AR(1) $r_t = \mu + \alpha r_{t-1} + \varepsilon_t \quad \varepsilon_t \overset{iid}{\sim} (0, \sigma^2)$

$\rho_k = \alpha^k \,, \quad \rho_k^* = \begin{cases} \rho_1 = \alpha & k = 1 \\ 0 & k \geq 2 \end{cases}$

$|\alpha| < 1$ STATIONARITY COND.

$E(r_t) = \dfrac{\mu}{1-\alpha} \,, \; V(r_t) = \dfrac{\sigma^2}{1-\alpha^2}$

$r_t = \mu + \varepsilon_t - \vartheta \varepsilon_{t-1}$ MA(1) $\quad \varepsilon_t \overset{iid}{\sim} (0, \sigma^2)$

MOVING AVERAGE.

$E(r_t) = \mu + \underbrace{E(\varepsilon_t)}_{0} - \vartheta \underbrace{E(\varepsilon_{t-1})}_{0} = \mu$

$V(r_t) = V(\mu + \varepsilon_t - \vartheta \varepsilon_{t-1}) = V(\varepsilon_t - \vartheta \varepsilon_{t-1}) =$

$$= V(\varepsilon_t) + V(-\vartheta\varepsilon_{t-1}) + 2\,Cov(\varepsilon_t, -\vartheta\varepsilon_{t-1}) =$$

$$= \sigma^2 + \vartheta^2\sigma^2 - 2\vartheta \underbrace{Cov(\varepsilon_t, \varepsilon_{t-1})}_{=0}$$

$$\Rightarrow V(r_t) = \sigma^2(1+\vartheta^2)$$

$$r_t = \mu + \varepsilon_t - \vartheta\varepsilon_{t-1} \Rightarrow$$

$$\Rightarrow Cov(r_t, r_{t-k}) = Cov(\mu + \varepsilon_t - \vartheta\varepsilon_{t-1}, r_{t-k})$$

$$= Cov(\varepsilon_t - \vartheta\varepsilon_{t-1}, r_{t-k}) = \overset{=0}{Cov(\varepsilon_t, r_{t-k})}$$

$$-\vartheta\,Cov(\varepsilon_{t-1}, r_{t-k}) = -\vartheta\,Cov(\varepsilon_{t-1}, r_{t-k})$$

$$\Rightarrow \gamma_k = -\theta \, cov(\varepsilon_{t-1}, r_{t-k})$$

$$k=1 \quad \gamma_1 = -\theta \, cov(\varepsilon_{t-1}, r_{t-1})$$

$$\Rightarrow \gamma_1 = -\theta \, cov(\varepsilon_{t-1}, \mu + \varepsilon_{t-1} - \theta \varepsilon_{t-2})$$

$$\Rightarrow \gamma_1 = -\theta \, \underbrace{cov(\varepsilon_{t-1}, \varepsilon_{t-1})}_{= V(\varepsilon_{t-1})} + \theta^2 \, \overbrace{cov(\varepsilon_{t-1}, \varepsilon_{t-2})}^{=0}$$

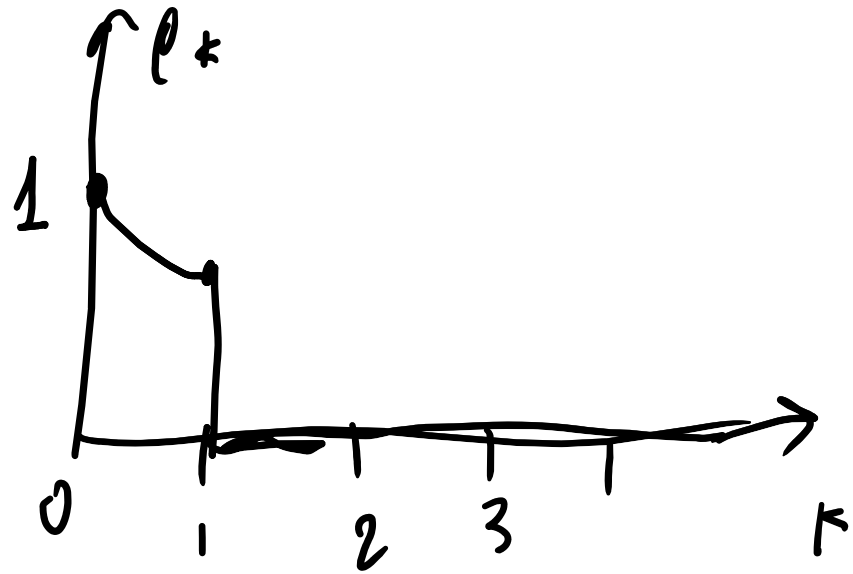
$$\Rightarrow \gamma_1 = -\theta \sigma^2$$

$$k \geq 2 \quad cov(\varepsilon_{t-1}, r_{t-k}) = 0 \quad \Rightarrow \gamma_k = 0$$

$$\gamma_k = \begin{cases} (1+\theta^2)\sigma^2 & k=0 \\ -\theta\sigma^2 & k=1 \\ 0 & k \geq 2 \end{cases}$$

