Please answer all questions. Everything covered in class can be freely referred to.

PROBLEM 1

a) Consider a concave function $U \xrightarrow{f} \mathbb{R}$ defined on a convex subset U of $\mathbb{R}^n$. Define $B_f = \{x \in U : f(x) \geq 0\}$. Is B_f a convex set? (Proof or counterexample)

b) Consider a maximization problem (f, S). The feasible set $S \subseteq \mathbb{R}^n$ is convex, and the objective function $S \xrightarrow{f} \mathbb{R}$ is concave. Define

$$M = \text{Set of all global maxima of } (f, S) \tag{1}$$

Is the set M convex? (Proof or counterexample)

c) Consider a maximization problem (f, S). The feasible set $S \subseteq \mathbb{R}^n$ is convex, and the objective function $S \xrightarrow{f} \mathbb{R}$ is strictly concave, i.e.

$$f(tx + (1-t)y) > tf(x) + (1-t)f(y), \text{for all } x \neq y, 0 < t < 1 \tag{2}$$

Define

$$M = \text{Set of all global maxima of } (f, S) \tag{3}$$

Show that the set M cannot contain two distinct elements

PROBLEM 2

Consider concave functions $U \xrightarrow{g_i} \mathbb{R}, i = 1, 2, \dots, L$ defined on a convex subset U of $\mathbb{R}^n$. Define

$$B_i = \{x \in U : g_i(x) \geq 0\} \tag{4}$$

a) is the intersection $\bigcap_{i=1}^{L} B_i$ a convex set? (Proof or counterexample)

b) is the union $\bigcup_{i=1}^{L} B_i$ a convex set? (Proof or counterexample)

c) is the cartesian product $\prod_{i=1}^{L} B_i$ a convex set? (Proof or counterexample)

d) is the sum $\sum_{i=1}^{L} B_i$ a convex set? (Proof or counterexample)

PROBLEM 3

a) For all allowed values of the parameters, find all global maxima of the following maximization problem, or show that none exist.

$$
\boxed{\begin{array}{ll}
\boxed{\text{maximization problem}} & \\
\text{Objective function} & \mathbb{R} \xrightarrow{f} \mathbb{R},\ f(x) = x \\
\text{Constraints} & x \le 2s + t - 5,\ x \le s + 2t - 3,\ x \ge 0 \\
\text{Variables} & x \\
\text{Parameters} & s, t \\
\text{conditions on parameters} & s \ge 0, t \ge 0
\end{array}} \tag{5}
$$

b) The solution of the maximization problem (5) is a function $x = F(s,t)$ .Is the function F concave?

PROBLEM 4

a) Consider the function

$$\mathbb{R}^3 \xrightarrow{f} \mathbb{R},\ f(x_1, x_2, x_3) = \alpha x_2 x_3 + \beta x_2^2 - c x_1^2 + c_1 x_1 + c_2 x_2 \tag{6}$$

For which values of the parameters $\alpha, \beta, c, c_1, c_2$ is this function concave?

b) Consider the function

$$\mathbb{R}^3 \xrightarrow{g} \mathbb{R}, g(x_1, x_2, x_3) = \alpha x_1^2 x_3 + b x_1^2 + d x_2^2 \tag{7}$$

For which values of the parameters α, b, d is this function concave?

c)Consider the function $h = 2f + 3g$,with f given by (6),and g given by (7). For which values of the parameters is h concave?

d)Consider the function $h = \min\{5f, 6g\}$, with f given by (6),and g given by (7). For which values of the parameters is h concave?

PROBLEM 5

Consider the following

maximization problem

Objective function $\mathbb{R}_{+}^{L+1} \xrightarrow{f} \mathbb{R},\ f(x_0, x_1, .., x_L) = x_0 + 2\sqrt{x_1 \dots x_L}$

Constraints $px_0 + x_1 + \ldots + x_L \le m, x_0 \ge 0, x_1 \ge 0, \ldots, x_L \ge 0$ (8)

Variables $x_0, x_1, \ldots, x_L$

Parameters p, m, L

conditions on parameters $p > 0, m > 0, L \ge 1, L$ integer

a) compute the largest value of L for which the objective function is concave, and solve (8) for this particular value only.

b) compute the smallest value of L for which the objective function is not concave, and solve (8) for this particular value only.

c) compare the global maxima in the two cases.

