PROBLEM 1

Consider the following subset of R^6

$$W = \{x \in R^6 : x_2 + x_4 - x_5 = 0\} \quad (1)$$

1.show that W is closed under linear combinations, hence a subspace of R^6

2.Find a linear map $R^6 \xrightarrow{T} R^6$ such that $W = \text{nullspace}(T)$ .Can this linear map T be one-to-one? onto?

3.Find a linear map $R^6 \xrightarrow{T} R^6$ such that $W = \text{Range}(T)$ .Can this linear map T be one-to-one? onto?

4. Find a basis of W

Answer

A basis of W is $b = [b_1, b_2, b_3, b_4, b_5] = [e_1, e_2 + e_5, e_3, e_4 + e_5, e_6]$ .It extends to a basis $\gamma = b \cup \{e_2\}$ of $\mathbb{R}^6$

A linear map $R^6 \xrightarrow{T} R^6$ with $W = \text{range}(T)$ is defined by $T(e_2) = 0, T(x) = x, \forall x \in b$ .No such map can be one-to-one, hence by the dimension theorem it cannot be onto either.

A linear map with $W = \text{nullspace}(T)$ is defined by $T(e_2) = e_2, T(x) = 0, \forall x \in b$ .No such map can be onto, hence by the dimension theorem it cannot be one-to-one either.

PROBLEM 2

For all allowed values of the parameters, find all global maxima of the following maximization problem, or show that none exist.

Objective function $p_1 y_1 + p_2 y_2$

Constraints

$$y_1 + 3y_2 \le 4$$
$$2y_1 + 3y_2 \le 1$$
$$y_1 + y_2 \le 0$$

variables y_1, y_2

parameters p_1, p_2

conditions on parameters $p_1 > 0, p_2 > 0$

answer

$$\boxed{\text{global maxima}} \quad \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{cases} \begin{bmatrix} -1 \\ 1 \end{bmatrix} & \text{if} \quad p_2 \le \frac{3p_1}{2} \\ \begin{bmatrix} -3 \\ 7/3 \end{bmatrix} & \text{if} \quad \frac{3p_1}{2} \le p_2 \le 3p_1 \\ \text{none} & \text{if} \quad p_2 > 3p_1 \end{cases} \tag{2}$$

PROBLEM 3

For all allowed values of the parameters, find all global maxima of the following maximization problem, or show that none exist.

Objective function $(A_1)^8 A_2$

Constraints

$$2B_1 - A_2 \ge \theta$$
$$A_1 + B_1 \le \delta$$
$$A_2 \le \delta$$
$$A_1 \ge 0, B_1 \ge 0, A_2 \ge 0$$

variables A_1, B_1, A_2

parameters δ, θ

conditions on parameters $\delta > 0$

answer

$$\boxed{\text{global maxima}} \quad [A_1 \quad A_2 \quad B_1] = \begin{cases} [\delta, \delta, 0] & if \quad \theta \le -\delta \\ [\delta, -\theta, 0] & if \quad -\delta < \theta \le -\delta/4 \\ [\frac{8\delta}{9} - \frac{4\theta}{9}, \frac{2\delta}{9} - \frac{\theta}{9}, \frac{\delta}{9} + \frac{4\theta}{9}] & if \quad -\delta/4 < \theta \le 2\delta \\ none & if \quad \theta > 2\delta \end{cases} \tag{3}$$

PROBLEM 4

For all allowed values of the parameters, find all global maxima of the following maximization problem, or show that none exist.

Objective function $2k\theta A - \frac{1}{2}A^2 - \frac{\theta^2}{2}L^2$

Constraints

$$0 \le A \le wL$$
$$0 \le L \le \gamma$$

variables A, L

parameters k, θ, γ, w

conditions on parameters $k > 0, \theta > 0, \gamma > 0, w > 0$

answer

global maxima

$$\begin{bmatrix} A \\ L \end{bmatrix} = \begin{cases} \begin{bmatrix} \gamma w \\ \gamma \end{bmatrix} & \text{if} \quad \gamma \le \frac{2k\theta w}{\theta^2 + w^2} \\ \begin{bmatrix} \frac{2k\theta w^2}{\theta^2 + w^2} \\ \frac{2k\theta w}{\theta^2 + w^2} \end{bmatrix} & \text{if} \quad \gamma \ge \frac{2k\theta w}{\theta^2 + w^2} \end{cases} \qquad (4)$$

PROBLEM 5

For all allowed values of the parameters, find all global maxima of the following maximization problem, or show that none exist.