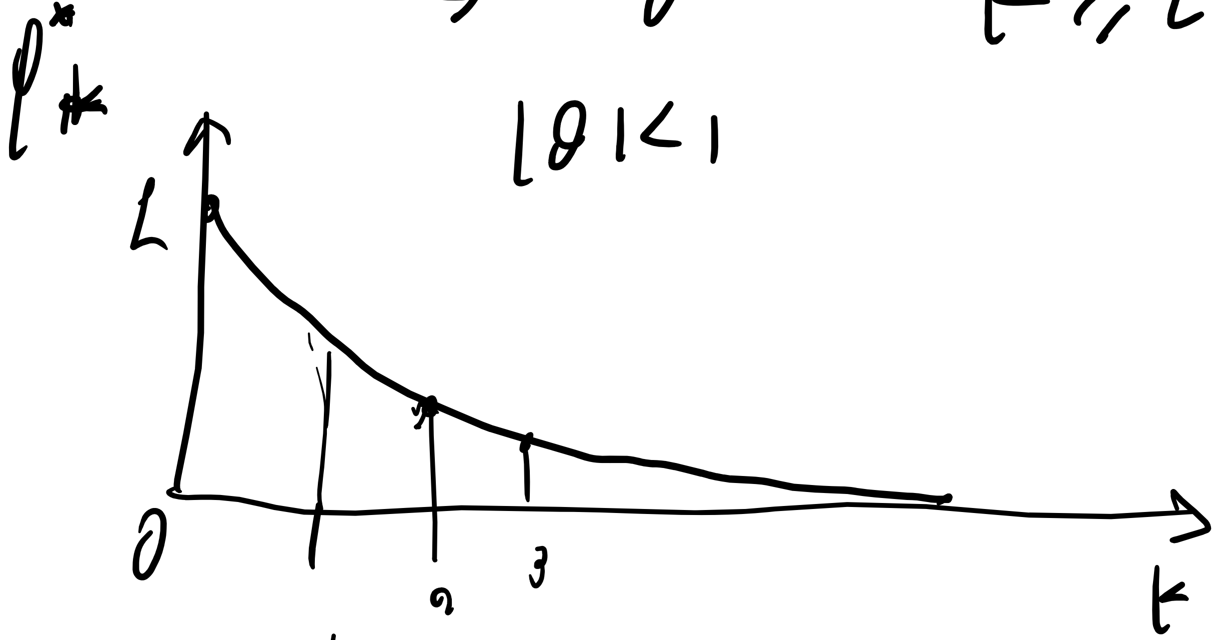
$$\rho_1 = -\frac{\theta\sigma^2}{(1+\theta^2)\sigma^2} = -\frac{\theta}{1+\theta^2}$$

$$\rho_k = 0 \qquad k \geq 2$$

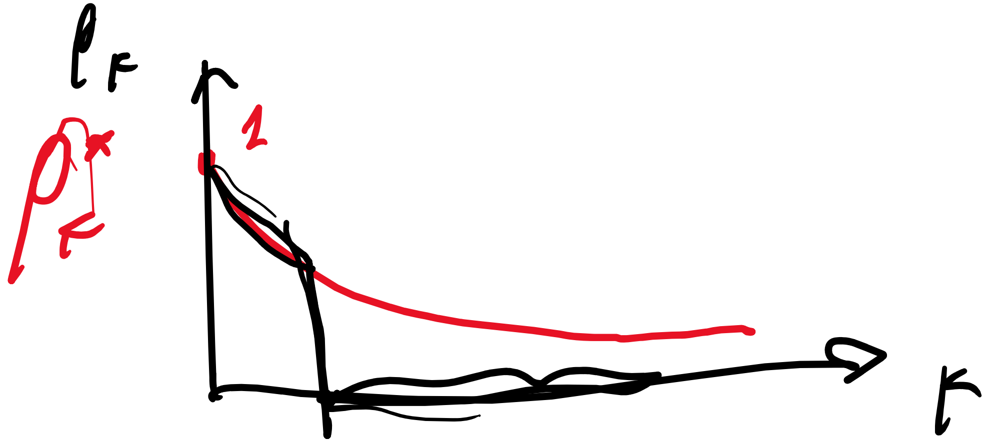


$$\rho^*_k = \begin{cases} \rho_1 = -\dfrac{\theta}{1+\theta^2} & k=1 \\ \sim \theta^k & k \geq 2 \end{cases}$$