PROBLEM 6

For all allowed values of the parameters, find all global maxima of the following maximization problem, or show that none exist.

maximization problem

Objective function $\mathbb{R} \xrightarrow{f} \mathbb{R},\ f(x) = x$

Constraints $x \ge 0, ax^2 + bx + c \ge 0$ (9)

Variables x

Parameters a, b, c

conditions on parameters $a \ne 0, b \ne 0, c \ne 0$

PROBLEM 7

For all allowed values of the parameters, find all global maxima of the following maximization problem, or show that none exist.

maximization problem

Objective function $\mathbb{R}^2 \xrightarrow{f} \mathbb{R},\ f(x_1, x_2) = x_1 + \alpha x_2$

Constraints $0 \le x_1 \le 4, 0 \le x_2 \le 4, (x_1 - 2)^2 - (x_2)^2 \ge 0$ (10)

Variables x_1, x_2

Parameters α

conditions on parameters $a > 0$

answers

Saturday, March 15, 2025 12:36

PROBLEM 1

a) Consider a concave function $U \xrightarrow{f} \mathbb{R}$ defined on a convex subset U of $\mathbb{R}^n$. Define $B_f = \{x \in U : f(x) \geq 0\}$. Is B_f a convex set? (Proof or counterexample)

8 B_f IS CONVEX, BECAUSE FOR ANY x, y in B_f

AND ANY $0 < t < 1$

$f(tx + (1-t)y) \geq$ (JENSEN INEQUALITY)

$t f(x) + (1-t) f(y) \geq$ $(x, y \in B_f)$

$t \cdot 0 + (1-t) \cdot 0 = 0$ i.E.

$f(tx + (1-t)y) \geq 0$ i.E

$tx + (1-t)y \in B_f$

b) Consider a maximization problem (f, S). The feasible set $S \subseteq \mathbb{R}^n$ is convex, and the objective function $S \xrightarrow{f} \mathbb{R}$ is concave. Define

$$M = \text{Set of all global maxima of } (f, S) \tag{1}$$

Is the set M convex? (Proof or counterexample)

M IS CONVEX

WE WILL SHOW THAT $b, b' \in M$ AND $0 < t < 1$ IMPLY

$tb + (1-t)b' \in M$

FOR ANY TWO b, b' IN M

$\boxed{1}$ $b \in S$ | $\boxed{1'}$ $b' \in S$

$\boxed{2}$ $f(b) \geq f(x)$ | $\boxed{2'}$ $f(b') \geq f(x)$ $\forall x \in S$

$\boxed{E}$ FOR $x = b'$ IMPLIES $f(b) \geq f(b')$

$\boxed{E'}$ FOR $x = b$ IMPLIES $f(b') \geq f(b)$

HENCE

$f(b) = f(b')$ (3)

FOR ANY $0 < t < 1$,

$tb + (1-t)b' \in S$ (4)

BECAUSE $b, b' \in S$ AND S IS CONVEX

THEN BY (4) AND $\boxed{E}$

$f(b) \geq f(tb + (1-t)b')$ (5)

AND BY JENSEN'S INEQUALITY

$f(tb + (1-t)b') \geq tf(b) + (1-t)f(b') \overset{3}{=} f(b)$ ie

$f(tb + (1-t)b') \geq f(b)$ (6)

BY (5) (6)

$f(tb + (1-t)b') = f(b)$ (7)

BY (7) $\boxed{E}$

$f(tb + (1-t)b') \geq f(x), \ \forall x \in S$ (8)

BY (8) (3) $tb + (1-t)b' \in M$ QED

c) Consider a maximization problem (f, S). The feasible set $S \subseteq \mathbb{R}^n$ is convex, and the objective function $S \xrightarrow{f} \mathbb{R}$ is strictly concave, i.e.

$$f\left(tx+(1-t)y\right) > tf(x)+(1-t)f(y), \text{ for all } x \neq y, 0<t<1 \qquad (2)$$

