An agreement to exchange cash flows in the future according to a prearranged formula.

Swap Rate The fixed rate in an interest rate swap that causes the swap to have a value of zero.

Swaption An option to enter into an interest rate swap where a specified fixed rate is exchanged for floating.

Swing Option Energy option in which the rate of consumption must be between a minimum and maximum level. There is usually a limit on the number of times the option holder can change the rate at which the energy is consumed.

Synthetic CDO A CDO created by selling credit default swaps.

Synthetic Option An option created by trading the underlying asset.

Systematic Risk Risk that cannot be diversified away.

Tailing the Hedge A procedure for adjusting the number of futures contracts used for hedging to reflect daily settlement.

Tail Loss *See* Conditional Value at Risk.

Take-and-Pay Option *See* Swing Option.

Term Structure of Interest Rates The relationship between interest rates and their maturities.

Terminal Value The value at maturity.

Theta The rate of change of the price of an option or other derivative with the passage of time.

Time Decay *See* Theta.

Time Value The value of an option arising from the time left to maturity (equals an option's price minus its intrinsic value).

Timing Adjustment Adjustment made to the forward value of a variable to allow for the timing of a payoff from a derivative.

Total Return Swap A swap where the return on an asset such as a bond is exchanged for LIBOR plus a spread. The return on the asset includes income such as coupons and the change in value of the asset.

Tranche One of several securities that have different risk attributes. Examples are the tranches of a CDO or CMO.

Transaction Costs The cost of carrying out a trade (commissions plus the difference between the price obtained and the midpoint of the bid–offer spread).

Treasury Bill A short-term non-coupon-bearing instrument issued by the government to finance its debt.

Treasury Bond A long-term coupon-bearing instrument issued by the government to finance it debt.

Treasury Bond Futures A futures contract on Treasury bonds.

Treasury Note *See* Treasury Bond. (Treasury notes have maturities of less than 10 years.)

Treasury Note Futures A futures contract on Treasury notes.

Tree Representation of the evolution of the value of a market variable for the purposes of valuing an option or other derivative.

Trinomial Tree A tree where there are three branches emanating from each node. It is used in the same way as a binomial tree for valuing derivatives.

Triple Witching Hour A term given to the time when stock index futures, stock index options, and options on stock index futures all expire together.

Underlying Variable A variable on which the price of an option or other derivative depends.

Unsystematic Risk *See* Nonsystematic Risk.

Up-and-In Option An option that comes into existence when the price of the underlying asset increases to a prespecified level.

Up-and-Out Option An option that ceases to exist when the price of the underlying asset increases to a prespecified level.

Uptick An increase in price.

Value at Risk A loss that will not be exceeded at some specified confidence level.

Variance–Covariance Matrix A matrix showing variances of, and covariances between, a number of different market variables.

Variance-Gamma Model A pure jump model where small jumps occur often and large jumps occur infrequently.

Variance Rate The square of volatility.

Variance Reduction Procedures Procedures for reducing the error in a Monte Carlo simulation.

Variance Swap Swap where the realized variance rate during a period is exchanged for a fixed variance rate. Both are applied to a notional principal.

Variation Margin An extra margin required to bring the balance in a margin account up to the initial margin when there is a margin call.

Vega The rate of change in the price of an option or other derivative with volatility.

Vega-Neutral Portfolio A portfolio with a vega of zero.

Vesting Period Period during which an option cannot be exercised.

VIX Index Index of the volatility of the S&P 500.

Volatility A measure of the uncertainty of the return realized on an asset.

Volatility Skew A term used to describe the volatility smile when it is nonsymmetrical.

Volatility Smile The variation of implied volatility with strike price.

Volatility Surface A table showing the variation of implied volatilities with strike price and time to maturity.

Volatility Swap Swap where the realized volatility during a period is exchanged for a fixed volatility. Both percentage volatilities are applied to a notional principal.

Volatility Term Structure The variation of implied volatility with time to maturity.

Warrant An option issued by a company or a financial institution. Call warrants are frequently issued by companies on their own stock.

Weather Derivative Derivative where the payoff depends on the weather.

Wiener Process A stochastic process where the change in a variable during each short period of time of length Δt has a normal distribution with a mean equal to zero and a variance equal to Δt.

Wild Card Play The right to deliver on a futures contract at the closing price for a period of time after the close of trading.

Writing an Option Selling an option.

Yield A return provided by an instrument.

Yield Curve *See* Term Structure.

Zero-Coupon Bond A bond that provides no coupons.

Zero-Coupon Interest Rate The interest rate that would be earned on a bond that provides no coupons.

Zero-Coupon Yield Curve A plot of the zero-coupon interest rate against time to maturity.

Zero Curve *See* Zero-Coupon Yield Curve.

Zero Rate *See* Zero-Coupon Interest Rate.

Major Exchanges Trading Futures and Options

American Stock Exchange	AMEX	www.amex.com
Australian Stock Exchange	ASX	www.asx.com.au
Bolsa de Mercadorias y Futuros, Brazil	BM&F	www.bmf.com.br
Bursa Malaysia	BM	www.bursamalaysia.com
Chicago Board of Trade	CBOT	www.cbot.com
Chicago Board Options Exchange	CBOE	www.cboe.com
Chicago Mercantile Exchange	CME	www.cme.com
Eurex	EUREX	www.eurexchange.com
Euronext	EURONEXT	www.euronext.com
Hong Kong Futures Exchange	HKFE	www.hkex.com.hk
Intercontinental Exchange	ICE	www.theice.com
International Petroleum Exchange, London	IPE	www.ipe.uk.com
International Securities Exchange	ISE	www.iseoptions.com
Kansas City Board of Trade	KCBT	www.kcbt.com
London Metal Exchange	LME	www.lme.co.uk
MEFF Renta Fija and Variable, Spain	MEFF	www.meff.es
Mexican Derivatives Exchange	MEXDER	www.mexder.com
Minneapolis Grain Exchange	MGE	www.mgex.com
Montreal Exchange	ME	www.me.org
New York Board of Trade	NYBOT	www.nybot.com
New York Mercantile Exchange	NYMEX	www.nymex.com
New York Stock Exchange	NYSE	www.nyse.com
Nordic Exchange	OMX	www.omxgroup.com
Osaka Securities Exchange	OSE	www.ose.or.jp
Philadelphia Stock Exchange	PHLX	www.phlx.com
Singapore Exchange	SGX	www.ses.com.sg
Sydney Futures Exchange	SFE	www.sfe.com.au
Tokyo Grain Exchange	TGE	www.tge.or.jp
Tokyo Financial Exchange	TFX	www.tfx.co.jp
Winnipeg Commodity Exchange	WCE	www.wce.ca

There has been a great deal of international consolidation of derivatives exchanges in the last few years. For example, in October 2006, CBOT and CME announced their intention to merge to form the world's largest derivatives exchange; EURONEXT and NYSE announced their intention to merge in June 2006; ASX merged with SFE in July 2006; ICE agreed to acquire NYBOT in September 2006 and IPE in June 2001; EUREX is jointly operated by Deutsche Börse AG and SWX Swiss Exchange; EURONEXT owns the London International Financial Futures Exchange (LIFFE) as well as two French exchanges; NYSE acquired the Pacific Exchange in September 2005. No doubt the consolidation has been largely driven by economies of scale that lead to lower trading costs.

Table for $N(x)$ When $x \leqslant 0$

This table shows values of $N(x)$ for $x \leqslant 0$. The table should be used with interpolation. For example,

$$N(-0.1234) = N(-0.12) - 0.34[N(-0.12) - N(-0.13)]$$
$$= 0.4522 - 0.34 \times (0.4522 - 0.4483)$$
$$= 0.4509$$

x	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
−0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641
−0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
−0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
−0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
−0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
−0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
−0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
−0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
−0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
−0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
−1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
−1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
−1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
−1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
−1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
−1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
−1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
−1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
−1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
−1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
−2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
−2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
−2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
−2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
−2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
−2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
−2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
−2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
−2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
−2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
−3.0	0.0014	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
−3.1	0.0010	0.0009	0.0009	0.0009	0.0008	0.0008	0.0008	0.0008	0.0007	0.0007
−3.2	0.0007	0.0007	0.0006	0.0006	0.0006	0.0006	0.0006	0.0005	0.0005	0.0005
−3.3	0.0005	0.0005	0.0005	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0003
−3.4	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0002
−3.5	0.0002	0.0002	0.0002	0.0002	0.0002	0.0002	0.0002	0.0002	0.0002	0.0002
−3.6	0.0002	0.0002	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001
−3.7	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001
−3.8	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001
−3.9	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
−4.0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Table for $N(x)$ When $x \geqslant 0$

This table shows values of $N(x)$ for $x \geqslant 0$. The table should be used with interpolation. For example,

$$\begin{aligned} N(0.6278) &= N(0.62) + 0.78[N(0.63) - N(0.62)] \\ &= 0.7324 + 0.78 \times (0.7357 - 0.7324) \\ &= 0.7350 \end{aligned}$$

x	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9986	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998
3.5	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998
3.6	0.9998	0.9998	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.7	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.8	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.9	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
4.0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

PRACTICAL FINANCIAL OPTIMIZATION

Decision making for financial engineers

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List of Models

NOTATION

Sets and Indices

$U = \{1, 2, \ldots, n\}$ index set of available financial instruments or asset classes.

$\mathcal{T} = \{0, 1, \ldots, \tau, \ldots T\}$ set of time periods, from today (0) until maturity (T). Unless stated otherwise in the text all time periods are of equal duration which is typically taken to be one month.

$\mathcal{K} = \{1, 2, \ldots, \kappa, \ldots, K\}$ index set of risk factors.

$\Sigma_t = \{1, 2, \ldots, S_t\}$ index set of states at period t.

$\Omega = \{1, 2, \ldots, N\}$ index set of scenarios.

i index of instrument or asset class from the set U.

t index of time step starting from today, $t = 0$, to the end of the horizon T.

j index of risk factor from the set $\mathcal{K}$.

l index of scenario from the set Ω.

Variables and Parameters

x vector of investments in assets with elements x_i. The units are in percentages of the total asset value or amounts in face value; the choice of units depends on the model and is made clear in the text.

b_0 vector of initial holdings with elements b_{0i}.

v_t^+ cash invested in short-term deposits at period t.

v_t^- cash borrowed at short-term rates at period t.

v_0 initial holdings in risk free asset (cash).

p^l statistical probability assigned to scenario l.

$\tilde{r}$ random vector of returns of assets with elements $\tilde{r}_i$.

r^l returns of assets in scenario l with elements r_i^l.

$\tilde{r}_t$ random vector of returns of asset at period t with elements $\tilde{r}_{ti}$.

r_t^l returns of asset at period t in scenario l with elements r_{ti}^l.

r_{ft} spot rate of return of the risk free asset for t periods.

$\tilde{F}$ random vector of cashflows from the assets with elements $\tilde{F}_i$.

F^l cashflows from the assets in scenario l with elements F_i^l.

$\tilde{F}_t$ random vector of cashflows at period t with elements $\tilde{F}_{ti}$.

F_t^l cashflows from the assets at period t in scenario l with elements F_{ti}^l.

$\tilde{P}$ random vector of prices of the assets with elements $\tilde{P}_i$.

P^l prices of the assets in scenario l with elements P_i^l.

$\tilde{P}_t$ random vector of prices at period t with elements $\tilde{P}_{ti}$.

P_t^l prices of the assets at period t in scenario l with elements P_{ti}^l.

$\tilde{P}_t^a$ random vectors of ask prices at period t with elements $\tilde{P}_{ti}^a$. In order to buy an instrument the buyer has to pay the price asked by traders.

$\tilde{P}_t^b$ random vectors of bid prices at period t with elements $\tilde{P}_{ti}^b$. In order to sell an instrument the owner must accept the price at which traders are bidding.

P_t^{al} vectors of ask prices at period t in scenario l with elements P_{ti}^{al}.

P_t^{bl} vectors of bid prices at period t in scenario l with elements P_{ti}^{bl}.

$\tilde{I}$ random variable of the total return of a benchmark portfolio, the index, with value I^l in scenario l.

$\tilde{L}_t$ random variable liability due at period t with elements L_t^l in scenario l.