$|\theta| < 1$



MA(1)



MA(1) STATIONARY $\forall \theta \in \mathbb{R}$

NO STATIONARITY RESTRICTION.

$$r_t = \mu + \varepsilon_t - \theta \varepsilon_{t-1}$$

$$\varepsilon_t = r_t - \mu + \theta \varepsilon_{t-1}$$

$$\varepsilon_{t-1} = r_{t-1} - \mu + \theta \varepsilon_{t-2}$$

$\Rightarrow$

$$\varepsilon_t = r_t - \mu + \theta r_{t-1} - \theta \mu + \theta^2 \varepsilon_{t-2}$$

$$\varepsilon_t = -\mu - \theta \mu - \theta^2 \mu - \dots + r_t + \theta r_{t-1} + \theta^2 r_{t-2} + \dots$$

$$\underline{r_t} = \mu + \theta \mu + \theta^2 \mu + \dots - \theta r_{t-1} - \theta^2 r_{t-2} - \dots + \varepsilon_t$$

r AR(∞) IFF $|\theta| < 1$

MA(1) $\rightarrow$ AR(∞) INVERTIBILITY IFF $|\theta| < 1$

MA(K)

$$r_t = \mu + \varepsilon_t - \theta_1 \varepsilon_{t-1} - \theta_2 \varepsilon_{t-2} - \dots - \theta_K \varepsilon_{t-K}$$

$$= \mu + \varepsilon_t (1 - \theta_1 L - \theta_2 L^2 - \dots - \theta_K L^K)$$

ALWAYS STATIONARY

INVERTIBLE IFF ROOTS OF are outside UNIT CYRCLE.

ARMA(1,1)

$$r_t = \mu + \alpha r_{t-1} + \varepsilon_t - \theta \varepsilon_{t-1}$$

STATIONARY IFF $|\alpha| < 1$

INVERTIBILITY CONDITIONS $|\alpha| < 1$ AND $|\theta| < 1$

$$\rho_k = \rho_1 \alpha^{k-1}$$

$$\rho_k^* = \begin{cases} \rho_1 & k = 1 \\ \sim \theta^k & k \geq 2 \end{cases}$$

$$\gamma_k = \operatorname{cov}(r_t, r_{t-k}) =$$

$$= \operatorname{cov}(\mu + \alpha r_{t-1} + \varepsilon_t - \theta \varepsilon_{(t-1)}, \ r_{t-k})$$

$$\Rightarrow \gamma_k = \alpha \operatorname{cov}(r_{t-1}, r_{t-k}) + \underbrace{\operatorname{cov}(\varepsilon_t, r_{t-k})}_{=0} - \theta \operatorname{cov}(\varepsilon_{t-1}, r_{t-k})$$

$$\Rightarrow \gamma_k = \alpha \gamma_{k-1} - \theta \underbrace{\operatorname{cov}(\varepsilon_{t-1}, r_{t-k})}_{=0}$$

$$k \geq 2 \quad \Rightarrow \boxed{\gamma_k = \alpha \gamma_{k-1}}$$

$k = 1$

$$\gamma_1 = \alpha \gamma_0 - \vartheta \operatorname{Cov}(\varepsilon_{t-1}, r_{t-1})$$

$$\operatorname{Cov}(\varepsilon_{t-1}, r_{t-1}) = \operatorname{Cov}(\varepsilon_{t-1}, \mu + \alpha r_{t-2} + \varepsilon_{t-1} - \vartheta \varepsilon_{t-2})$$

$$= \operatorname{Cov}(\varepsilon_{t-1}, \varepsilon_{t-1}) = V(\varepsilon_{t-1}) = \sigma^2$$