Define

$$M = \text{Set of all global maxima of } (f, S) \qquad (3)$$

Show that the set M cannot contain two distinct elements

SUPPOSE, FOR CONTRADICTION, THAT M CONTAINS TWO DISTINCT ELEMENTS b, b'. THEN AS IN PART (b)

$$b, b' \in S \qquad (1)$$

$$f(b) = f(b') \geq f(x) \quad \forall x \in S \qquad (2)$$

$$tb + (1-t)b' \in S \qquad (3)$$

$$f(tb + (1-t)b') > tf(b) + (1-t)f(b') = f(b)$$

ie

$$f(tb + (1-t)b') > f(b) \qquad (4)$$

BY (3) AND (2)

$$f(b) \geq f(tb + (1-t)b') \qquad (5)$$

(4) AND (5) CONTRADICT EACH OTHER

PROBLEM 2

Consider concave functions $U \xrightarrow{g_i} \mathbb{R}, i = 1,2,\ldots,L$ defined on a convex subset U of $\mathbb{R}^n$ .Define

$$B_i = \{x \in U : g_i(x) \geq 0\} \tag{4}$$

a) is the intersection $\bigcap_{i=1}^{L} B_i$ a convex set? (Proof or counterexample)

b) is the union $\bigcup_{i=1}^{L} B_i$ a convex set? (Proof or counterexample)

c) is the cartesian product $\prod_{i=1}^{L} B_i$ a convex set? (Proof or counterexample)

d) is the sum $\sum_{i=1}^{L} B_i$ a convex set? (Proof or counterexample)

THE SETS B_i ARE CONVEX BY PROBLEM 1

a) $\cap B_i$ IS CONVEX, BECAUSE INTERSECTIONS PRESERVE CONVEXITY

c) ΠB_i IS CONVEX, BECAUSE CARTESIAN PRODUCTS PRESERVE CONVEXITY

(d) ΣB_i IS CONVEX, BECAUSE SUMS PRESERVE CONVEXITY

(b) $\cup B_i$ MAY NOT BE CONVEX

$\mathbb{R} \xrightarrow{g_1} \mathbb{R}$, $g_1(x) = x - 1$

$\mathbb{R} \xrightarrow{g_2} \mathbb{R}$, $g_2(x) = -x$

$B_1 = \{x \in \mathbb{R} : g_1(x) \geq 0\} = [1, \infty)$

$B_2 = \{x \in \mathbb{R} : g_2(x) \geq 0\} = (-\infty, 0]$

$B_1 \cup B_2 = (-\infty, 0] \cup [1, \infty)$ IS NOT CONVEX

PROBLEM 3

a) For all allowed values of the parameters, find all global maxima of the following maximization problem, or show that none exist.

maximization problem	
Objective function	$\mathbb{R} \xrightarrow{f} \mathbb{R}, f(x) = x$
Constraints	$x \leq 2s + t - 5, x \leq s + 2t - 3, x \geq 0$
Variables	x
Parameters	s, t
conditions on parameters	$s \geq 0, t \geq 0$

(5)

b) The solution of the maximization problem (5) is a function $x = F(s,t)$ .Is the function F concave?

LET $\alpha = \min\{2s+t-5,\ s+2t-3\}$ (1)

THEN THE FEASIBLE SET C IS

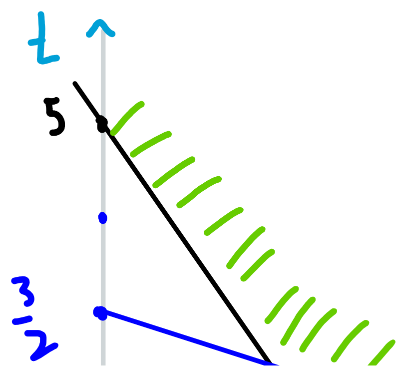
$$C = \begin{cases} \emptyset & \text{if } \alpha < 0 \\ [0,\alpha] & \text{if } \alpha \geq 0 \end{cases} \quad (2)$$

