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PROBLEM 1

Let $V \xrightarrow{T} W$ be a linear map between finite-dimensional vector spaces, $\mathrm{B} = [\beta_1, ..., \beta_k]$ a list of n vectors in V, and $T(\mathrm{B}) = [T(\beta_1), ..., T(\beta_k)]$ the image of this list under T.

Please answer the following questions with a proof, if your answer is yes, or a counterexample, if your answer is no.

(a) If B spans V, does $T(\mathrm{B})$ span $\mathrm{image}(T)$?

yes

(b) If $T(\mathrm{B})$ is linearly independent in W, is it true that B linearly independent in V?

yes

(c) If B is linearly independent in V, and $V \xrightarrow{T} W$ is a monomorphism, is it true that $T(\mathrm{B})$ is linearly independent in W ?

yes

PROBLEM 2

Let H be a hyperplane in $\mathbb{R}^n$,b a vector in one of the half-spaces of H, and c a vector in the other(opposite) half-space of H. Let L be the line uniquely defined by the points b, c. Is the set $L \cap H$ nonempty? Is it a flat? What is its dimension?

<u>Answer</u>: if $b \in H, c \in H$ then $L \cap H = L = \text{Flat of dimension } 1$,otherwise $L \cap H = \text{a single point=flat of dimension zero}$

PROBLEM 3

Let b, c be vectors in $\mathbb{R}^n$,and let $\beta > 0, \gamma > 0$ .Show that the open balls $B_\beta(b), B_\gamma(c)$ are equal if and only if $b = c$ and $\beta = \gamma$

Answer: suppose $B_\beta(b) = B_\gamma(c)$

show that $\beta = \gamma$

Consider the points $x = b + \beta e_1, y = b - \beta e_1, e_1 = [1, 0, ..., 0]$ Then $x, y \in \overline{B}_\beta(b)$,and therefore $x, y \in \overline{B}_\gamma(c)$ .Then

$2\beta = |b+\beta e_1 - (b-\beta e_1)| = |x-y| = |x-c+c-y| \le |x-c|+|c-y| \le \gamma+\gamma = 2\gamma$,and therefore $\beta \le \gamma$,and similarly $\gamma \le \beta$

show that b=c.

If not, then the points $x = b+\beta\frac{b-c}{|b-c|}, y = c-\beta\frac{b-c}{|b-c|}$ belong to $\overline{B}_\beta(b)$,and we have

$$|x-y| \le 2\beta \tag{1}$$

$$|x-y| = \left|b-c+2\beta\left(\frac{b-c}{|b-c|}\right)\right| = \left(1+\frac{2\beta}{|b-c|}\right)|b-c| = |b-c|+2\beta > 2\beta \tag{2}$$

(1) and (2) are contradictory.

PROBLEM 4

Let A be a nonempty flat in $\mathbb{R}^n$ .Show that A is not bounded.

Answer: If A is singleton, then it is bounded. Suppose, for contradiction, that A contains at least two elements and is bounded. Then there exists a $\delta > 0$ such that $A \subseteq B_\delta(0)$ .Let x, y be two distinct elements of A. Then for all $t \in \mathbb{R}$,$tx+(1-t)y \in A$,and therefore $|tx+(1-t)y| < \delta, \forall t \in \mathbb{R}$ .Then we obtain the following contradiction

$|t||x-y| = |t(x-y)+y-y| \le |t(x-y)+y|+|y| = |tx+(1-t)y|+|y| \le \delta+|y|, \forall t \in \mathbb{R}$

PROBLEM 5

Which of these functions are concave? (Proof or counterexample). The domain of all these functions is $\mathbb{R}^2_{++}$

- $f(x,y) = x+\sqrt{y}$

Sum of concave functions=concave function

- $f(x,y) = x+y^2$

Not concave, its hessian matrix is $\begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$

- $f(x,y) = x-\frac{1}{y}$

Sum of concave functions=concave function

- $f(x,y)=\min(x+1,\sqrt{y}+2)$

Lower envelope of concave functions=concave function

- $f(x,y)=\max(x+1,y+2)$

Let $A=[2,0],B=[1,1],t=3/4$ .Then

$f(A)=f(B)=3, f(tA+(1-t)B)=f(7/4,1/4)=7/4$

$f(tA+(1-t)B)=7/4<3=tf(A)+(1-t)f(B)$

i.e., f is not concave because it violates Jensen's inequality for concave functions.