Q a conformable covariance matrix.

$\sigma_{ii'}$ covariance of random variables indexed by i and i'.

$\rho_{ii'}$ correlation of random variables indexed by i and i'.

$\bar{x}_i$ maximum holdings in asset i.

$\underline{x}_i$ minimum holdings in asset i.

Glossary of Symbols

$\mathcal{E}[\tilde{r}]$ expectation of the random variable or vector $\tilde{r}$ with respect to the statistical probabilities p^l assigned to scenarios $l \in \Omega$.

$\mathcal{E}_P[\tilde{r}]$ or $\mathcal{E}_\lambda[\tilde{r}]$ expectation of the random variable or vector $\tilde{r}$ with respect to the probability distribution P or the probabilities $\lambda \in P$.

$\mathcal{U}(a)$ utility function with arguments over the real numbers a.

$\bar{r}$ mean value of a random variable or vector $\tilde{r}$.

$R(x; \tilde{r})$ portfolio return as a function of x with parameters $\tilde{r}$.

$V(x; \tilde{P})$ portfolio value as a function of x with parameters $\tilde{P}$.

$\max[a, b]$ the maximum of a and b.

I a conformable identity matrix.

$\mathbf{1}$ conformable vector with all components equal to 1.

Abbreviations

CVaR Conditional Value-at-Risk.

LTCM Long-Term Capital Management.

MAD Mean absolute deviation.

PFO *Practical Financial Optimization: Decision making for financial engineers.*

SPDA Single Premium, Deferred Annuities.

VaR Value-at-Risk.

Chapter 5

Scenario Optimization

5.1 Preview

Scenario optimization provides powerful and flexible models for risk management in both equities and fixed income assets. In this chapter we develop models that tradeoff reward against risk, when both measures are computed from scenario data. The scenarios can be quite general describing market, credit, liquidity, actuarial and other types of risk. Fixed income, equities and derivative assets can be optimized in the same framework, and liabilities can also be incorporated. The relationships between several models are revealed.

5.2 Mean Absolute Deviation Models

We start with a model that trades off the mean absolute deviation measure of risk, see Definition 2.3.7, against the portfolio reward. When asset returns are given by a discrete and finite scenario set the model is formulated as a linear program. Large scale portfolios can be optimized using linear programming software that is widely available, robust, and efficient. We also know from Theorem 2.3 that, when returns are normally distributed, the variance and mean absolute deviation risk measures are equal, within a constant. In this case the solution of a mean absolute deviation model is equivalent to the solution of the Markowitz mean-variance model. Identical frontiers are generated by these two models to trade off risk and reward.

We define the optimization model using portfolio values $V(x; \tilde{P})$ instead of portfolio returns $R(x; \tilde{r})$. This allows us to model instruments in the portfolio with zero cost, such as futures. It is also consistent with the commonly used measure of Value-at-Risk which looks at portfolio value as opposed to returns. Note, however, the relationship $V(x; \tilde{P}) = R(x; \tilde{r})V_0$ between value and return, where V_0 is the initial value of the portfolio. A portfolio that consists of positions x_i with current prices P_{0i} has an initial value

$$V_0 = \sum_{i=1}^{n} P_{0i} x_i. \tag{5.1}$$

This is a *budget constraint* stipulating that the total holdings in the portfolio at current market prices are the initial endowment. The future value of the portfolio is given by

$$V(x; \tilde{P}) = \sum_{i=1}^{n} \tilde{P}_i x_i, \tag{5.2}$$

and its mean value is

$$V(x; \bar{P}) = \sum_{i=1}^{n} \bar{P}_i x_i. \tag{5.3}$$

The portfolio value is a linear function of the positions, with coefficients the asset prices. Similarly, portfolio return is a linear function with coefficients the asset returns, $R(x; \tilde{r}) = \sum_{i=1}^{n} \tilde{r}_i x_i$. (Note that the asset allocations x are in percentage of total wealth when calculating portfolio return, and in nominal amounts when calculating total portfolio value.)

In the scenario setting we have $V(x; P^l) = \sum_{i=1}^{n} P_i^l x_i$ and $R(x; r^l) = \sum_{i=1}^{n} r_i^l x_i$. For now we do not specify the precise time when the future value is realized. In single period models the investment horizon is one future time period. This is the *risk horizon* of the model, and is implicitly assumed to be equal to the time horizon T. A time index t for future prices and returns is used only for multi-period models.

Some constraints may be imposed on the asset allocation, of the form $x \in X$, where X denotes the set of feasible solutions. This constraint is specified by policy and regulatory considerations. Other constraints may include limits on the maximum holdings in each asset, dictated by liquidity considerations, or constraints imposed for diversification or tradeability of the portfolio. Such constraints were discussed in the context of the fixed income portfolios in Sections 4.2.4–4.2.5. A constraint that is always present in the models of this section is the budget constraint (5.1), and the non-negativity of the variables x, so that short sales are not allowed.

The following model trades off the portfolio mean absolute deviation against its expected value. It requires that the expected value of the portfolio exceeds μV_0 parametrized by the target return μ.

$$\text{Minimize} \quad \mathcal{E}\left[\left| V(x; \tilde{P}) - V(x; \bar{P}) \right| \right] \tag{5.4}$$

$$\text{subject to} \quad V(x; \bar{P}) \geq \mu V_0, \tag{5.5}$$

$$\sum_{i=1}^{n} P_{0i} x_i = V_0, \tag{5.6}$$

$$x \in X. \tag{5.7}$$

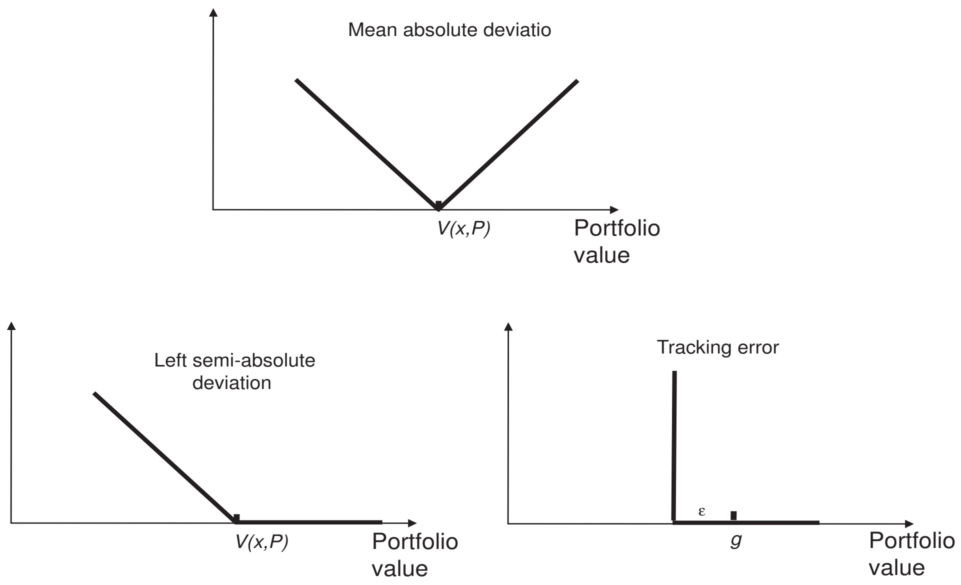

Figure 5.1: The mean absolute deviation function and its variants: the left semi-absolute deviation and function for the tracking model.

In the discrete scenario setting the objective function takes the form $\sum_{l\in\Omega} p^l \left|V(x;P^l) - V(x;\bar{P})\right|$, and we can write model (5.4)–(5.7) as:

$$\text{Minimize} \quad \sum_{l\in\Omega} p^l \left|V(x;P^l) - V(x;\bar{P})\right| \tag{5.8}$$

$$\text{subject to} \quad \sum_{i=1}^{n} \bar{P}_i x_i \geq \mu V_0, \tag{5.9}$$

$$\sum_{i=1}^{n} P_{0i} x_i = V_0, \tag{5.10}$$

$$x \in X. \tag{5.11}$$

The objective function is piecewise linear with slope 1 when $V(x;P^l) > V(x;\bar{P})$ and slope -1 when $V(x;P^l) < V(x;\bar{P})$, see Figure 5.1. Viewed along the axis x_i the slope of the objective function is P_i^l when $P_i^l > \bar{P}_i$, and $-P_i^l$ when $P_i^l < \bar{P}_i$. Since this function is not linear the solution of the model is not possible with linear programming. This difficulty can be overcome with a re-formulation of the model.

We introduce variables $\tilde{y}_+$ and $\tilde{y}_-$ to measure, respectively, the positive and negative deviations of the portfolio value from its mean, and write the deviation function as

$$V(x;\tilde{P}) - V(x;\bar{P}) = \tilde{y}_+ - \tilde{y}_- \tag{5.12}$$

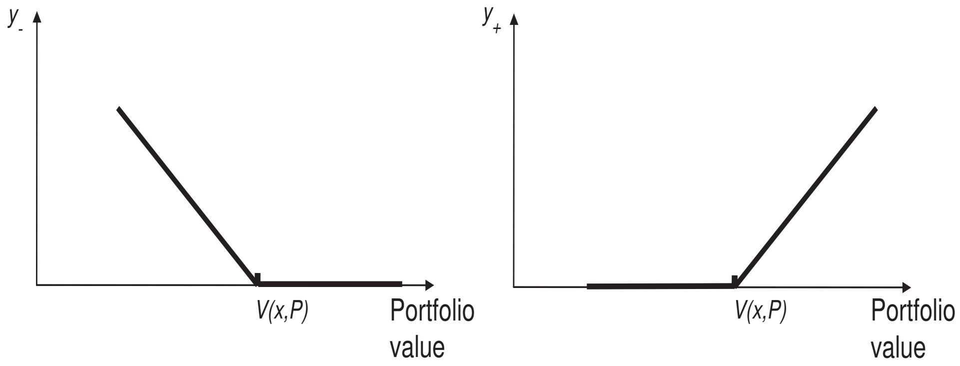

Figure 5.2: The functions $\tilde{y}_+$ and $\tilde{y}_-$ have the same payoff as a European call option and a short position in a European put option, respectively.

where

$$\tilde{y}_+ = \max[0, V(x;\tilde{P}) - V(x;\bar{P})], \tag{5.13}$$

$$\tilde{y}_- = \max[0, V(x;\bar{P}) - V(x;\tilde{P})]. \tag{5.14}$$

Figure 5.2 illustrates these two functions. It follows that $\tilde{y}_+$ is non-zero in those scenarios when the portfolio value exceeds its mean value. This variable measures the *upside* potential of the portfolio in outperforming its mean value. $\tilde{y}_-$ is non-zero when the portfolio value is less than its mean, and it measures the *downside* risk of the portfolio in under-performing its mean value. In the discrete scenario setting we express the definitions above as a system of inequalities:

$$y_+^l \geq V(x;P^l) - V(x;\bar{P}),\ y_+^l \geq 0, \text{ for all } l \in \Omega, \tag{5.15}$$

$$y_-^l \geq V(x;\bar{P}) - V(x;P^l),\ y_-^l \geq 0, \text{ for all } l \in \Omega. \tag{5.16}$$

The upside potential of the portfolio has the same payoff as a European call option with a strike price equal to the mean value of the portfolio. The maturity of the option is the risk-horizon of the model. Similarly, the downside risk of the portfolio is equivalent to a short position in a European put option.