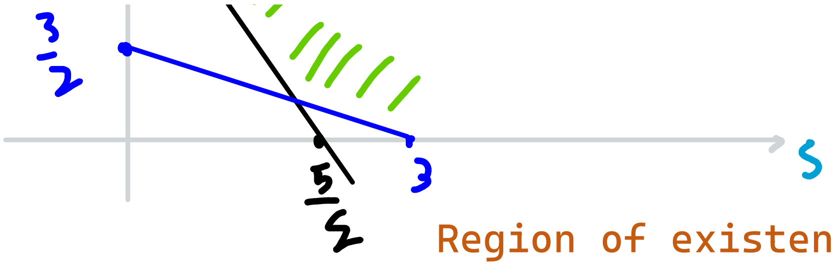
THEN BY (1), $\alpha \geq 0$ IFF THE PARAMETERS s,t SATISFY

$$\boxed{2s+t \geq 5,\ s+2t \geq 3,\ s \geq 0,\ t \geq 0}$$

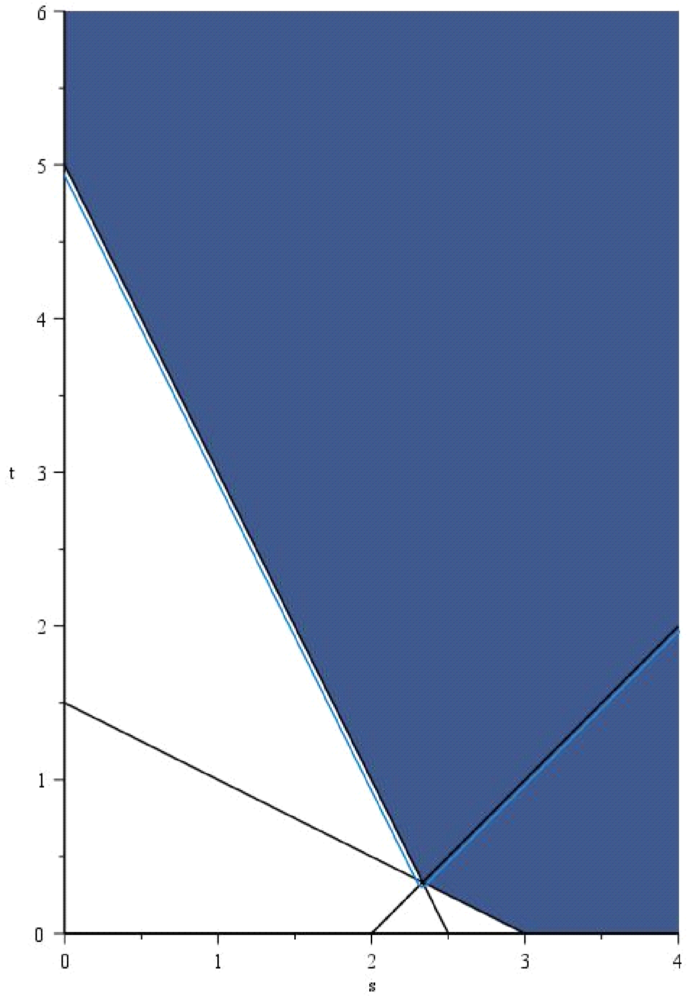
i.e IF

$$t \geq \max\left\{0, \frac{3-s}{2}, 5-2s\right\} = \beta(s)$$





Region of existence $t \geq \beta(s)$



$$C = \begin{cases} \emptyset & t < \beta(s) \\ [0, \alpha] & \text{OTHERWISE} \end{cases} \qquad (3)$$

AND THE GLOBAL MAXIMUM $F(s,t)$ SATISFIES

$$F(s,t) = \begin{cases} \text{NONE} & \text{IF } t < \beta(s) \\ \alpha & \text{OTHERWISE} \end{cases} \qquad (4)$$

FROM NOW ON $t \geq \beta(s)$

FROM NOW ON $t \geq \beta(s)$

THEN $F(s,t) \overset{1}{\underset{4}{=}} \min\{2s+t-5,\ s+2t-3\} =$

= LOWER ENVELOPE OF CONCAVE FUNCTIONS

= CONCAVE FUNCTION, AND

$$F(s,t) = \begin{cases} 2s+t-5 & \text{if } s \leq t+\varepsilon \\ \text{GLOBAL MAXIMA WHEN } t \geq \beta(s) & \\ s+2t-3 & \text{if } s \geq t+\varepsilon \end{cases}$$

PROBLEM 4

a) Consider the function

$$\mathbb{R}^3 \xrightarrow{f} \mathbb{R}, f(x_1,x_2,x_3) = \alpha x_2 x_3 + \beta x_2^2 - c x_1^2 + c_1 x_1 + c_2 x_2 \quad (6)$$

For which values of the parameters $\alpha, \beta, c, c_1, c_2$ is this function concave?

