Takehome 2015

EXERCISE 1

Solve the following problem for all values of the parameters

$$\max \Pi = p x_1^{\alpha} x_2^{\beta} - w_1 x_1 - w_2 x_2 \text{ subject to } x_1 \geq 0, x_2 \geq 0$$
$$\text{variables } x_1, x_2 \text{ parameters } \alpha > 0, \beta > 0, p > 0, w_1 > 0, w_2 > 0$$

Answer exercises 2 and 3 only for those parameter values for which global maxima exist.

EXERCISE 2

Let $\Pi(p, w_1, w_2)$ be the value of the profit function at a global maximum, as computed in exercise 1.

Show that $\Pi(p, w_1, w_2)$ is a convex function of prices (p, w_1, w_2)

EXERCISE 3

Compute the partial derivatives of the profit function $\Pi(p, w_1, w_2)$,as derived in exercise 2, with respect to p, w_1, w_2 .

Compare your answer to the solution of exercise 1.

EXERCISE 4

Let $R_+^n \xrightarrow{f} R$ be a function defined for all $x \in R^n, x \geq 0$ .Assume that

1. $p \in R, p > 0, w \in R^n, w_i > 0\ \forall i=1,..,n$.
2. $F(0) = 0, F(tx) > tF(x), \forall t > 1, \forall x \neq 0$
3. There exists a point $a \in R^n, a \geq 0$ such that $pF(a) - wa > 0$

(We use the inner product notation $wx = w_1x_1 + w_2x_2 + ... + w_nx_n$)

Solve the problem

$$\max \Pi = pF(x) - wx, \text{subject to } x \geq 0$$
$$\text{variables } x, \text{parameters } p, w$$

ANSWERS

Exercise 1

If $\alpha+\beta>1$ then there is NO GLOBAL MAXIMUM, because by setting $x_1 = x_2 = t$ the objective function becomes

$$\pi = p\, t^{\alpha+\beta} - (w_1 + w_2)\, t = t\left[p\, t^{\alpha+\beta-1} - w_1 - w_2\right] \rightarrow \text{infinity}$$

as $t \to$ infinity

If $\alpha+\beta<1$ then the objective function is concave, hence any solution of the necessary conditions is a global max.

The equations $\partial\pi/\partial x_i = 0$, $i=1,2$ have the (unique) solution

$$\boxed{\begin{aligned} x_1 &= \left(\frac{\alpha^{1-\beta}\,\beta^{\beta}\,p}{w_1^{1-\beta}\,w_2^{\beta}}\right)^{\frac{1}{1-\alpha-\beta}} \\ x_2 &= \left(\frac{\alpha^{\alpha}\,\beta^{1-\alpha}\,p}{w_1^{\alpha}\,w_2^{1-\alpha}}\right)^{\frac{1}{1-\alpha-\beta}} \qquad (1) \end{aligned}}$$

GLOBAL MAX WHEN $\alpha+\beta<1$. These are the input demand functions of the firm

The value of the objective function π at the maximum is

$$\boxed{\begin{aligned} \pi &= A\left(\frac{p}{w_1^{\alpha}w_2^{\beta}}\right)^{\frac{1}{1-\alpha-\beta}} \qquad (2) \\ A &= (1-\alpha-\beta)\,\alpha^{\frac{\alpha}{1-\alpha-\beta}}\,\beta^{\frac{\beta}{1-\alpha-\beta}} \end{aligned}}$$

VALUE FUNCTION WHEN $\alpha+\beta<1$

If $\alpha+\beta=1$, then the objective function is concave, hence any solution of the necessary conditions is a global max.

$$\pi(x_1,x_2) = p\,x_1^{\alpha}x_2^{1-\alpha} - w_1x_1 - w_2x_2$$

This is problem 6 in the notes with $\alpha = 1/2$, hence similar results obtain. First, note that there is no maximum with $(x_1>0,\ x_2=0)$, or $(x_1=0,\ x_2>0)$ because

$\pi(x_1, 0) = -w_1 x_1 < 0 = \pi(0,0)$.