Now we note that the absolute value $|a|$ of a real number a is the minimum value b, such that $b \geq a$ and $b \geq -a$. We define the *auxiliary* variable y^l such that

$$y^l \geq y_+^l, \text{ and } y^l \geq y_-^l, \text{ for all } \ l \in \Omega\ . \tag{5.17}$$

Hence, the objective function of the mean absolute deviation model can be written as $\sum_{l\in\Omega} p^l y^l$, and using the definitions (5.15)–(5.16) we obtain the term $\left|V(x;P^l) - V(x;\bar{P})\right|$ by solving

$$\text{Minimize} \quad y^l \tag{5.18}$$

$$\text{subject to} \quad y^l \geq V(x;P^l) - V(x;\bar{P}) \tag{5.19}$$

$$y^l \geq V(x;\bar{P}) - V(x;P^l) \tag{5.20}$$

$$x \in X. \tag{5.21}$$

Hence, using the auxiliary variable y^l we formulate the mean absolute deviation model above as the following linear program:

Model 5.2.1 Minimization of mean absolute deviation

$$\text{Minimize} \quad \sum_{l \in \Omega} p^l y^l \tag{5.22}$$

$$\text{subject to} \quad \sum_{i=1}^{n} \bar{P}_i x_i \geq \mu V_0, \tag{5.23}$$

$$y^l \geq V(x;P^l) - V(x;\bar{P}), \qquad \text{for all } l \in \Omega, \tag{5.24}$$

$$y^l \geq V(x;\bar{P}) - V(x;P^l), \qquad \text{for all } l \in \Omega, \tag{5.25}$$

$$\sum_{i=1}^{n} P_{0i} x_i = V_0, \tag{5.26}$$

$$x \in X. \tag{5.27}$$

The model is given below in the equivalent formulation for maximizing expected portfolio value with a limit on the risk.

Model 5.2.2 Portfolio optimization with absolute deviation constraints

$$\text{Maximize} \quad \sum_{i=1}^{n} \bar{P}_i x_i \tag{5.28}$$

$$\text{subject to} \quad \sum_{l \in \Omega} p^l y^l \leq \omega, \tag{5.29}$$

$$y^l \geq V(x;P^l) - V(x;\bar{P}), \qquad \text{for all } l \in \Omega, \tag{5.30}$$

$$y^l \geq V(x;\bar{P}) - V(x;P^l), \qquad \text{for all } l \in \Omega, \tag{5.31}$$

$$\sum_{i=1}^{n} P_{0i} x_i = V_0, \tag{5.32}$$

$$x \in X. \tag{5.33}$$

Varying the parameter ω we trace the frontier of mean absolute deviation versus expected value. This is identical to the frontier traced by varying the parameter μ in Model 5.2.1.

5.2.1 Semi-absolute deviation

It is possible to modify the mean absolute deviation models above to differentiate the penalty for upside potential from that for downside risk. Constraints (5.24) set the values of the auxiliary variable y^l equal to the upside potential, corresponding to scenarios in which y^l_+ takes nonnegative values, and y^l_- is zero. Similarly, constraints (5.25) set the values of the auxiliary variable equal to the downside risk corresponding to nonnegative y^l_- and zero y^l_+. Introducing nonnegative parameters λ_u and λ_d, scaled such such that $\lambda_u + \lambda_d = 1$, we differentially penalize upside potential from downside risk with the following modification of Model 5.2.1. Figure 5.1 illustrates the semi-absolute deviation function.

Model 5.2.3 Upside potential and downside risk in mean absolute deviation

$$\text{Minimize} \quad \sum_{l\in\Omega} p^l y^l \tag{5.34}$$

$$\text{subject to} \quad \sum_{i=1}^{n} \bar{P}_i x_i \geq \mu V_0, \tag{5.35}$$

$$y^l \geq \lambda_u \left(V(x; P^l) - V(x; \bar{P}) \right), \quad \text{for all } l \in \Omega, \tag{5.36}$$

$$y^l \geq \lambda_d \left(V(x; \bar{P}) - V(x; P^l) \right), \quad \text{for all } l \in \Omega, \tag{5.37}$$

$$\sum_{i=1}^{n} P_{0i} x_i = V_0, \tag{5.38}$$

$$x \in X. \tag{5.39}$$

In the limit we may set $\lambda_u = 0$ and $\lambda_d = 1$, in which case we obtain a model with only one half the constraints of Model 5.2.1. This model minimizes the expected downside risk only for a target expected value and no constraints on the upside potential; it minimizes the left semi-absolute deviation risk measure of Definition 2.3.8. We will see in the next section that the mean absolute deviation and the semi-absolute deviation models are equivalent. Hence, one can solve a simplified version of Model 5.2.3, eliminating either (5.36) or (5.37) by setting, respectively, λ_u or λ_d equal to zero, instead of solving Model 5.2.1.

5.2.2 Equivalence of mean absolute deviation with semi-absolute deviation

In this section we show that the mean absolute deviation risk measure is equivalent to both the right and left semi-absolute deviation measures. Consider a random variable $\tilde{r}$ with mean value $\bar{r}$, and let $\text{prob}(\tilde{r} = r) = p(r)$.

We have

$$\mathcal{E}(\tilde{r} - \bar{r}) = \int_{-\infty}^{+\infty} (r - \bar{r})p(r)dr$$
$$= \int_{-\infty}^{\bar{r}} (r - \bar{r})p(r)dr + \int_{\bar{r}}^{+\infty} (r - \bar{r})p(r)dr. \tag{5.40}$$

Using the property of random variables that

$$\mathcal{E}(\tilde{r} - \bar{r}) = \mathcal{E}(\tilde{r}) - \bar{r} = 0,$$

we obtain from (5.40)

$$-\int_{-\infty}^{\bar{r}} (r - \bar{r})p(r)dr = \int_{\bar{r}}^{+\infty} (r - \bar{r})p(r)dr. \tag{5.41}$$

Consider now the mean absolute deviation function

$$\mathcal{E}(\mid \tilde{r} - \bar{r} \mid) = \int_{-\infty}^{+\infty} \mid r - \bar{r} \mid p(r)dr \tag{5.42}$$
$$= \int_{-\infty}^{\bar{r}} -(r - \bar{r})p(r)dr + \int_{\bar{r}}^{+\infty} (r - \bar{r})p(r)dr \tag{5.43}$$
$$= 2\int_{\bar{r}}^{+\infty} (r - \bar{r})p(r)dr. \tag{5.44}$$

Equation (5.43) follows from (5.42) because of the properties of absolute value functions and the fact that

$$(r - \bar{r}) \leq 0 \text{ for all } r \in (-\infty, \bar{r}),$$
$$(r - \bar{r}) \geq 0 \text{ for all } r \in (\bar{r}, +\infty), \text{and}$$
$$p(r) \geq 0 \text{ for all } r.$$

Equation (5.44) follows from (5.43) because of (5.41). Similarly, we can show that

$$\mathcal{E}(\mid \tilde{r} - \bar{r} \mid) = -2\int_{-\infty}^{\bar{r}} (r - \bar{r})p(r)dr. \tag{5.45}$$

Therefore,

$$\frac{1}{2}\mathcal{E}(\mid \tilde{r} - \bar{r} \mid) = \int_{\bar{r}}^{+\infty} (r - \bar{r})p(r)dr$$
$$= \int_{-\infty}^{\bar{r}} -(r - \bar{r})p(r)dr,$$

and the semi-absolute deviation (right or left) are equivalent to the mean absolute deviation.

We note that the equivalence of the risk measures holds when the deviations are measured against the mean. When we measure deviations against some exogenous random target, such as a market index, no such equivalence property holds. Measuring deviations against a random target is addressed next.

5.2.3 Tracking model

We consider the semi-absolute deviation model with only downside risk, i.e., of Model 5.2.3 with $\lambda_u = 0$ and $\lambda_d = 1$. Instead of measuring risk using a linear function of the downside deviation, as was done in Model 5.2.3, the new model imposes an infinite penalty for any deviations that are more than a user specified parameter ϵV_0 below the mean. Deviations that are within ϵV_0 below the mean—or greater than the mean—do not contribute to the risk measure, see Figure 5.1. The model maximizes expected return while it restricts the downside deviations to remain within $-\epsilon V_0$ from the target mean value.

Model 5.2.4 Portfolio optimization with limits on maximum downside risk

$$\text{Maximize} \quad \sum_{i=1}^{n} \bar{P}_i x_i \tag{5.46}$$

$$V(x; P^l) \geq V(x; \bar{P}) - \epsilon V_0, \qquad \text{for all } l \in \Omega, \tag{5.47}$$

$$\sum_{i=1}^{n} P_{0i} x_i = V_0, \tag{5.48}$$

$$x \in X. \tag{5.49}$$

In this model downside risk is defined with respect to the portfolio mean $V(x; \bar{P})$. The model can be defined with respect to some other target random variable $\tilde{g}$. For instance, $\tilde{g}$ may be set equal to the value of an investment V_0 in some broadly defined market index or alternative investment opportunity—also called the *numeraire*—with growth rate $\tilde{I}$. Or $\tilde{g}$ may be set equal to a fixed value μV_0 in which case the target is the value we want the portfolio to achieve. When adopting an integrated financial product management perspective the target of the portfolio is the liability value.

Assuming that the random target $\tilde{g}$ takes values g^l under scenario $l \in \Omega$ we modify Model 5.2.4 into the following *tracking* model.

Model 5.2.5 Tracking model

$$\text{Maximize} \quad \sum_{i=1}^{n} \bar{P}_i x_i \tag{5.50}$$

$$\text{subject to} \quad V(x; P^l) \geq g^l - \epsilon V_0, \qquad \text{for all } l \in \Omega, \tag{5.51}$$

$$\sum_{i=1}^{n} P_{0i} x_i = V_0, \tag{5.52}$$

$$x \in X. \tag{5.53}$$

The term “tracking” indicates that the portfolio value stays with a margin $-\epsilon V_0$ of the target. Parameter ϵ is user specified with smaller values of ϵ leading to closer tracking but perhaps at the sacrifice of excess returns. The same tracking model can be obtained from the regret models of the next section.

5.3 Regret Models

We now turn to the minimization of the regret function of Definition 2.3.9. As in the rest of this chapter the model is defined using portfolio values $V(x;\tilde{P})$ instead of portfolio returns $R(x;\tilde{r})$. Letting $\tilde{g}$ be the random target value for the portfolio we write the regret function as

$$G(x;\tilde{P},\tilde{g}) = V(x;\tilde{P}) - \tilde{g}. \tag{5.54}$$

The regret function takes positive values when the portfolio outperforms the target and negative values when the portfolio underperforms. In the former case we have gains and in the later losses. Both gains and losses are measured *vis-à-vis* a target which serves as the benchmark.

Following the development of the previous section we introduce variables $\tilde{y}_+$ and $\tilde{y}_-$ to measure, respectively, the positive and negative deviations of the portfolio value from the target, and write the regret function as

$$V(x;\tilde{P}) - \tilde{g} = \tilde{y}_+ - \tilde{y}_-, \tag{5.55}$$

where

$$\tilde{y}_+ = \max[0, V(x;\tilde{P}) - \tilde{g}], \tag{5.56}$$

$$\tilde{y}_- = \max[0, \tilde{g} - V(x;\tilde{P})]. \tag{5.57}$$

It follows that $\tilde{y}_+$ is non-zero in those scenarios when the portfolio value exceeds the target. This variable measures the *upside regret* when the portfolio outperforms the target. $\tilde{y}_-$ is non-zero when the portfolio value is less than the target, and it measures the *downside regret.* With these definitions the complementarity condition is satisfied. In the discrete scenario setting we can express the definitions above as systems of inequalities:

$$y^l_+ \geq V(x;P^l) - g^l,\ y^l_+ \geq 0, \text{ for all } l \in \Omega, \tag{5.58}$$

$$y^l_- \geq V(x;g^l) - V(x;P^l),\ y^l_- \geq 0, \text{ for all } l \in \Omega. \tag{5.59}$$

It is customary to think of downside regret as the measure of risk, and of upside regret as a measure of the portfolio reward. In Section 5.6 we develop models that trace an efficient frontier of risk versus reward using the downside and upside regret, respectively.

The probability that regret does not exceed threshold value ζ is given by

$$\Psi(x;\zeta) = \sum_{\{l\in\Omega | G(x;P^l,g^l)\leq\zeta\}} p^l. \tag{5.60}$$

The *perfect regret* has a probability distribution function given by

$$\Psi(x^*;\zeta) = \begin{cases} 0 \text{ if } \zeta < 0 \\ 1 \text{ if } \zeta \geq 0. \end{cases} \tag{5.61}$$

It follows that $\Psi(x;\zeta) \leq \Psi(x^*;\zeta)$ and the distribution of perfect regret provides a useful criterion for comparing portfolios.