THE HESSIAN MATRIX OF f IS

$$H = \begin{bmatrix} -2c & 0 & 0 \\ 0 & 2\beta & \alpha \\ 0 & \alpha & 0 \end{bmatrix}$$

f IS CONCAVE IFF H IS NEGATIVE SEMIDEFINITE $\forall x \in \mathbb{R}^3$, I.E. IFF

$\alpha = 0$ AND $\beta \leq 0 \leq c$

b) Consider the function

$$\mathbb{R}^3 \xrightarrow{g} \mathbb{R},\, g(x_1, x_2, x_3) = \alpha x_1^2 x_3 + b x_1^2 + d x_2^2$$

For which values of the parameters α, b, d is this function concave?

THE HESSIAN MATRIX OF g IS

$$H = \begin{bmatrix} 2\alpha x_3 + 2\beta & 0 & 2\alpha x_1 \\ 0 & 2d & 0 \\ 2\alpha x_1 & 0 & 0 \end{bmatrix}$$

g IS CONCAVE IFF H IS NEGATIVE SEMIDEFINITE FOR ALL $x \in \mathbb{R}^3$, I.E IFF

$\alpha = 0,\ \beta \le 0,\ d \le 0$

c)Consider the function $h = 2f + 3g$,with f given by (6),and g given by (7). For which values of the parameters is h concave?

THE HESSIAN OF h IS

$$H = \begin{bmatrix} 6\alpha x_3 + 6\beta - 4c & 0 & 6\alpha x_1 \\ 0 & 4\beta + 6d & 2\alpha \\ 6\alpha x_1 & 2\alpha & 0 \end{bmatrix}$$

h IS CONCAVE IFF H IS NEGATIVE SEMIDEFINITE $\forall x \in \mathbb{R}^3$, I.E IFF

$\alpha = 0,\ \beta \le \frac{2c}{3},\ \beta \le -\frac{3d}{2}$

d)Consider the function $h = \min\{5f, 6g\}$, with f given by (6),and g given by (7).

For which values of the parameters is h concave?

SINCE LOWER ENVELOPES PRESERVE CONCAVITY

$\alpha = 0, \beta \le 0 \le c, d \le 0 \Rightarrow h$ CONCAVE

ONE WAY

PROBLEM 5

Consider the following

maximization problem	
Objective function	$\mathbb{R}_+^{L+1} \xrightarrow{f} \mathbb{R}, f(x_0, x_1, .., x_L) = x_0 + 2\sqrt{x_1 ... x_L}$
Constraints	$px_0 + x_1 + ... + x_L \le m, x_0 \ge 0, x_1 \ge 0, ..., x_L \ge 0$
Variables	$x_0, x_1, ..., x_L$
Parameters	p, m, L
conditions on parameters	$p > 0, m > 0, L \ge 1, L$ integer

(8)

a) compute the largest value of L for which the objective function is concave, and solve (8) for this particular value only.

b) compute the smallest value of L for which the objective function is not concave, and solve (8) for this particular value only.

c) compare the global maxima in the two cases.

(a) L = 2 IS THE LARGEST VALUE FOR WHICH f IS CONCAVE

L = 3 IS THE SMALLEST VALUE FOR WHICH f IS NOT CONCAVE

L = 1

THE FUNCTIONS x_0 AND $2\sqrt{x_1}$ ARE CONCAVE, BY THE SECOND DERIVATIVE TEST. THEN

$f = x_0 + 2\sqrt{x_1}$ = CONCAVE + CONCAVE = CONCAVE

L = 2

x_0 is CONCAVE

$2\sqrt{x_1 x_2}$ is CONCAVE (COBB-DOUGLAS WITH SUM OF EXPONENTS $\frac{1}{2} + \frac{1}{2} = 1$)