$$\gamma_1 = \alpha \gamma_0 - \vartheta \sigma^2$$

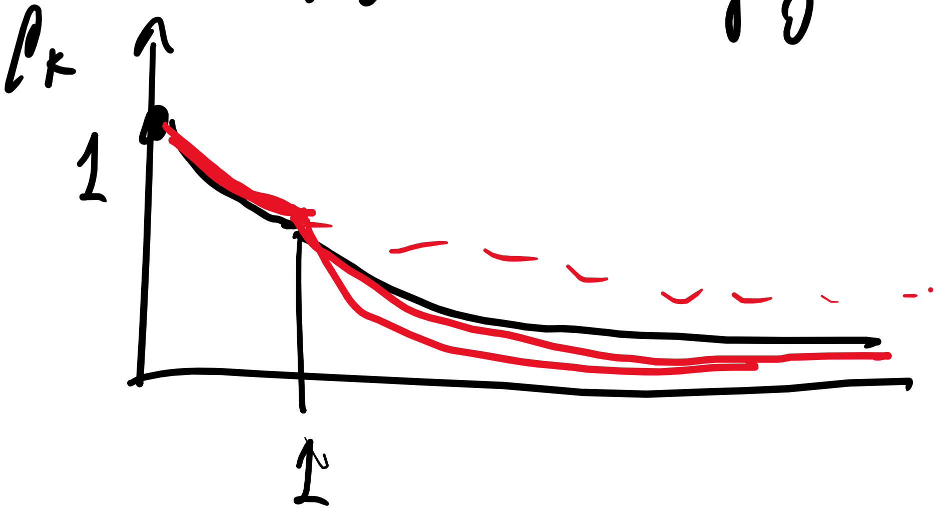
$$\gamma_0 = V(r_t) = V(\mu + \alpha r_{t-1} + \varepsilon_t - \vartheta \varepsilon_{t-1})$$

$$\Rightarrow \cancel{\gamma_0}\ V(r_t) = \alpha^2 V(r_{t-1}) + V(\varepsilon_t) + \vartheta^2 V(\varepsilon_{t-1})$$
$$+ \alpha \operatorname{Cov}(r_{t-1}, \varepsilon_t) \pm \alpha \vartheta \operatorname{Cov}(r_{t-1}, \varepsilon_{t-1}) + \operatorname{Cov}(\varepsilon_t, \varepsilon_{t-1})$$

$$V(r_t) = \alpha^2 V(r_{t-1}) + \sigma^2 + \vartheta^2 \sigma^2 - \alpha \vartheta \sigma^2$$

$$\Rightarrow V(r_t) = \sigma^2 \frac{1 + \sigma^2 - \alpha \vartheta}{1 + \alpha^2} = \gamma_0$$

$$\rho_1 = \frac{\gamma_1}{\gamma_0} = \frac{\alpha \gamma_0 - \vartheta \sigma^2}{\gamma_0} = \alpha - \vartheta \frac{\sigma^2}{\gamma_0}$$



$$r_t = \mu + \alpha r_{t-1} + \varepsilon_t - \theta \varepsilon_{t-1} \quad \text{ARMA}(1,1)$$

$\theta = 0$, AR(1)

$$r_t = \mu + \alpha r_{t-1} + \varepsilon_t$$

$$\hat{\alpha} = \frac{\sum (r_t - \bar{r})(r_{t-1} - \bar{r})}{\sum (r_{t-1} - \bar{r})^2} \quad \text{OLS}$$

$$\hat{\mu} = \bar{r}_t - \hat{\alpha}\, \bar{r}_{t-1} \Rightarrow \hat{\mu} = \bar{r}(1 - \hat{\alpha}) \quad \text{OLS}$$

LIKELIHOOD $\doteq f(r_1, r_2, \ldots, r_n; \mu, \alpha, \sigma^2)$

JOINT DENSITY

$$= f(\mu, \alpha, \sigma^2 \mid r_1, r_2, \ldots, r_n)$$

$r_1, r_2, \ldots, r_n \quad r_t = \mu + \alpha r_{t-1} + \varepsilon_t \quad \varepsilon_t \overset{iid}{\sim} N(0, \sigma^2)$

AN $x_i \overset{iid}{\sim}$ INDEP. $(i = 1, 2, \ldots, n)$

$$f(x_1, x_2, \ldots, x_n) = f(x_1) f(x_2) \cdots f(x_n)$$

AR(1) NOT APPLICABLE.