Consider now the minimization of expected downside regret, with the requirement that the expected value of the portfolio exceeds μV_0, and constraints such as those imposed when minimizing the mean absolute deviation risk function. We have the following model that trades off the risk measure of expected downside regret against expected value.

$$\text{Minimize} \quad \mathcal{E}[\tilde{y}_-] \tag{5.62}$$

$$\text{subject to} \quad \mathcal{E}[V(x;\bar{P})] \geq \mu V_0, \tag{5.63}$$

$$\sum_{i=1}^{n} P_{0i}x_i = V_0, \tag{5.64}$$

$$x \in X. \tag{5.65}$$

In the discrete scenario setting we use the definition for y_-^l in eqn. (5.59) and formulate the regret minimization problem as the following linear program.

Model 5.3.1 Minimization of expected downside regret

$$\text{Minimize} \quad \sum_{l\in\Omega} p^l y_-^l \tag{5.66}$$

$$\text{subject to} \quad \sum_{i=1}^{n} \bar{P}_i x_i \geq \mu V_0, \tag{5.67}$$

$$y_-^l \geq g^l - V(x;P^l), \qquad \text{for all } l \in \Omega, \tag{5.68}$$

$$y_-^l \geq 0, \qquad \text{for all } l \in \Omega, \tag{5.69}$$

$$\sum_{i=1}^{n} P_{0i}x_i = V_0, \tag{5.70}$$

$$x \in X. \tag{5.71}$$

Note the similarity between the mean absolute deviation Model 5.2.3 and the regret minimization model. In particular, if we set $\lambda_0 = 0$ and $\lambda_d = 1$ in Model 5.2.3 we obtain Model 5.3.1 when the random target is set equal to the portfolio expected value, i.e., $g^l = V(x;\bar{P})$, for all scenarios. The regret model, however, is modelled with an exogenously given target, whereas the mean absolute deviation model the target is endogenous, depending on the portfolio.

5.3.1 ϵ-regret models

Let us now relax the notion of downside regret to measure only those losses that exceed some threshold ϵV_0. When the portfolio value underperforms the target by an amount less than a nonnegative value ϵV_0 we consider downside regret as zero. For example, we may consider any underperformance that does not exceed 1% of the initial capital V_0 as acceptable. To model this setting we introduce the ϵ-regret function.

Definition 5.3.1 ϵ-regret function. *The difference between the target random variable $\tilde{g}$ shifted by $-\epsilon V_0$, and the portfolio value random variable is the ϵ-regret given by*

$$G_\epsilon(x; \tilde{P}, \tilde{g}) = V(x; \tilde{P}) - (\tilde{g} - \epsilon V_0), \tag{5.72}$$

and for the discrete scenario case by

$$G_\epsilon(x; P^l, g^l) = V(x; P^l) - (g^l - \epsilon V_0). \tag{5.73}$$

◇

Positive values indicate that the portfolio value outperforms the target or it underperforms by no more than ϵV_0. Negative values indicate that the portfolio underperforms the target by more than ϵV_0. The *perfect ϵ-regret* has a probability distribution function given by

$$\Psi_\epsilon(x^*; \zeta) = \begin{cases} 0 \text{ if } \zeta < -\epsilon V_0 \\ 1 \text{ if } \zeta \geq -\epsilon V_0. \end{cases} \tag{5.74}$$

The linear programming model for minimizing downside ϵ-regret is defined similarly to Model 5.3.1 as follows.

Model 5.3.2 Minimization of expected downside ϵ-regret

$$\text{Minimize} \quad \sum_{l \in \Omega} p^l y_-^l \tag{5.75}$$

$$\text{subject to} \quad \sum_{i=1}^{n} \bar{P}_i x_i \geq \mu V_0, \tag{5.76}$$

$$y_-^l \geq (g^l - \epsilon V_0) - V(x; P^l), \quad \text{for all } l \in \Omega, \tag{5.77}$$

$$y_-^l \geq 0, \quad \text{for all } l \in \Omega, \tag{5.78}$$

$$\sum_{i=1}^{n} P_{0i} x_i = V_0, \tag{5.79}$$

$$x \in X. \tag{5.80}$$

The regret models can be written in equivalent formulations that maximize expected value with constraints on the downside regret. For Model 5.3.2 this formulation is as follows.

Model 5.3.3 Portfolio optimization with ϵ-regret constraints

$$\text{Maximize} \quad \sum_{i=1}^{n} \bar{P}_i x_i \tag{5.81}$$

$$\text{subject to} \quad \sum_{l \in \Omega} p^l y_-^l \leq \omega, \tag{5.82}$$

$$y_-^l \geq (g^l - \epsilon V_0) - V(x; P^l), \qquad \text{for all } l \in \Omega, \tag{5.83}$$

$$y_-^l \geq 0, \qquad \text{for all } l \in \Omega, \tag{5.84}$$

$$\sum_{i=1}^{n} P_{0i} x_i = V_0, \tag{5.85}$$

$$x \in X. \tag{5.86}$$

We can now note an equivalence between the tracking Model 5.2.5 and ϵ-regret optimization. In particular if we set $\omega = 0$ in Model 5.3.3 we restrict the downside ϵ-regret to zero for all scenarios. In this case $y_-^l = 0$ and substituting for $y_-^l = 0$ in (5.83)–(5.84) we get (5.51). Hence, the tracking model can be obtained as a special case of optimization with ϵ-regret constraints.

5.4 Conditional Value-at-Risk Models

Consider now the minimization of Conditional Value-at-Risk (CVaR). In Definition 2.6.3 CVaR was given as the expected value of the losses, conditioned on the losses being in excess of VaR. Recall from eqn. (2.60) that the losses in portfolio value are given by the loss function

$$L(x; \tilde{P}) = V_0 - V(x; \tilde{P}). \tag{5.87}$$

Here we should note that positive values of the loss function correspond to downside risk, while negative values of the loss function correspond to gains. Negative values measure the upside potential. This convention is at odds with the convention of using negative values to measure downside risk in the mean absolute deviation models, the regret models, and the put/call efficient frontier models. This difference in notation is due to historical reasons as the literature on CVaR has its roots in the calculation of expected shortfall in insurance, where losses are the focus in studying insurance risk and are by convention positive. The literature on mean absolute deviation, regret, and related models has its origins in asset management, where gains are the focus of study, and gains are by convention positive.

If ζ is the VaR at the $100\alpha\%$ probability level, then the Conditional Value-at-Risk is given by the expression:

$$\text{CVaR}(x;\alpha) = \mathcal{E}[L(x;P^l) \mid L(x;P^l) > \zeta], \tag{5.88}$$

and in the discrete scenario setting we have

$$\text{CVaR}(x;\alpha)\frac{\sum_{\{l\in\Omega|L(x;P^l)>\zeta\}} p^l L(x;P^l)}{\sum_{\{l\in\Omega|L(x;P^l)>\zeta\}} p^l}, \tag{5.89}$$

Here ζ is the α-VaR,and its value depends on α. Under a technical condition that the probability of scenarios with losses strictly greater than ζ is exactly equal to $1-\alpha$, i.e., $\Psi(x;\zeta) = \alpha$ (cf. eqn. 2.62) we have

$$\text{CVaR}(x;\alpha) = \frac{\sum_{\{l\in\Omega|L(x;P^l)\geq\zeta\}} p^l L(x;P^l)}{1-\alpha}. \tag{5.90}$$

We will see that this function can be optimized using a linear programming model. The formulation of a linear model is facilitated with the use of auxiliary variables similar to those used in the mean absolute deviation, tracking and regret models. Let

$$\tilde{y}_+ = \max\left[0, L(x;\tilde{P}) - \zeta\right]. \tag{5.91}$$

$\tilde{y}_+$ is equal to zero when the losses are less than the Value-at-Risk, ζ, and it is equal to the excess loss when the losses exceed ζ.

In the discrete scenario setting we have

$$y^l_+ = \max\left[0, L(x;P^l) - \zeta\right], \text{ for all } \ l \in \Omega\ . \tag{5.92}$$

Using this definition of y^l_+ we write:

$$\begin{aligned}
\sum_{l\in\Omega} p^l y^l_+ &= \sum_{\{l\in\Omega|L(x;P^l)<\zeta\}} p^l y^l_+ + \sum_{\{l\in\Omega|L(x;P^l)\geq\zeta\}} p^l y^l_+ \\
&= 0 + \sum_{\{l\in\Omega|L(x;P^l)\geq\zeta\}} p^l \left(L(x;P^l) - \zeta\right) \\
&= \sum_{\{l\in\Omega|L(x;P^l)\geq\zeta\}} p^l L(x;P^l) - \zeta \sum_{\{l\in\Omega|L(x;P^l)\geq\zeta\}} p^l \\
&= \sum_{\{l\in\Omega|L(x;P^l)\geq\zeta\}} p^l L(x;P^l) - \zeta(1-\alpha)
\end{aligned}$$

Dividing both sides by $(1-\alpha)$ and rearranging terms we get

$$\zeta + \frac{\sum_{l\in\Omega} p^l y^l_+}{1-\alpha} = \frac{\sum_{\{l\in\Omega|L(x;P^l)\geq\zeta\}} p^l L(x;P^l)}{1-\alpha}. \tag{5.93}$$

The term on the right is CVaR$(x;\alpha)$ (eqn. 5.90). It can be optimized using linear programming to minimize the linear function on the left.

We minimize CVaR subject to the condition that the expected value of the portfolio exceeds some target μV_0, and the constraints imposed on the scenario models of previous sections. The model trades off the risk measure CVaR against expected value. It is written as

$$\begin{aligned} \text{Minimize} \quad & \text{CVaR}(x;\alpha) && (5.94)\\ \text{subject to} \quad & \mathcal{E}[V(x;\tilde{P})] \geq \mu V_0, && (5.95)\\ & \sum_{i=1}^{n} P_{0i}x_i = V_0, && (5.96)\\ & x \in X. && (5.97) \end{aligned}$$

Using the definition of CVaR from (5.93) we write this model as follows.

Model 5.4.1 Minimization of CVaR

$$\begin{aligned} \text{Minimize} \quad & \zeta + \frac{\sum_{l\in\Omega} p^l y_+^l}{1-\alpha} && (5.98)\\ \text{subject to} \quad & \sum_{i=1}^{n} \bar{P}_i x_i \geq \mu V_0, && (5.99)\\ & y_+^l \geq L(x;P^l) - \zeta, \quad \text{for all } l \in \Omega, && (5.100)\\ & y_+^l \geq 0, \quad \text{for all } l \in \Omega, && (5.101)\\ & \sum_{i=1}^{n} P_{0i}x_i = V_0, && (5.102)\\ & x \in X. && (5.103) \end{aligned}$$

When the loss function $L(x;P^l)$ is linear the model is a linear program. This is the case with the loss function (2.60) which in the scenario setting is

$$\begin{aligned} L(x;P^l) &= V_0 - V(x;P^l) && (5.104)\\ &= \sum_{i=1}^{n} (P_{0i} - P_i^l)x_i. && (5.105) \end{aligned}$$

Solution of this model gives us the minimum CVaR, and the VaR value ζ^* corresponding to the minimum CVaR portfolio. (Recall that CVaR$\geq$VaR.)

A frontier trading off expected shortfall against expected value can be traced by varying the parameter μ. The frontier can also be traced using an equivalent formulation to Model 5.4.1 that maximizes expected portfolio value subject to a constraint on CVaR.

Model 5.4.2 Portfolio optimization with CVaR constraints

$$\text{Maximize} \quad \sum_{i=1}^{n} \bar{P}_i x_i \tag{5.106}$$

$$\text{subject to} \quad \zeta + \frac{\sum_{l\in\Omega} p^l y_+^l}{1-\alpha} \le \omega, \tag{5.107}$$

$$y_+^l \ge L(x; P^l) - \zeta, \qquad \text{for all } l \in \Omega, \tag{5.108}$$

$$y_+^l \ge 0, \qquad \text{for all } l \in \Omega, \tag{5.109}$$

$$\sum_{i=1}^{n} P_{0i} x_i = V_0, \tag{5.110}$$

$$x \in X. \tag{5.111}$$

5.5 Expected Utility Maximization

The scenario optimization modes have thus far—following the same line of development as Chapter 2—optimized a pre-specified measure of risk against a pre-specified measure of reward, such as expected return. However, investor are restricted in their choices to the selection of a model and to the setting of a target return. The expected utility maximization model we formulate in this section allows users to optimize according to their own criteria when trading off risk and rewards. In particular, while the models of the previous sections trace frontiers of efficient portfolios—from which the user has to select one—the expected utility maximization model allows the user to select a unique portfolio that optimizes the user's utility function.