THEN

$f = x_0 + 2\sqrt{x_1 x_2}$ = CONCAVE + CONCAVE = CONCAVE

$L \geq 3 \Rightarrow f$ IS NOT CONCAVE

BECAUSE f VIOLATES JENSEN'S INEQUALITY

$$A = [0, 0, \ \ , 0] \quad \Big| \quad f(A) = 0$$

$$B = [0, 2, \ \ , 2] \quad \Big| \quad f(B) = 2^{L/2}$$

$$C = \frac{1}{2}A + \frac{1}{2}B = [0, 1, \ \ , 1], \ f(C) = 2$$

$$f\left(\frac{1}{2}A + \frac{1}{2}B\right) = 2 < 2^{\frac{L}{2}} = \frac{1}{2}f(A) + \frac{1}{2}f(B)$$

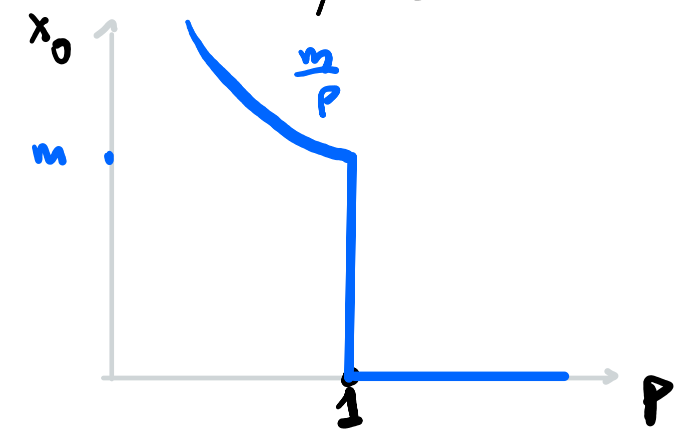
VIOLATION OF JENSEN'S INEQUALITY FOR CONCAVE FUNCTIONS

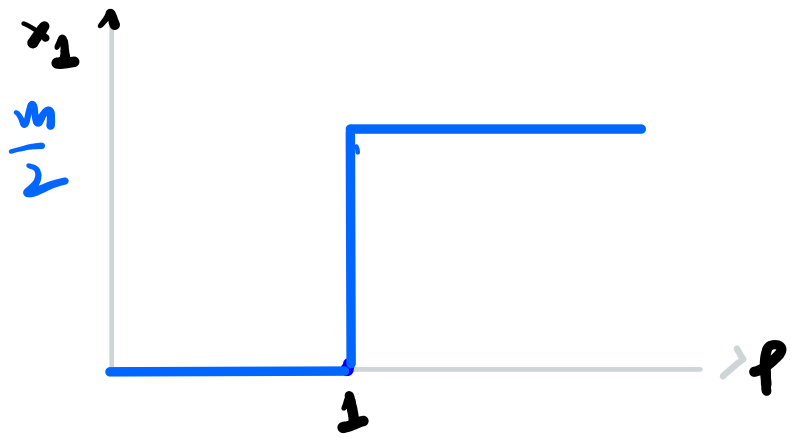
(b) WHEN $L=2$ AND $x = [x_0, x_1, x_2]$ THEN

THE GLOBAL MAXIMA ARE

$$x = \begin{cases} \left[\frac{m}{p}, 0, 0\right] & \text{if } p < 1 \\ [m - 2t, t, t], \ 0 \leq t \leq \frac{m}{2} & \text{if } p = 1 \end{cases}$$

$$x = \begin{cases} [m-2t,\ t,\ t],\ 0 \le t \le \frac{m}{2} & \text{if } p = 1 \\ [0,\ \frac{m}{2},\ \frac{m}{2}] & \text{if } p > 1 \end{cases}$$