$\forall X, Y$ T.M.

$$f(X, Y) = f(X|Y)\, f(Y)$$
$$= f(Y|X) \cdot f(X)$$

$$f(r_1, r_2, \ldots, r_n) = f(r_n | r_{n-1}, \ldots, r_2, r_1) \cdot$$

$$f(r_{n-1}, r_{n-2}, \ldots r_2, r_1) =$$

$$= f(r_n|r_{n-1}, \ldots, r_1)\; f(r_{n-1}|r_{n-2}, \ldots, r_1) \cdot f(r_{n-2}|r_{n-3}, \ldots r_1)$$
$$\ldots f(r_2|r_1) \cdot f(r_1)$$

$$= \prod_{t=2}^{n} f(r_t|r_{t-1}, \ldots, r_1) \cdot \underbrace{f(r_1)}_{}$$

$r_t \mid r_{t-1}, \dots, r_1$

$I_{t-1} = \{ r_{t-1}, \dots, r_1 \}$ INFORMATION SET.

$\subseteq$

$I_t = \{ r_t, r_{t-1}, \dots, r_1 \}$

$I_{t-2} = \{ r_{t-2}, \dots, r_1 \}$

$r_t = \mu + \alpha r_{t-1} + \varepsilon_t \qquad \varepsilon_t \overset{iid}{\sim} N(0, \sigma^2) \qquad t = 1, 2, \dots, n$

$r_t \mid I_{t-1} \sim N(\cdot, \cdot)$ ΓΙΟΤΙ $\varepsilon_t \sim N$

$$E(r_t) = \frac{\mu}{1-\alpha}, \quad V(r_t) = \frac{\sigma^2}{1-\alpha^2}$$

$$E(r_t \mid I_{t-1}) = E(\mu + \alpha r_{t-1} + \varepsilon_t \mid I_{t-1}) =$$

$$= \mu + \alpha E(r_{t-1} \mid I_{t-1}) + E(\varepsilon_t \mid I_{t-1})$$

$= 0$

$$= \mu + \alpha r_{t-1} + \underbrace{E(\varepsilon_t)}_{=0}$$

ΔΙΟΤΙ $I_{t-1} = \{$ΑΝΕΞΑΡΤΗΤ. ΤΟΥ $\varepsilon_t\}$

$$= \mu + \alpha r_{t-1}$$

$$V(r_t | I_{t-1}) = E\left[\left(r_t - \underbrace{E(r_t | I_{t-1})}\right)^2 \Big| I_{t-1}\right]$$

$$= E\left[(r_t - \mu - \alpha r_{t-1})^2 \big| I_{t-1}\right) =$$

$$= E(\varepsilon_t^2 | I_{t-1}) = E(\varepsilon_t^2) = \sigma^2$$

ΑΡΑ

0×1 ΠΛΗΦ. ΓΙΑ ε_t

$$r_t | I_{t-1} \sim N(\cancel{\mu} + \alpha r_{t-1}, \sigma^2)$$

$$f(r_t \mid I_{t-1}) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(r_t - \mu - \alpha r_{t-1})^2}{2\sigma^2}}$$

$$\ell = \ln L(\alpha, \mu, \sigma^2 \mid \sim\sim)$$

$$= \ln \left[\left(\frac{1}{\sqrt{2\pi\sigma^2}} \right)^{n-1} e^{-\sum_{t=2}^{n} \frac{(r_t - \mu - \alpha r_{t-1})^2}{2\sigma^2}} \cdot f(r_1) \right]$$

$$r_1 \sim N\left(\frac{\mu}{1-\alpha}, \frac{\sigma^2}{1-\alpha^2}\right)$$

$$\ell = -\frac{n-1}{2}\ln 2\pi - \frac{n-1}{2}\ln\sigma^2 - \frac{1}{2\sigma^2}\sum_{t=2}^{n}(r_t-\mu-\alpha r_{t-1})^2$$

$$+ \underbrace{\ln f(r_1)}$$

ΣΤΑΘΕΡΟ Ή ΑΜΕΛΕΙΤΑΙΟ.

$$\ell_{AP.} = -\frac{n-1}{2}\ln 2\pi - \frac{n-1}{2}\ln\sigma^2 - \frac{1}{2\sigma^2}\sum_{t=2}^{n}(r_t-\mu-\alpha r_{t-1})^2$$

$$\max_{\alpha} \ell_{AP} \equiv \min_{\alpha} \sum_{t=2}^{n}(r_t-\mu-\alpha r_{t-1})^2 \qquad \hat{\alpha}_{MLE} = \hat{\alpha}_{OLS}$$

ΠΕΤΑΩ 1 ΠΑΡΑΤΗΡ.