Hence if a maximum exists, it will either be

$x_1 = 0$ and $x_2 = 0$, with $\pi = 0$, OR

$x_1 > 0$ and $x_2 > 0$.

If $x_1 > 0, x_2 > 0$ is a global maximum then it satisfies

$$\frac{\partial \pi}{\partial x_1} = 0, \quad \frac{\partial \pi}{\partial x_2} = 0 \quad \text{ie}$$

$$\frac{x_2}{x_1} = \frac{1-\alpha}{\alpha}\frac{w_1}{w_2}, \quad p = \frac{w_1^{\alpha} w_2^{1-\alpha}}{\alpha^{\alpha}(1-\alpha)^{1-\alpha}}, \quad \pi = 0$$

Since Arrow-Enthoven holds, we have

If $\alpha + \beta = 1$ and $p = \dfrac{w_1^{\alpha} w_2^{1-\alpha}}{\alpha^{\alpha}(1-\alpha)^{1-\alpha}}$

$\dfrac{x_2}{x_1} = \dfrac{1-\alpha}{\alpha}\dfrac{w_1}{w_2}$ is a global maximum, and $\pi = 0$

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The other two cases are as in problem 6, ie

$\alpha + \beta = 1$, $p > \dfrac{w_1^{\alpha} w_2^{1-\alpha}}{\alpha^{\alpha}(1-\alpha)^{1-\alpha}}$ $\Rightarrow$ NO MAXIMUM

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$\alpha + \beta = 1$, $p < \dfrac{w_1^{\alpha} w_2^{1-\alpha}}{\alpha^{\alpha}(1-\alpha)^{1-\alpha}}$ $\Rightarrow$ $x_1 = 0 = x_2$ is maximum and $\pi = 0$

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Exercise 2

$\pi = 0$ is CONVEX, AS A CONSTANT, This takes care of the cases $\alpha+\beta=1$

In the case $\alpha+\beta<1$

$$\pi(p, w_1, w_2) = A\left(\frac{p}{w_1^{\alpha} w_2^{\beta}}\right)^{\frac{1}{1-\alpha-\beta}}$$ is convex because its Hessian is positive-semidefinite

Exercise 3

In the case $\alpha+\beta<1$

$$\boxed{\begin{aligned} \frac{\partial \pi}{\partial w_1} &= -\left(\frac{\alpha^{1-\beta}\beta^{\beta}\,p}{w_1^{1-\beta}\,w_2^{\beta}}\right)^{\frac{1}{1-\alpha-\beta}} \\ \frac{\partial \pi}{\partial w_2} &= -\left(\frac{\alpha^{\alpha}\beta^{1-\alpha}\,p}{w_1^{\alpha}\,w_2^{1-\alpha}}\right)^{\frac{1}{1-\alpha-\beta}} \\ \frac{\partial \pi}{\partial p} &= \left(\frac{\alpha^{\alpha}\beta^{\beta}\,p^{\alpha+\beta}}{w_1^{\alpha}\,w_2^{\beta}}\right)^{\frac{1}{1-\alpha-\beta}} \end{aligned} \quad (6)}$$

Comparing (6) with (1) we obtain Hotelling's lemma

$$x_1 = -\frac{\partial \pi}{\partial w_1}$$ DEMAND FUNCTION FOR INPUT 1

$$x_2 = -\frac{\partial \pi}{\partial w_2}$$ DEMAND FUNCTION FOR INPUT 2

$$x_1^{\alpha} x_2^{\beta} = \frac{\partial \pi}{\partial p}$$ SUPPLY FUNCTION

$$\boxed{x_1 \; x_2 = \frac{\partial \pi}{\partial p}}$$ SUPPLY FUNCTION

Exercise 4

There is no global maximum, because for $t > 1$

$$\begin{aligned}
\pi(ta) &= pF(ta) - w(ta) \\
&> t\,pF(a) - t(wa) \\
&= t\,(pF(a) - wa) \\
&= t\,\pi(a) \rightarrow \text{infinity as } t \rightarrow \text{infinity}
\end{aligned}$$