(c) WHEN $L=3$ AND $x = [x_0, x_1\ x_2, x_3]$

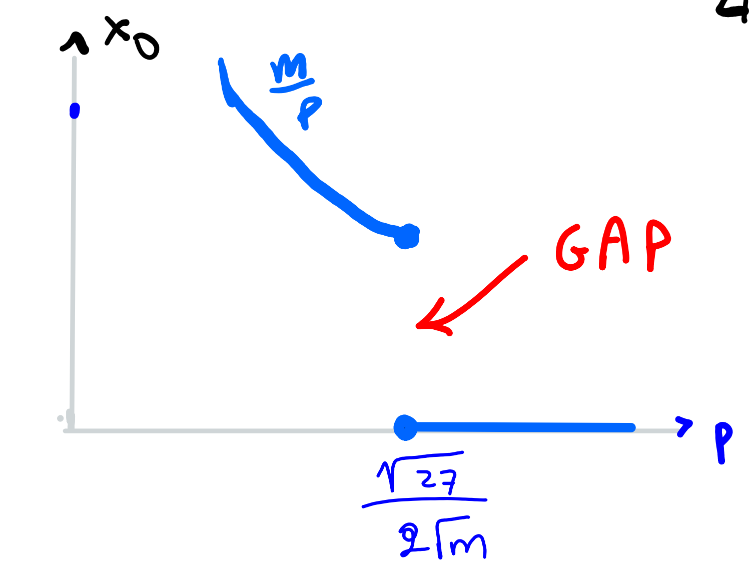
LET $A = [\frac{m}{p}, 0, 0, 0]$

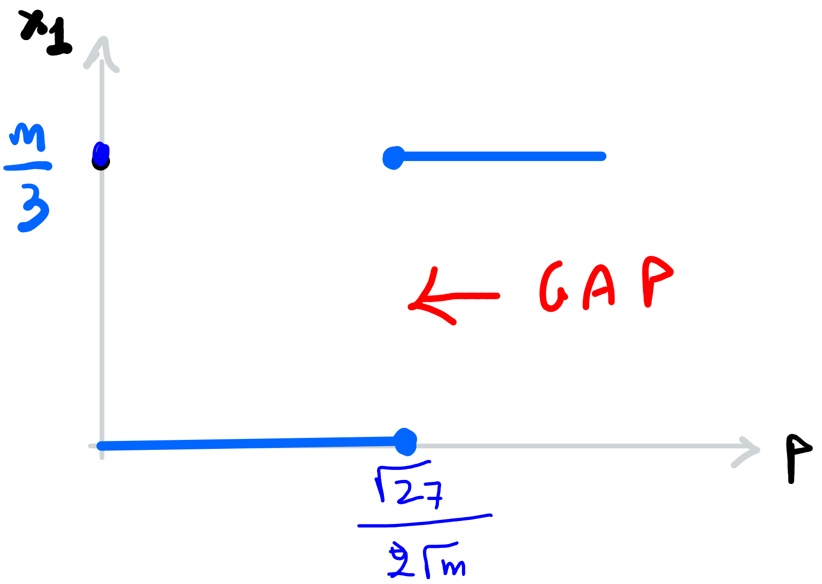
$B = [0, \frac{m}{3}, \frac{m}{3}, \frac{m}{3}]$. THEN

GLOBAL MAXIMA

$$x = \begin{cases} A & \text{if} \quad m < \frac{27}{4p^2} \\ \{A, B\} & \text{if} \quad m = \frac{27}{4p^2} \\ B & \text{if} \quad m > \frac{27}{4p^2} \end{cases}$$

L ... if $m > \frac{\ldots}{4p^2}$





THE MAIN DIFFERENCE BETWEEN THE TWO CASES IS THAT WHEN f IS NOT CONCAVE, THE DEMAND CORRESPONDENCE $x(p, m)$ IS NOT CONVEX-VALUED AT SOME (p, m), i.E. THE SET M OF GLOBAL MAXIMA AT $m = 27/(4p^2)$ IS NOT CONVEX

PROBLEM 6

For all allowed values of the parameters, find all global maxima of the following maximization problem, or show that none exist.

maximization problem	
Objective function	$\mathbb{R} \xrightarrow{f} \mathbb{R}, f(x) = x$
Constraints	$x \geq 0, ax^2 + bx + c \geq 0$
Variables	x
Parameters	a, b, c
conditions on parameters	$a \neq 0, b \neq 0, c \neq 0$

(9)