Using the notation introduced earlier we define the expected utility maximization model using portfolio return, where return is a linear function with coefficients the asset returns, $R(x;\tilde{r}) = \sum_{i=1}^{n} \tilde{r}_i x_i$. Recall that the asset allocations x are in percentage of total wealth when calculating portfolio return. In the scenario setting we have $R(x;r^l) = \sum_{i=1}^{n} r_i^l x_i$, and the model is written as follows:

Model 5.5.1 Expected Utility Maximization

$$\text{Maximize} \quad \sum_{l\in\Omega} p^l \mathcal{U}(R(x;r^l)) \tag{5.112}$$

$$\text{subject to} \quad R(x;r^l) = \sum_{i=1}^{n} r_i^l x_i, \tag{5.113}$$

$$\sum_{i=1}^{n} x_i = 1, \tag{5.114}$$

$$x \in X. \tag{5.115}$$

5.6 Put/Call Efficient Frontiers

The tracking and regret Models 5.2.5 and 5.3.1 measure risk as a piece-wise linear function of the portfolio deviations from a random target. The models trade off risk against the portfolio reward which is given by the expected value of the portfolio. The expected portfolio value is calculated independently of the random target. This view is inconsistent with enterprise wide risk management, since the reward of the portfolio must be measured *vis-à-vis* the random target. For instance, if the portfolio value is always equal to a target liability then the portfolio has neither risk nor reward. Reward is manifested when portfolio values exceed the target, and risk is manifested when portfolio values are below the target. In this section we formulate a model that trades off the portfolio downside (risk) against the portfolio upside (reward) taking into account the random target. By explicitly incorporating the liability in measuring both risk and reward the model takes an integrated view of the financial intermediation process.

The upside potential has identical payoffs to a call option on the future portfolio value relative to the target. When the portfolio value is below the target there is zero upside potential, and the call option is out-of-the-money. When the portfolio value exceeds the target the upside potential is precisely the payoff of a call that is in-the-money. Similarly, the downside payoffs are identical to those of a short position in a put option on the future portfolio value relative to the target. The portfolio *call value* is the expected upside and the *put value* is the expected downside. Portfolios that achieve the higher call value for a given put value are called *put/call efficient*. Alternatively, we define a portfolio as put/call efficient if it achieves the lowest put value for a given call value. The models in this section generate put/call efficient frontiers.

The deviations of the portfolio value from the random target $\tilde{g}$ are expressed using variable $\tilde{y}_+$ and $\tilde{y}_-$ as

$$V(x;\tilde{P}) - \tilde{g} = \tilde{y}_+ - \tilde{y}_-, \tag{5.116}$$

where

$$\tilde{y}_+ = \max[0, V(x;\tilde{P}) - \tilde{g}], \tag{5.117}$$

$$\tilde{y}_- = \max[0, \tilde{g} - V(x;\tilde{P})]. \tag{5.118}$$

This is the regret function of Section 5.3 from which now we make an interesting observation. $\tilde{y}_+$ measures the upside potential of the portfolio to outperform the target. $\tilde{y}_-$ measures the downside risk and has the same payoff as a short position in a European put option.

We can now separate the upside potential of the portfolio from the downside risk. The following model traces the put/call efficient frontier for put values parametrized by ω.

$$\text{Maximize} \quad \mathcal{E}\,[\tilde{y}_+] \tag{5.119}$$

$$\text{subject to} \quad \mathcal{E}[\tilde{y}_-] \leq \omega, \tag{5.120}$$

$$\sum_{i=1}^{n} P_{0i} x_i = V_0, \tag{5.121}$$

$$x \in X. \tag{5.122}$$

We start with a linear programming formulation for this model without any constraints of the form $x \in X$. We will see that the put/call efficient frontier in the unconstrained case is a straight line through the origin with slope at least one. Liquidity constraints will then be added to the model and liquidity premia can be calculated from the optimal solution.

5.6.1 Put/Call efficient frontiers without constraints

We formulate a linear program for tracing the efficient frontier in model (5.119)–(5.122). The linear expression for the portfolio value is written explicitly as

$$V(x;P^l) = \sum_{i=1}^{n} P_i^l x_i. \tag{5.123}$$

The budget constraint $\sum_{i=1}^{n} P_{0i} x_i = V_0$ is now eliminated in order to formulate an unconstrained problem, taking also $X = \mathbb{R}^n$. Let I^l be the total return of the benchmark portfolio, also called the *numeraire*. Then our random target is $g^l = V^0 I^l$ and substituting for V^0 from the budget constraint $V^0 = \sum_{i=1}^{n} P_{0i} x_i$ we get

$$g^l = \sum_{i=1}^{n} P_{0i} I^l x_i. \tag{5.124}$$

Hence, equation (5.116) can be expressed in the following discrete form which implicitly includes the budget constraint

$$\sum_{i=1}^{n} P_i^l x_i - \sum_{i=1}^{n} P_{0i} I^l x_i = y_+^l - y_-^l \quad , \text{ for all } l \in \Omega. \tag{5.125}$$

This is a tracking equation that measures the deviations of the portfolio from the benchmark, and it is imposed as a constraint in model (5.119)–(5.122) which is formulated as follows.

Model 5.6.1 Put/Call efficient portfolio

$$\text{Maximize} \quad \sum_{l \in \Omega} p^l y_+^l \tag{5.126}$$

$$\text{subject to} \quad \sum_{l \in \Omega} p^l y_-^l \le \omega, \tag{5.127}$$

$$y_+^l - y_-^l - \sum_{i=1}^{n} (P_i^l - P_{0i} I^l) x_i = 0, \qquad \text{for all } l \in \Omega, \tag{5.128}$$

$$y_+^l, y_-^l \ge 0, \qquad \text{for all } l \in \Omega. \tag{5.129}$$

We define now the following dual prices for this problem.

π_ω: the dual price for the expected downside constraint (5.127),

π^l: the dual price associated with the tracking constraint (5.128) in scenario $l \in \Omega$.

The dual price associated with a constraint measures the change in the objective value per unit change of the right hand side of the constraint. The dual price π_ω represents the marginal tradeoff between expected upside and expected downside. If the allowable expected downside ω increases by a small amount $\epsilon > 0$ the expected upside will increase by $\pi_\omega \epsilon$. It follows that the slope of the put/call efficient frontier traced by varying ω is a straight line with slope

$$\pi_\omega = \frac{\sum_{l \in \Omega} p^l y_+^l}{\sum_{l \in \Omega} p^l y_-^l}. \tag{5.130}$$

For $\omega = 0$ we have the trivial solution $x = 0$, and the efficient frontier goes through the origin. The efficient frontier is a straight line through the origin with slope π_ω. In the unconstrained case the dual price remains constant as we increase the allowable downside ω. We can tradeoff unlimited expected upside for unlimited expected downside at a marginal rate π_ω. It will be established next that the slope of the frontier is at least 1, which means that a put/call efficient portfolio will generate at least as much upside potential as it has downside risk.

In the constrained case the frontier may become piecewise linear. A new line segment starts when the dual price π_ω changes as ω increases. The dual price will decrease with increasing ω thus producing a concave frontier. This non-increasing property of dual prices is a standard property of linear programs. In the context of the financial application it has an intuitive explanation. The most attractive securities—those that have the highest upside for a given downside—are used first. The rate of increase of upside potential (call value) per unit of downside risk (put value) diminishes, or it remains constant if no constraint, such as liquidity, restricts the trading of the most attractive security in the portfolio.

Dual problem for put/call efficient frontiers

Additional insights about the problem are gained from the *dual* formulation of the *primal* linear program in Model 5.6.1. When the primal problem has a finite solution it is identical to the solution of the dual, which is formulated using the duality theory from Appendix A as follows:

Model 5.6.2 Dual problem for put/call efficient portfolio

$$\text{Minimize} \quad \omega\pi_\omega \tag{5.131}$$

$$\text{subject to} \quad \sum_{l\in\Omega}(P_i^l - P_{0i}I^l)\pi^l = 0, \qquad \text{for all } i \in U, \tag{5.132}$$

$$p^l\pi_\omega - \pi^l \geq 0, \qquad \text{for all } l \in \Omega, \tag{5.133}$$

$$\pi^l \geq p^l, \qquad \text{for all } l \in \Omega, \tag{5.134}$$

$$\pi_\omega \geq 0. \tag{5.135}$$

From (5.133)–(5.134) we have $p^l \leq \pi^l \leq \pi_\omega p^l$ and it follows that $\pi_\omega \geq 1$. That is, each additional unit of downside results in at least one unit of upside, and the call value of the portfolio is at least as large as its put value. Since the dual price of the expected downside constraint (5.127) is positive this constraint is always active at optimality. All allowable downside is used, as expected, since the objective function is to maximize the upside potential.

Constraint (5.132) is a tracking constraint in the dual space defined for each security. It is analogous to the tracking constraint (5.128) defined in the primal problem for each scenario. We write the upside and downside, respectively, of each security in scenario l as

$$z_{+i}^l = \max[0, P_i^l - P_{0i}I^l] \tag{5.136}$$

$$z_{-i}^l = \max[0, P_{0i}I^l - P_i^l]. \tag{5.137}$$

These quantities measure the upside potential and the downside risk we assume by investing one unit in a security i instead of selling the security at the current market price P_{0i} and investing the proceeds in the alternative investment. With these definitions we write

$$P_i^l - P_{0i}I^l = z_{+i}^l - z_{-i}^l. \tag{5.138}$$

In the special case when $\pi_\omega = 1$ we get $\pi^l = p^l$. It follows from (5.132) that

$$\sum_{l\in\Omega}(P_i^l - P_{0i}I^l)p^l = 0, \text{ for all } i \in U, \tag{5.139}$$

and substituting (5.138) and rearranging we get

$$\frac{\sum_{l\in\Omega} p^l z_{+i}^l}{\sum_{l\in\Omega} p^l z_{-i}^l} = 1, \text{ for all } i \in U. \tag{5.140}$$

This relation shows that a unit of upside is achieved for each additional unit of downside. The actual values of x are irrelevant since securities equally trade off expected upside for expected downside. Scenario probabilities that satisfy $p^l = \pi^l$ are *put/call neutral*. Under these probabilities the securities are neutral in terms of their contribution to the expected upside (call value) and to the expected downside (put value).

Benchmark neutral probabilities

When the solution to the model does not satisfy $\pi_\omega = 1$ it is possible to recover a set of put/call neutral probabilities from the dual prices π^l. We normalize the dual prices by $D_f = \sum_{l\in\Omega}\pi^l$ to obtain

$$\psi^l = \frac{\pi^l}{D_f}. \tag{5.141}$$

D_f is called the *risk-free discount factor* for reasons that are explained in Section 5.7. Now, $\sum_{l\in\Omega}\psi^l = 1$ and the normalized dual prices can be interpreted as probabilities. Scaling equation (5.132) by D_f, substituting in this (5.138), and rearranging we obtain

$$\frac{\sum_{l\in\Omega} \psi^l z_{+i}^l}{\sum_{l\in\Omega} \psi^l z_{-i}^l} = 1, \text{ for all } i \in U. \tag{5.142}$$

Hence, the expected upside of each security computed with probabilities ψ^l is equal to the expected downside. Exactly as we had in the special case of $\pi_\omega = 1$, a unit of expected upside is achieved for each unit of expected downside, when expectations are computed with respect to the probabilities ψ^l. Under these probabilities the securities are put/call neutral. The probabilities obtained by scaling the dual prices of the put/call efficient portfolio model are called *put/call neutral* or *benchmark neutral*.