Objective function $x - \frac{1}{2y^2}$

Constraints

$$px + qy \le m$$
$$x \ge 1, y \ge 0$$

variables x, y

parameters p, q, m

conditions on parameters $p > 0, q > 0, m > 0$

answer

$$\boxed{\begin{array}{l}\text{global maxima} \\ \begin{bmatrix} x \\ y \end{bmatrix} = \begin{cases} \begin{bmatrix} m/p \\ 0 \end{bmatrix} & \text{if} \quad m \geq p \\ \text{none} & \text{if} \quad m < p \end{cases}\end{array}} \qquad (5)$$

PROBLEM 6

For all allowed values of the parameters, find all global minima of the following minimization problem, or show that none exist.

Objective function $w_1 x_1 + w_2 x_2$

Constraints

$$\max\{x_1, x_2\} \geq q$$
$$x_1 \geq 0, x_2 \geq 0$$

variables x_1, x_2

parameters q, w_1, w_2

conditions on parameters $q > 0, w_1 > 0, w_2 > 0$

answer

global minima

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{cases} \begin{bmatrix} q \\ 0 \end{bmatrix} & \text{if} \quad w_1 < w_2 \\ \left\{ \begin{bmatrix} q \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ q \end{bmatrix} \right\} & \text{if} \quad w_1 = w_2 \\ \begin{bmatrix} 0 \\ q \end{bmatrix} & \text{if} \quad w_1 > w_2 \end{cases} \tag{6}$$

PROBLEM 7

For all allowed values of the parameters, find all global maxima of the following maximization problem, or show that none exist.

Objective function $pF(x) - wx$

constraints $x \in \mathbb{R}^n, x \geq 0$

variables $x = \begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix} \in \mathbb{R}^n$

parameters

$$p \in \mathbb{R}$$
$$w = \begin{bmatrix} w_1 & w_2 & \dots & w_n \end{bmatrix} \in \mathbb{R}^n$$
$$\mathbb{R}_+^n \xrightarrow{F} \mathbb{R}_+$$

conditions on parameters

$$p > 0, w_1 > 0, w_2 > 0, \dots, w_n > 0.$$
$$F(0) = 0.$$
$$\text{there exists a vector } b \in \mathbb{R}_+^n \text{ such that } pF(b) - wb > 0.$$
$$F(tx) > tF(x), \text{for all } t > 1 \text{ and for all } x \neq 0.$$

Answer

No global maximum exists, because the vectors $a_n = (n+1)b, n = 1, 2, \dots.$ are all feasible, and the objective function $\Pi(x) = pF(x) - wx$ satisfies

$$\Pi(x_n) = pF(a_n) - wa_n = pF((n+1)b) - w(n+1)b$$
$$> p(n+1)F(b) - w(n+1)b = (n+1)(pF(b) - wb) \to \infty \text{ as } n \to \infty$$

PROBLEM 8

For all allowed values of the parameters, find all global maxima of the following maximization problem, or show that none exist.

Objective function

$$U(x,y) = \begin{cases} x + A & \text{if} \quad y \le x + A \\ y & \text{otherwise} \end{cases}$$

constraints

$$x + py \le m$$
$$x \ge 0, y \ge 0$$

variables x, y

parameters A, p, m

conditions on parameters $A > 0, p > 0, m > 0$

answer

global maxima

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{cases} \begin{bmatrix} m/p \\ 0 \end{bmatrix} & \text{if} \quad m/p + A > m/q \\ \left\{\begin{bmatrix} m/p \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ m/q \end{bmatrix}\right\} & \text{if} \quad m/p + A = m/q \\ \begin{bmatrix} 0 \\ m/q \end{bmatrix} & \text{if} \quad m/p + A < m/q \end{cases} \tag{7}$$

PROBLEM 9

Find all global maxima and all global minima of the following problem or show that none exist.

Objective function $x_1^3 - x_2 2^{x_1}$

Constraints

$$(x_1+1)(x_2-1)(x_1-7)(x_2-5)=0$$
$$\left((x_2)^2-1\right)\left((x_3)^2-1\right)=0$$
$$\left((x_1)^3+1\right)(x_3-1)(x_3-4)=0$$

variables x_1, x_2, x_3

answer

No global maximum exists, because the vectors $a^n=[7,-n,1], n=1,2,...$ are all feasible, and the objective function $f(x_1,x_2,x_3)=x_1^3-x_2 2^{x_1}$ satisfies

$f(a^n)=7^3+n2^7 \to \infty$ as $n \to \infty$

PROBLEM 10

For all allowed values of the parameters, find all global maxima of the following maximization problem, or show that none exist.

Objective function $49-\frac{1}{2}(x_1-1)^2-\frac{1}{2}(x_2-2)^2$

Constraints

$$2x_1+3x_2 \le 10$$
$$x_1 \ge 0, x_2 \ge 0$$

variables x_1, x_2

answer

$x=[1,2]$