MAX $f(x) = x$

SUBJECT TO $x \geq 0$, $\alpha x^2 + \beta x + \gamma \geq 0$

- CASE $\alpha > 0$: NO GLOBAL MAX EXISTS BECAUSE THE FEASIBLE SET IS UNBOUNDED FROM ABOVE AND $\lim_{x \to \infty} f(x) = \infty$
- CASE $\alpha < 0$, $\gamma < 0$
  THEN THE FEASIBLE SET IS EMPTY, AND NO GLOBAL MAX EXISTS
- CASE $\alpha < 0$, $\gamma > 0$
  THEN THE FEASIBLE SET IS $[0, \theta]$
  WHERE $\theta = \dfrac{-\beta - \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha}$
  AND THE GLOBAL MAXIMUM IS θ

$$x = \begin{cases} \theta & \text{IF } \alpha < 0, \gamma > 0 \\ & \text{GLOBAL MAXIMA} \\ \text{NONE} & \text{OTHERWISE} \end{cases}$$

PROBLEM 7

For all allowed values of the parameters, find all global maxima of the following maximization problem, or show that none exist.

maximization problem	
Objective function	$\mathbb{R}^2 \xrightarrow{f} \mathbb{R}, f(x_1, x_2) = x_1 + \alpha x_2$
Constraints	$0 \le x_1 \le 4, 0 \le x_2 \le 4, (x_1 - 2)^2 - (x_2)^2 \ge 0$
Variables	x_1, x_2
Parameters	α
conditions on parameters	$a > 0$

(10)

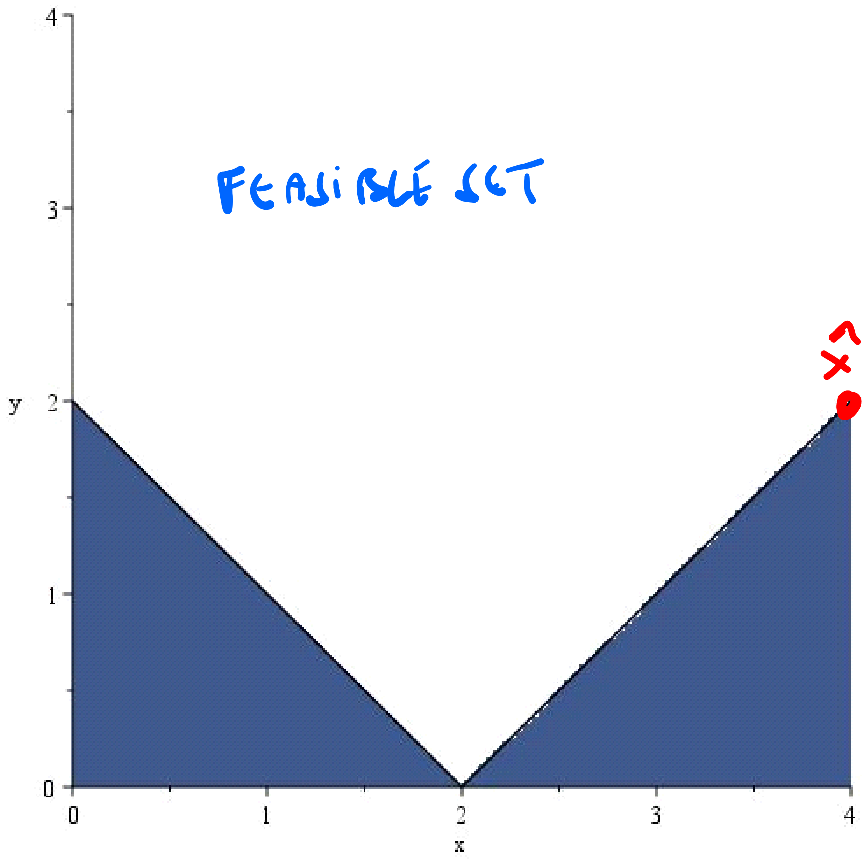
THE UNIQUE GLOBAL MAXIMUM IS $\hat{x} = [4, 2]$ FOR ALL $\alpha > 0$.

BECAUSE $\hat{x}$ IS FEASIBLE, AND FOR ANY OTHER FEASIBLE x, $x \le \hat{x}$

SINCE f IS STRICTLY INCREASING, WE HAVE:

x FEASIBLE $\Rightarrow x \le \hat{x} \Rightarrow f(x) \le f(\hat{x})$

AND $x \neq \hat{x} \Rightarrow f(x) < f(\hat{x})$


FEASIBLE SET
$\hat{x}$
y
x
0
1
2
3
4