Infinite liquidity benchmark neutral prices

Further insights can be obtained by writing the dual tracking constraint (5.132) as

$$P_{0i} \sum_{l \in \Omega} \pi^l I^l = \sum_{l \in \Omega} \pi^l P_i^l. \tag{5.143}$$

Dividing throughout by D_f and rearranging we get

$$P_{0i} = \frac{\sum_{l \in \Omega} \psi^l P_i^l}{\sum_{l \in \Omega} \psi^l I^l}. \tag{5.144}$$

Under infinite liquidity the price of security i is the expected payoff discounted at the rate $\sum_{l \in \Omega} \psi^l I^l$. This is the infinite liquidity benchmark-neutral price of the security. In the case of a complete market under infinite liquidity the benchmark is the future value of the portfolio and $\tilde{I} = 1$. The equation above gives the risk neutral price of the security.

5.6.2 Put/call efficient frontiers with finite liquidity

We now develop put/call efficient portfolios in the presence of constraints. In particular finite liquidity restricts the amount we can invest in a security without affecting its price. Large investments will incur a liquidity premium. The put/call efficient portfolios will allow us to price the liquidity premia. Finite liquidity bounds the maximum amount that can be invested in i by $\bar{x}_i$ and the minimum amount by $\underline{x}_i$.

The bounds can be used to limit the holdings in tranches of a security trading at a given price. For instance, x_i could be holdings in a 10-year corporate bond trading at market price P_{0i} for amounts up to $\bar{x}_i$, and $x_{i'}$ holdings in the same 10-year corporate that trades at a higher price $P_{i'0}$ for amounts up to $\bar{x}_{i'}$. The higher price reflects a liquidity premium. An investor that finds the 10-year corporate attractive to hold in order to meet the portfolio targets will first buy as much as possible in the cheaper tranche i before purchasing additional amounts in the more expensive tranch i'.

The primal model for put/call efficient portfolios with finite liquidity is an extension of the unconstrained Model 5.6.1 to include bounds on the holdings.

Model 5.6.3 Put/Call efficient portfolio with finite liquidity

$$\text{Maximize} \quad \sum_{l\in\Omega} p^l y_+^l \tag{5.145}$$

$$\text{subject to} \quad \sum_{l\in\Omega} p^l y_-^l \leq \omega, \tag{5.146}$$

$$y_+^l - y_-^l - \sum_{i=1}^{n}(P_i^l - P_{0i}I^l)x_i = 0, \quad \text{for all } l \in \Omega, \tag{5.147}$$

$$-x_i \leq -\underline{x}_i, \quad \text{for all } i \in U, \tag{5.148}$$

$$x_i \leq \bar{x}_i, \quad \text{for all } i \in U, \tag{5.149}$$

$$y_+^l, y_-^l \geq 0, \quad \text{for all } l \in \Omega. \tag{5.150}$$

We define now dual prices for the liquidity constraints.

$\underline{\pi}_i$**:** the dual price for the lower bound constraints (5.148),

$\bar{\pi}_i$**:** the dual price for the upper bound constraints (5.149).

The dual problem is the following linear program.

Model 5.6.4 Dual problem for put/call efficient portfolio with finite liquidity

$$\text{Minimize} \quad \omega\pi_\omega - \sum_{i=1}^{n} \underline{x}_i\underline{\pi}_i + \sum_{i=1}^{n} \bar{x}_i\bar{\pi}_i \tag{5.151}$$

$$\text{subject to} \quad \underline{\pi}_i - \bar{\pi}_i + \sum_{l\in\Omega}(P_i^l - P_{0i}I^l)\pi^l = 0, \quad \text{for all } i \in U, \tag{5.152}$$

$$p^l\pi_\omega - \pi^l \geq 0, \quad \text{for all } l \in \Omega, \tag{5.153}$$

$$\pi^l \geq p^l, \quad \text{for all } l \in \Omega, \tag{5.154}$$

$$\pi_\omega \geq 0 \quad \text{and} \quad \underline{\pi}_i, \bar{\pi}_i \geq 0, \quad \text{for all } i \in U. \tag{5.155}$$

The dual problem always has a feasible solution obtained by setting $\pi^l = p^l$, $\pi_\omega = 1$, and

$$\bar{\pi}_i = \max\left[0, \sum_{l\in\Omega}(P_i^l - P_{0i}I^l)\pi^l\right] \tag{5.156}$$

$$\underline{\pi}_i = \max\left[0, \sum_{l\in\Omega}(P_{0i}I^l - P_i^l)\pi^l\right]. \tag{5.157}$$

Feasibility of the dual implies that the primal solution is bounded. It is therefore not possible to trade infinite expected upside for a given expected downside since the amount traded in the most attractive securities is bounded.

Can we obtain a put/call neutrality condition for the finite liquidity case similar to condition (5.140) for the unconstrained case? Consider the dual constraint

$$\underline{\pi}_i - \bar{\pi}_i + \sum_{l \in \Omega} (P_i^l - P_{0i} I^l) \pi^l = 0, \quad \text{for all } i \in U, \tag{5.158}$$

For the special case of $\pi_\omega = 1$ and substituting in this equation $\pi^l = p^l$ and the definitions of z_{+i}^l and z_{-i}^l we get

$$\sum_{l \in \Omega} p^l z_{+i}^l - \sum_{l \in \Omega} p^l z_{-i}^l = \bar{\pi}_i - \underline{\pi}_i. \tag{5.159}$$

Only securities that have $\bar{\pi}_i = \underline{\pi}_i = 0$ are put/call neutral, trading equal expected upside for equal expected downside. We know from linear programming theory that the dual prices are zero when the corresponding constraints are inactive. Hence, put/call neutral securities are those that trade strictly between their bounds. (In degenerate cases a security may trade at one of its bounds and the corresponding dual price can still be zero.) For securities that are at the bounds—and assuming non-degenerate cases—the corresponding dual price will be nonnegative. The expected upside will increase if the active bound is relaxed and the dual prices for the bounds reflect the cost of liquidity. In particular $\underline{\pi}_i$ reflects a liquidity premium and $\bar{\pi}_i$ a liquidity discount.

Put/call neutral valuation of liquidity

We can now use the analysis of both the constrained and unconstrained models to estimate the cost of liquidity. Normalize the dual prices π^l by $D_f = \sum_{l \in \Omega} \pi^l$ to obtain

$$\psi^l = \frac{\pi^l}{D_f}, \tag{5.160}$$

and scale equation (5.159) by D_f to get

$$\sum_{l \in \Omega} \psi^l z_{+i}^l - \sum_{l \in \Omega} \psi^l z_{-i}^l = \frac{\bar{\pi}_i}{D_f} - \frac{\underline{\pi}_i}{D_f}. \tag{5.161}$$

Let us define the auxiliary variable $z_i^l = z_{+i}^l - z_{-i}^l$, which is the deviation of the security i with respect to the benchmark in scenario l. (To be precise this is a gain with respect to the benchmark when $z_i^l > 0$ and a loss when $z_i^l < 0$.) The expected deviation of security i under the probabilities ψ^l is given by

$$\mathcal{E}_\lambda[z_i^l] = \frac{\bar{\pi}_i}{D_f} - \frac{\underline{\pi}_i}{D_f}. \tag{5.162}$$

Securities that trade strictly between their bounds have $\bar{\pi}_i = \underline{\pi}_i = 0$ and their expected deviation is zero. These securities are put/call neutral with respect to the probabilities ψ^l. When a security trades at the upper bound $x_i = \bar{x}_i$ we have $\underline{\pi}_i = 0$, $\bar{\pi}_i \geq 0$ and $\mathcal{E}_\lambda[z_i^l] > 0$. (Equality holds in degenerate cases.) The security contributes more to the upside than the downside and, therefore, its current price is at a discount.

We can estimate the liquidity discount as follows. For each security that trades at the upper bound set $\underline{\pi}_i = 0$ in (5.158), scale by D_f and rearrange to get

$$P_{0i} = \frac{\sum_{l\in\Omega} \psi^l P_i^l}{\sum_{l\in\Omega} \psi^l I^l} - \frac{\bar{\pi}_i}{\sum_{l\in\Omega} \pi^l I^l}. \tag{5.163}$$

Recall from equation (5.144) that $\frac{\sum_{l\in\Omega} \psi^l P_i^l}{\sum_{l\in\Omega} p^l I^l}$ is the infinite liquidity price of the security, and therefore the term $\frac{\bar{\pi}_i}{\sum_{l\in\Omega} \pi^l I^l}$ is the liquidity discount.

Similarly we obtain the following pricing equation for for securities trading at their lower bound

$$P_{0i} = \frac{\sum_{l\in\Omega} \psi^l P_i^l}{\sum_{l\in\Omega} \psi^l I^l} + \frac{\underline{\pi}_i}{\sum_{l\in\Omega} \pi^l I^l}. \tag{5.164}$$

The term $\frac{\underline{\pi}_i}{\sum_{l\in\Omega} \pi^l I^l}$ is the liquidity premium.

5.7 Asset Valuation using Scenario Optimization

We have seen in the previous section that dual prices from a portfolio optimization model allow us to estimate liquidity premia. In this section we will see that dual prices can be used for the valuation of new securities. Under assumptions of *complete markets*, as defined below, the optimization model and its dual prices are not essential in deriving security prices. However, in incomplete markets there is no unique price for securities, and prices depend on assumptions we make about investor preferences towards risk. In this setting an optimization formulation that trades of the reward from holding a security against its risks is essential. We discuss the use of scenario optimization models for the valuation of securities in both complete and incomplete markets.

Definition 5.7.1 Complete markets. *When the number of independent securities n is equal to the number of states N reached by the scenarios then the market is said to be complete.* ◇

An elementary form of independent securities are the so called Arrow-Dubreu securities that pay 1 in state l, and 0 in all other states. Arrow-Dubreu securities are indexed by l, denoting the unique dependence of each security's payoff matrix to the states.

An arbitrary security i is characterized by its payoff vector $(P_i^l)_{l=1}^N$ denoting the payments made by this security in each state $l \in \Omega$. P_i^l is known as *state-* or *scenario-dependent* price, as it is the price of security in state l. In the case of complete markets, the payoff of any arbitrary security can be replicated by a portfolio of Arrow-Dubreu securities. In particular, it is easy to see that a portfolio consisting of holdings P_i^l in the lth Arrow-Dubreu, security for all $l \in \Omega$, will replicate the payoff vector of the security. This observation plays an important role in the valuation of the security: in the absence of arbitrage the ith security should have the same price as the portfolio of Arrow-Dubreu securities. This observation is made precise next, and the absence of arbitrage is the the only requirement for pricing securities in complete markets.

5.7.1 Optimization models of arbitrage

Consider securities $i = 1, 2, \ldots, n$, with market prices P_{0i} and payoff vectors $(P_i^l)_{l=1}^N$. Arbitrageurs act by solving a linear programming model to create portfolios of minimal cost with non-negative payoff for all possible future states at the end of the horizon. The following single-period stochastic model applies:

$$\text{Minimize} \quad \sum_{i=1}^n P_{0i} x_i \tag{5.165}$$

$$\text{subject to} \quad \sum_{i=1}^n P_i^l x_i \geq 0, \qquad \text{for all} \quad l \in \Omega\ , \tag{5.166}$$

$$x \ \text{unrestricted.} \tag{5.167}$$

The solution of this model determines whether arbitrage exists, or if the prices are at equilibrium. We distinguish the following three cases.