- $f(x,y,z)=\min(\log x+2y-z,2x+3\sqrt{y}-z^2,x,y,z)$

Lower envelope of concave functions=concave function

- $f(x,y,z)=\min(x,y,\frac{x^2+y^2}{8})$

Let $A=[1,2],B=[2,1],t=1/2$ .Then

$f(A)=f(B)=5/8, f(tA+(1-t)B)=f(3/2,3/2)=9/16$

$f(tA+(1-t)B)=9/16<5/8=tf(A)+(1-t)f(B)$

i.e., f is not concave because it violates Jensen's inequality for concave functions.

- $f(x,y)=-(x-3)^2-(y-3)^2$

Sum of concave functions=concave function

PROBLEM 6

For all allowed values of the parameters, find all global minima of the following minimization problem, or show that none exist

Objective function $C(x_1,x_2)=w_1x_1+w_2x_2$

constraints $x_1+x_2^2\geq q, x_1\geq 0, x_2\geq 0$

variables x_1,x_2

parameters q,w_1,w_2

conditions on parameters $q>0,w_1>0,w_2>0$

The solution is

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{cases} \begin{bmatrix} q \\ 0 \end{bmatrix} & \text{if} \quad w_1\sqrt{q} < w_2 \\ \begin{bmatrix} q \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \sqrt{q} \end{bmatrix} & \text{if} \quad w_1\sqrt{q} = w_2 \\ \begin{bmatrix} 0 \\ \sqrt{q} \end{bmatrix} & \text{if} \quad w_1\sqrt{q} > w_2 \end{cases} \quad (3)$$

PROBLEM 7

For all allowed values of the parameters, find all global maxima of the following maximization problem, or show that none exist

Objective function $f(x_1, x_2, x_3) = \sqrt{x_1 x_2 x_3} - w_1 x_1 - w_2 x_2 - w_3 x_3$

constraints $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$

variables x_1, x_2, x_3

parameters w_1, w_2, w_3

conditions on parameters $w_1 > 0, w_2 > 0, w_3 > 0$

There is no global maximum, because
$f(t,t,t) = t\left[t^{1/2} - (w_1 + w_2 + w_3)\right] \to \infty$ as $t \to \infty$

PROBLEM 8

For all allowed values of the parameters, find all global maxima of the following maximization problem, or show that none exist

Objective function $U(x, y) = x + \theta \log(y + A)$

constraints $x + py \leq m, x \geq 0, y \geq 0$

variables x, y

parameters θ, A, p, m

conditions on parameters $\theta > 0, A > 0, p > 0, m > 0$

The solution is

$$[x,y]=\begin{cases}[m,0] & \text{if} \quad \theta \le Ap \\ \left[Ap+m-\theta,\dfrac{\theta}{p}-A\right] & \text{if} \quad Ap \le \theta \le Ap+m \\ \left[0,\dfrac{m}{p}\right] & \text{if} \quad Ap+m \le \theta \end{cases} \tag{4}$$

PROBLEM 9

For all allowed values of the parameters, find all global maxima of the following maximization problem, or show that none exist

Objective function $U(x_1,x_2)=x_1x_2$

constraints $x_1+px_2 \le 1+p, x_1 \ge 2, x_2 \ge 0$

variables x_1,x_2

parameters p

conditions on parameters $p>0$

The solution is

$$[x_1,x_2]=\begin{cases}\text{none} & \text{if} \quad p<1 \\ [2,\dfrac{p-1}{p}] & \text{if} \quad 1 \le p \le 3 \\ [\dfrac{p+1}{2},\dfrac{p+1}{2p}] & \text{if} \quad p>3 \end{cases} \tag{5}$$

PROBLEM 10

For all allowed values of the parameters, find all global maxima of the following maximization problem, or show that none exist

Objective function $f(x)=bx=\sum_{i=1}^{n} b_i x_i$

constraints $|x| \le \theta$

variables $x \in \mathbb{R}^n$

parameters $b \in \mathbb{R}^n, \theta \in \mathbb{R}$

conditions on parameters $\theta > 0$.

NOTATION: $|x| = \sqrt{\sum_{i=1}^{n} x_i^2} = \text{euclidean norm of vector } x$

HINT: solve first for n=2, try to guess the solution without using derivatives, then generalize.

The (unique) global maximum is $x = \theta \frac{b}{|b|}$