Market equilibrium

Under market equilibrium condition no arbitrage opportunities for are available, and the portfolio x^* created by the model has zero cost today, and neither payoff nor any obligation at any state at the end of the horizon. The solution of the model is bounded, and it satisfies

$$\sum_{i=1}^n P_{01} x_i^* = 0, \tag{5.168}$$

$$\sum_{i=1}^n P_i^l x_i^* = 0, \ \text{for all} \quad l \in \Omega\ . \tag{5.169}$$

First-order arbitrage

Under first-order arbitrage it is possible to extract cash from the market today, without creating any future obligations. The portfolio created by the model has an unbounded solution, while the constraints are feasible. The solution of the arbitrageur's problem satisfies:

$$\sum_{i=1}^{n} P_{01} x_i^* = -\infty, \tag{5.170}$$

$$\sum_{i=1}^{n} P_i^l x_i^* \geq 0, \text{ for all } \; l \in \Omega \; . \tag{5.171}$$

Second-order arbitrage

Under second-order arbitrage it is possible to create a portfolio of zero cost today, without any future obligations and with a non-zero probability of positive payoff at some future state. The portfolio created by the model satisfies:

$$\sum_{i=1}^{n} P_{01} x_i^* = 0, \tag{5.172}$$

$$\sum_{i=1}^{n} P_i^l x_i^* > 0, \text{ for some } l \in \Omega. \tag{5.173}$$

In the absence of first-order arbitrage the arbitrageur's linear program is bounded. Hence, its dual problem is feasible (see Appendix A). If π^l denotes the dual variable for the lth equality constraint in (5.166), then feasibility of the dual implies the following system of equations:

$$\sum_{l \in \Omega} P_i^l \pi^l = P_{0i}, \text{ for all } \; i = 1, 2, \ldots, n. \tag{5.174}$$

Furthermore, to eliminate second-order arbitrage the constraints (5.166) must be satisfied with equality. Hence the dual prices must be strictly positive, i.e.,

$$\pi^l > 0, \text{ for all } \; l \in \Omega \; . \tag{5.175}$$

Solving the system (5.174)–(5.175) we obtain a set of dual prices that are consistent with the absence of arbitrage. When the number of independent securities n is equal to the number of states N this system has a unique solution. The solution vector $\pi = (\pi^l)_{l=1}^N$ that satisfies this system is called the *state price* vector.

The standard interpretation of dual prices of linear programming will allow us to use the state price vector to price securities with arbitrary payoffs. For example if the arbitrageur wishes to receive one extra unit of payment at state l then the objective function will increase by π^l. Hence, the cost of an Arrow-Dubreu security that pays one unit at state l and zero otherwise is π^l.

5.7.2 Valuation in complete markets

In a complete market we can identify as many independent securities as there are states of the economy. Solving then the arbitrageur's linear program, or its dual, we obtain a unique state price vector. This vector gives us the prices of Arrow-Dubreu securities. We can then replicate the state-dependent payoff of any security using a linear combination of Arrow-Dubreau securities, and the security's value is equal to the total price of this replicating portfolio. With this approach we can price securities that generate any arbitrary state-dependent cashflow stream.

For instance, we can estimate the price of a risk-free security that pays 1 unit in each state. This security must be worth the price of a portfolio holding one of each Arrow-Dubreu security, i.e., $D_f = \sum_{l\in\Omega} \pi^l$. Equivalently we can calculate the risk-free rate of return associated with the time horizon of the linear program by:

$$1 + r_f = \frac{1}{\sum_{l\in\Omega} \pi^l}. \tag{5.176}$$

Note that $D_f = \frac{1}{1+r_f}$ and this is the *risk-free discount factor*.

Similarly, the price of a security i that pays P_i^l in the lth state is equal to the price of a portfolio holding P_i^l in the lth Arrow-Dubreu security that pays 1 in the lth state and zero otherwise. Hence, we have the following pricing equation:

$$P_{0i} = \sum_{l\in\Omega} P_i^l \pi^l. \tag{5.177}$$

Risk-neutral valuation

We note that the probabilities of the states, p^l, have not been used in our analysis. The price of an arbitrary payoff can be obtained without the need to know the expected payoff. This, of course, has been a consequence of the assumption about market completeness. Under this assumption the distribution of the payoff is not modelled explicitly, which also means that the investors preferences towards risk are irrelevant.

Our ability to do away with any assumptions on risk preferences leads to what is known as *risk-neutral valuation*. Unfortunately, this is a misnomer. It does not mean that investors are risk neutral, typically they are risk averse. What it means is that valuation can take place in a way that is neutral to investor risk attitudes.

It turns out, however, that we can create a world of probabilities of the states in which investors are indeed risk neutral. We denote these probabilities by ψ^l for each state l. In this hypothetical world investors will discount the expected value of the payoffs of an arbitrary security by the risk-free rate to get the security's price:

$$P_{0i} = \frac{1}{1+r_f}\left(\sum_{l\in\Omega}\psi^l P_i^l\right). \tag{5.178}$$

We can now verify that the above equation is true when ψ^l is given by

$$\psi^l = \frac{\pi^l}{\sum_{l\in\Omega}\pi^l} = \frac{\pi^l}{D_f}, \tag{5.179}$$

by substituting this value of ψ^l in (5.178), and recalling that $D_f = 1/(1+r_f)$ to obtain

$$D_f\left(\sum_{l\in\Omega}\frac{\pi^l}{D_f}P_i^l\right) = \sum_{l\in\Omega}\pi^l P_i^l = P_{0i}. \tag{5.180}$$

Here, the last equality follows from (5.177).

In conclusion, the risk-neutral valuation of a security in a complete market is the expected value of its payoff—calculated using the risk-neutral probabilities ψ^l—discounted at the risk-free rate. In this next sections we see how far this analysis can take us when the markets are incomplete.

5.7.3 Valuation in dynamically complete markets

When the number of states exceeds the number of independent securities it is not possible to use the arguments of the previous section. The arbitrageur's dual program has more variables than constraints and, hence, it does not have a unique solution. The dual prices are not uniquely determined, and it is not possible to determine the price of an arbitrary cashflow using the prices of Arrow-Debreu securities. Something must be done to ensure uniqueness of the Arrow-Debreu prices. Introducing more independent securities or reducing the number of states will work.

In the absence of more independent securities we can reduce the number of states by refining the time-step. If, for instance, we have two independent securities but three possible states at the end of the horizon T, we may halve the time step so that two states are possible during the first half of the horizon T_1, and an additional two states are possible from the end of the first half to the end of the horizon T_2. At each time step and for each state we introduce contingent Arrow-Debreu securities associated with each of the possible states at the next time step; the payout of the ith contingent security is denoted by P_{ti}^l. A contingent Arrow-Debreu security is not observed today in the current state, but in a future uncertain state. However, once the future state is realized then the security payoffs are known with certainty and the market is complete during the next time step.

This process is illustrated in Figure 5.3. At each intermediate step there are only two possible states, and the market is complete over each time

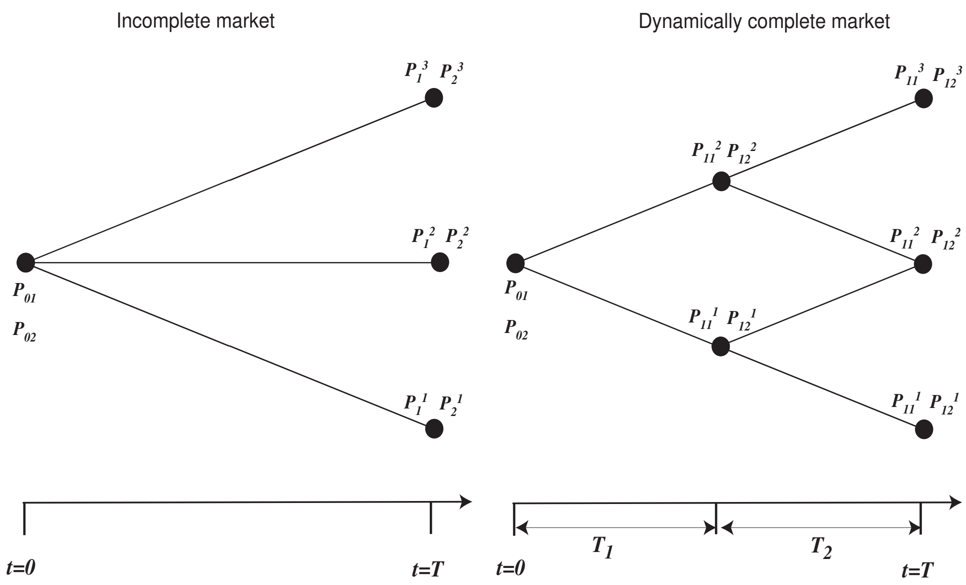

Figure 5.3: A three-state single-step model that needs three independent securities to be complete, can be made dynamically complete when refined into two-step model with two states at each step.

step as the number of states at each step does not exceed the number of independent securities. The market is *dynamically complete* over the time horizon T as it can be made complete at each time step. (Note that since we have three states at the end of the horizon, the states at T_2 recombine. However, this does not have to be always the case although the number of states of a non-recombining tree will grow exponentially with the number of time steps. Such growth is computationally intractable.)

5.7.4 Valuation in incomplete markets

We now turn to markets that can not be made dynamically complete. Such is the case when market imperfections—transaction costs, differential taxation regimes for different investors, portfolio constraints—do not allow us to replicate arbitrary payoff structures. In this setting we will assume some risk preference on the part of the investor and seek prices that leave the investor indifferent between holding the security or holding a market benchmark. The prices so obtained are not uniquely determined for all investors in the market. But they are *utility invariant* in that the utility value of the investor does not change for the given price. In a sense these are the worst possible prices for an investor, for if the price of a security would decrease her utility function she would then sell the security, while if the price of the security will increase the utility function than she would buy. Utility invariant prices are all called *benchmark neutral* in the sense that the investor is neutral between holding the benchmark or holding the security.

We set up now the utility maximization models that will allow us to obtain utility invariant prices for new securities. We first develop the primal and dual programs for an investor that selects portfolio from a set of priced securities to maximize a utility function. We then develop the programs for selecting portfolios from a universe of priced securities augmented by a new security whose price is not known. Finally, we develop the conditions on the price of the new security under which its inclusion in the portfolio will not change the utility function, and this will give the utility invariant price.

With the standard notation we have a portfolio of value $V(x;\tilde{P}) = \sum_{i=1}^{n} \tilde{P}_i x_i$ and a benchmark portfolio, such as for instance a market index, whose value is the random variable $\tilde{g}$. The current prices of the securities are given by $(P_{0i})_{i=1}^{n}$ and of the index by V_0. We will consider only portfolios created by an initial endowment V_0, so that the amount V_0 is either used to buy one unit of the benchmark, or to invest in traded securities so that $\sum_{i=1}^{n} P_{0i} x_i \leq V_0$.

We assume that the number of securities n is less than the number of states N, so that the markets are incomplete. Furthermore, portfolio constraints of the form

$$\sum_{i=1}^{n} a_{ji} x_i \leq b_j, \text{ for all } \ j = 1, 2, \ldots, K \tag{5.181}$$

create market imperfections.

In the discrete scenario setting the state-dependent portfolio value is given by

$$V(x; P^l) = \sum_{i=1}^{n} P_i^l x_i, \tag{5.182}$$

and the benchmark values is given by g^l. The deviations of the portfolio value from the benchmark are expressed using variable y_+^l and y_-^l as

$$V(x; P^l) - g^l = y_+^l - y_-^l, \tag{5.183}$$

where we define

$$y_+^l = \max[0, V(x; P^l) - g^l], \text{ for all } \ l \in \Omega \ , \tag{5.184}$$

$$y_-^l = \max[0, g^l - V(x; P^l), \text{ for all } \ l \in \Omega \ . \tag{5.185}$$

With these definitions the risk of the portfolio can be measured using Definition 2.3.10 for the expected downside as $\sum_{l\in\Omega} p^l y_-^l$. The reward of the portfolio can be measured using Definition 2.8.2 for the expected upside as $\sum_{l\in\Omega} p^l y_+^l$. Assuming a bi-linear utility function for our investor (see Definition 2.8.8) we can write down the portfolio selection problem as the following linear program.

$$\text{Maximize} \quad \sum_{l \in \Omega} p^l y_+^l - \lambda \sum_{l \in \Omega} p^l y_-^l \tag{5.186}$$

$$\text{subject to} \quad \sum_{i=1}^{n} P_i^l x_i - y_+^l + y_-^l = g^l, \qquad \text{for all } \ l \in \Omega \ , \tag{5.187}$$

$$\sum_{i=1}^{n} P_{0i} x_i \leq V_0, \tag{5.188}$$

$$\sum_{i=1}^{n} a_{ji} x_i \leq b_j, \qquad \text{for all } \ j = 1, 2, \ldots, K, \tag{5.189}$$

$$y_+^l, y_-^l \geq 0, \qquad \text{for all } \ l \in \Omega \ . \tag{5.190}$$

To write down the dual of this linear program we associate dual variables π^l, α and θ_j with constraints (5.187), (5.188), and (5.189), respectively. The dual program is now written as follows (see Appendix A):

$$\text{Minimize} \quad \sum_{l \in \Omega} g^l \pi^l + V_0 \alpha + \sum_{j=1}^{K} b_j \theta_j \tag{5.191}$$

$$\text{subject to} \quad \sum_{l \in \Omega} P_i^l \pi^l + P_{0i} \alpha + \sum_{j=1}^{K} a_{ji} \theta_j = 0, \qquad \text{for all } \ i = 1, 2, \ldots, n, \tag{5.192}$$

$$-\pi^l \geq p^l, \qquad \text{for all } \ l \in \Omega \ , \tag{5.193}$$

$$\pi^l \geq -\lambda p^l, \qquad \text{for all } \ l \in \Omega \ , \tag{5.194}$$

$$\alpha \geq 0, \theta_j \geq 0, \qquad \text{for all } \ j = 1, 2, \ldots, K. \tag{5.195}$$

In the absence of portfolio constraints (5.189)we can drop the term $\sum_{j=1}^{K} a_{ij}\theta_j$ from equation (5.192) and solve for P_{0i} to get

$$P_{0i} = -\frac{\sum_{l \in \Omega} P_i^l \pi^l}{\alpha}. \tag{5.196}$$

Comparing this with the pricing equation for complete markets (5.177) we note that the state price vector in a market with budget constraints is given by

$$\psi^l = \frac{\pi^l}{\alpha}. \tag{5.197}$$

In the absence of the budget constraint the corresponding dual variable α is dropped, and we recover the state price vector for Arrow-Dubreu securities in complete markets.

The effect of the portfolio constraints can be analyzed by solving equation (5.192) for P_{0i} to obtain

$$P_{0i} = -\frac{\sum_{l \in \Omega} P_i^l \pi^l}{\alpha} - \frac{\sum_{j=1}^{K} a_{ij}\theta_j}{\alpha}. \tag{5.198}$$

The first term on the right gives us the price of the security in markets with only budget constraints. The second term is an adjustment factor due to the portfolio trading constraints.

We consider now the pricing of a new security i' with payoff vector $(P_{i'}^l)_{l=1}^N$. This instrument will be traded in the portfolio if it does not violate the portfolio trading constraints. Indeed, assuming that the constraints will be inactive for this particular security, then the second term in the equation above will be zero. This suggests that the price of the new security can be estimated by

$$P_{0i'} = -\sum_{l\in\Omega} P_{i'}^l \psi^l. \tag{5.199}$$

We show next that this price is indeed utility invariant. That is, investors who are offered the security at this price are indifferent about adding it in their portfolio. If the security is offered at a higher price investors will sell, while if it is offered at a lower price they will buy. In either case the investor's utility will improve, but the investor's trading will affect the price. At the above price investors will not trade and therefore their actions will not affect the security price. This is an equilibrium price, alas it is an equilibrium price for those investors that have a given level of risk aversion denoted by the value of the parameter λ. We make these statements precise next.

5.7.5 Utility invariant pricing

We consider the primal problem of arbitrageurs that are offered a security i' with payoff vector $(P_{i'}^l)_{l=1}^N$ at a price $P_{0i'}$, in addition to the usual set of securities U. We denote holdings in the new security by the unrestricted (free) variable $x_{i'}$ and formulate the following linear program:

$$\text{Maximize} \quad \sum_{l\in\Omega} p^l y_+^l - \lambda \sum_{l\in\Omega} p^l y_-^l \tag{5.200}$$

$$\text{subject to} \quad \sum_{i=1}^n P_i^l x_i + P_{i'}^l x_{i'} - y_+^l + y_-^l = g^l, \qquad \text{for all } \ l \in \Omega\ , \tag{5.201}$$

$$\sum_{i=1}^n P_{0i} x_i + P_{0i'} x_{i'} \le V_0, \tag{5.202}$$

$$\sum_{i=1}^n a_{ji} x_i \le b_j, \qquad \text{for all } \ j = 1, 2, \ldots, K, \tag{5.203}$$

$$y_+^l, y_-^l \ge 0, \qquad \text{for all } \ l \in \Omega\ . \tag{5.204}$$

If the optimal solution to this program is equal to the optimal solution of program (5.186)–(5.190), which was formulated without security i', then the new security plays no role in the arbitrageur's portfolio choice. If this were the case the investor is indifferent to the new security. Under what conditions are then the solutions to (5.200)–(5.204) and (5.186)–(5.190) identical? The answer to this question can be obtained using duality theory.

The dual to the linear program (5.200)–(5.204) is given by:

$$\text{Minimize} \quad \sum_{l \in \Omega} g^l \pi^l + V_0 \alpha + \sum_{j=1}^{K} b_j \theta_j \tag{5.205}$$

$$\text{subject to} \quad \sum_{l \in \Omega} P_i^l \pi^l + P_{0i} \alpha + \sum_{j=1}^{K} a_{ji} \theta_j = 0, \qquad \text{for all } \ i = 1, 2, \ldots, n, \tag{5.206}$$

$$\sum_{l \in \Omega} P_{i'}^l \pi^l + P_{0i'} \alpha = 0, \qquad \text{for all } \ i = 1, 2, \ldots, n, \tag{5.207}$$

$$-\pi^l \geq p^l, \qquad \text{for all } \ l \in \Omega \ , \tag{5.208}$$

$$\pi^l \geq -\lambda p^l, \qquad \text{for all } \ l \in \Omega \ , \tag{5.209}$$

$$\alpha, \theta \geq 0. \qquad . \tag{5.210}$$

The solution to the primal problem and its dual satisfy the following Kuhn-Tucker optimality conditions (see Appendinx A):

$$\sum_{l \in \Omega} p^l y_+^l - \lambda \sum_{l \in \Omega} p^l y_-^l = \sum_{l \in \Omega} g^l \pi^l + V_0 \alpha + \sum_{j=1}^{K} b_j \theta_j \tag{5.211}$$

$$\sum_{i=1}^{n} P_i^l x_i + P_{i'}^l x_{i'} - y_+^l + y_-^l = g^l \tag{5.212}$$

$$\sum_{i=1}^{n} P_{0i} x_i + P_{0i'} x_{i'} \leq V_0 \tag{5.213}$$

$$\sum_{i=1}^{n} a_{ji} x_i \leq b_j, \text{for all } \ j = 1, 2, \ldots, K \tag{5.214}$$

$$\sum_{l \in \Omega} P_i^l \pi^l + P_{0i} \alpha + \sum_{j=1}^{\kappa} a_{ij} \theta_j = 0, \text{for all } \ i = 1, 2, \ldots, n \tag{5.215}$$

$$\sum_{l \in \Omega} P_{i'}^l \pi^l + P_{0i'} \alpha = 0, \text{for all } \ i = 1, 2, \ldots, n \tag{5.216}$$

$$p^l \leq -\pi^l \leq \lambda p^l \tag{5.217}$$

$$y_+^l, y_-^l \geq 0, \ \text{for all } \ l \in \Omega \tag{5.218}$$

$$\alpha \geq 0, \theta_j \geq 0, \ \text{for all } \ j = 1, 2, \ldots, K. \tag{5.219}$$

Similarly the solution to the primal problem (5.186)–(5.190) and its dual satisfy the following Kuhn-Tucker conditions:

$$\sum_{l\in\Omega} p^l y_+^l - \lambda \sum_{l\in\Omega} p^l y_-^l = \sum_{l\in\Omega} g^l \pi^l + V_0\alpha + \sum_{j=1}^{K} b_j\theta_j \tag{5.220}$$

$$\sum_{i=1}^{n} P_i^l x_i - y_+^l + y_-^l = g^l \tag{5.221}$$

$$\sum_{i=1}^{n} P_{0i} x_i \leq V_0 \tag{5.222}$$

$$\sum_{j=1}^{K} a_{ji} x_i \leq b_j, \text{for all} \;\; j = 1, 2, \ldots, K \tag{5.223}$$

$$\sum_{l\in\Omega} P_i^l \pi^l + P_{0i}\alpha + \sum_{j=1}^{K} a_{ji}\theta_j = 0, \text{for all} \;\; i = 1, 2, \ldots, n \tag{5.224}$$

$$p^l \leq -\pi^l \leq \lambda p^l \tag{5.225}$$

$$y_+^l, y_-^l \geq 0, \text{ for all} \;\; l \in \Omega \;, \tag{5.226}$$

$$\alpha \geq 0, \theta_j \geq 0, \text{ for all} \;\; j = 1, 2, \ldots, K. \tag{5.227}$$

A solution x^* that solves (5.220)–(5.227) is also a solution to (5.211)–(5.219) with $x_{i'}^* = 0$ if eqn. (5.216) is satisfied for any values of α and π^l. In order for this equation to hold we need the following condition, which establishes the utility invariant price of the new security:

$$P_{0i'} = -\frac{\sum_{l\in\Omega} P_{i'}^l \pi^l}{\alpha}. \tag{5.228}$$

5.8 Postview

This chapter develops portfolio optimization models when the uncertainty of security prices or returns is captured through a set of scenarios. A linear program for minimizing the mean absolute deviation of portfolio returns is given first, followed by tracking models that consider only downside risk against a random target which may represent, for instance, a random liability or a market index.

It then develops linear programs for minimizing the regret risk measure, and develops a special case for minimizing only downside regret, and a generalization that minimizes downside regret only when losses exceed a threshold. It is worth noting that for scenarios obtained when the asset return follow a multivariate normal distribution the mean absolute deviation model is equivalent to the mean-variance portfolio optimization. The regret optimization model is a generalization of mean absolute deviation optimization with a random target, and it is further generalized to ϵ-regret optimization from which we obtain the tracking model.

The linear programming model for optimizing the coherent risk measure of conditional Value-at-Risk, and the nonlinear program for expected utility maximization are given next.

Finally the chapter develops the linear programs for generating put/call efficient portfolios, and uses duality theory to obtain some interest results about the optimal portfolios generated from these models. As an outcome of this analysis we can use the linear programming models for the valuation of assets in both complete and incomplete markets.

5.9 Notes and References

Scenario optimization in the form presented in this chapter has emerged in the 1990s as a viable approach for risk management with complex financial products. For the early references on scenario optimization see Dembo (1991), Zenios (1993), Mulvey and Zenios (1994a,1994b), and for the use of scenario optimization in practice see Dembo et al. (2000). However, these models can be viewed as special and simpler cases of the dynamic stochastic programming models that were introduced for portfolio management much earlier; see the Notes and References of Chapter 6.

For the mean-absolute deviation model see Konno and Yamazaki (1991) and for the semi-absolute deviation and the tracking models see Zenios and Kang (1993) and Worzel, Vassiadou-Zeniou and Zenios (1994). For applications to fixed income securities see Zenios (1993), Zenios (1995).

The equivalence of mean absolute deviation models with semi-absolute deviation was established, in a more general form than the one presented here, by Kenyon, Savage and Ball (1999).

The regret minimization model in the framework of scenario optimization for portfolio management was first introduced by Dembo (1991) and further analyzed by Dembo and King (1992) who also introduced ϵ-regret. Dembo (1993) used the regret model to develop portfolio immunization strategies. The ϵ-regret optimization model was developed by Mauser and Rosen (1999) who applied it to study portfolios with credit risk.

For the optimization of conditional Value-at-Risk see Rockafellar and Uryasev (2000), Uryasev (2000), Palmquist, Uryasev and Krokhmal (1999), Andersson, Mauser, Rosen and Uryasev (2000) and Pflug (2000).

The expected utility maximization literature is vast; see Williams (1936), Kelly (1956), Breiman (1960, 1961), Mossin (1968), Samuelson (1971,1977) Hakkansson (1970) McLean, Blazenko and Ziemba (1992), Kallberg and Ziemba (1984), Hakansson and Ziemba (1995), or the book by Ingersoll (1987).

The Put/call efficient frontier model was developed by Dembo and Mausser (2000). Asset valuation using scenario optimization in complete markets is discussed in Aziz (1998a), and in dynamically completed markets in Aziz(1998b). The development is based on earlier work by Ross (1976) and its extensions by Dybvig and Ross (1986) and Prisman (1986). Aziz (1999, 2000), Dembo, Rosen and Saunders (2000), and Saunders (2002) discuss valuation in incomplete markets.